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NON-HOLONOMIC CONSTRAINED SYSTEMS
AS IMPLICIT DIFFERENTIAL EQUATIONS

Abstract. Non-holonomic constraints, both in the Lagrangian and Hamiltonian formalism, are discussed from the geometrical viewpoint of implicit differential equations. A precise statement of both problems is presented remarking the similarities and differences with other classical problems with constraints. In our discussion, apart from a constraint submanifold, a field of permitted directions and a system of reaction forces are given, the later being in principle unrelated to the constraint submanifold. An implicit differential equation is associated to a non-holonomic problem using the Tulczyjew’s geometrical description of the Legendre transformation. The integrable part of this implicit differential equation is extracted using an adapted version of the integrability algorithm. Moreover, sufficient conditions are found that guarantees the compatibility of the non-holonomic problem, i.e., that assures that the integrability algorithm stops at first step, and moreover it implies the existence of a vector field whose integral curves are the solutions to the problem. In addition this vector field turns out to be a second order differential equation. These compatibility conditions are shown to include as particular cases many others obtained previously by other authors. Several examples and further lines of development of the subject are also discussed.

1. Introduction

Non-holonomic constraints have been the subject of deep analysis (not exempt from some controversy) since the dawn of Analytical Mechanics. In fact D’Alembert’s principle of virtual work [10], [26], and Gauss principle of least constraint [17] can be considered to be the first solutions to the analysis of systems with constraints, holonomic or not. A golden age for the subject came with the contributions of O. Hölder [21], G. Hamel [20], P. Appell [1], E. Delassus [12], T. Levi–Civita [27], N. G. Chetaev [9], etc., when the discussion of (linear) non-holonomic constraints in Lagrangian mechanics was considered systematically. After the quantum revolution this classical problem was kept frozen in the limbo of the “postponed” problems. However, there has been papers that from time to time have addressed some of the weak points of the discussions as they were left in the
thirties, for instance, the existence and uniqueness of solutions, the inadequacy of Chetaev's conditions, etc. Some contributions in this transition era can be found in the papers by R.J. Eden [15], V. Valcovici [46], J. Neimark and N. Fufaev [37], R. van Dooren [14], V. Rumiantsev [41], Y. Pironneau [38], etc.

Almost simultaneously with this period a quite revolution was taking place regarding the foundations of the old discipline of Mechanics. Geometry was used in a systematic way to set up its foundations and new and old ideas from Geometry and Mechanics were fusioning in a harmonic and pleasant picture that has been evolving until today and is not completely finished yet. W. Tulczyjew [43] was one of the pioneers of this Geometric revolution and his ideas and insight have inspired many developments in the field as is reflected for instance by the variety of contributions to these Proceedings. Sooner or later the geometrical shock wave had to reach also the problem of non-holonomic constraints. In fact, the first geometrical description of non-holonomic constraints took place in relatively early times in a paper by L. D. Faddeev and A. M. Vershik [16]. The importance of this paper was not recognized until quite recently even if it contained the first general result on the existence of dynamics of Lagrangian systems with (not necessarily linear) non-holonomic constraints. Even previous to this, J. Klein had already addressed the problem of constraints using the geometry of Lagrangian systems [23]. Much recent are the papers by E. Massa et al discussing the geometrical meaning of Chetaev's conditions [34]; G. Giachetta [19] and M. de León et al [30] using jet bundle geometry techniques; J. Koiller [24], L. Bates et al [3] and Bloch et al [6] discussing non-holonomic constraints with symmetry. More contributions can be found in the papers by L. Cushman et al discussing the geometrical meaning of Chetaev's conditions; J. Cariñena & M. F. Rañada [8] geometrizing Lagrange's multipliers; R. Weber [48], A. Van der Shaft et al [47], P. Dazord [11] and C. M.-Marle [33] addressing the problem from the Hamiltonian viewpoint; Sarlet et al [42] using connection theory; de León et al [29] using almost product structures and projectors, F. Barone et al [2] using Tulczyjew's ideas to set the frame for generalized Lagrangian systems, F. Cardin et al combining the vakonomic and non-holonomic approach, and finally C.M.-Marle in these Proceedings revisiting Faddeev-Vershik conditions.

Our modest contribution to this old subject will consist in using Tulczyjew's idea of modelling mechanical systems as implicit differential equations [36], [32], [35], to discuss non-holonomic constraints. This approach was already taken by W. Tulczyjew himself [45, 25], and by S. Benenti [5] who was able to obtain a set of sufficient conditions for the existence of dynamics for linear non-holonomic constraints and apply it successfully to solve the problem of constrained geodesics. In our approach we will compare first the problem of Lagrangian systems with non-holonomic constraints with the problem of constrained Hamiltonian systems. We will set up the geometry of Lagrangian and Hamiltonian systems with non-holonomic constraints and after a brief discussion of implicit differential equations and the integrability algorithm, we will apply it to the non-holonomic problem. We will find that a small modification of the integrability algorithm is needed to encompass the
restrictions imposed by non-holonomic constraints to the solutions of the problem. From
the analysis of the adapted integrability algorithm some particular cases arise immediately.
The simplest non-trivial case is discussed thoroughly and a general condition for the
existence of a solution, that turns out to be a vector field, is discussed. This geometrical
condition contains as particular cases Faddeev-Vershik's condition, Chetaev's conditions,
Bates' regularity condition, Benenti's conditions, Marle's conditions, etc.

We will apply these ideas to discuss several examples that are inspired in old models
like Appell's machine, etc.

2. Non-holonomic constrained systems

The discussion to follow of non-holonomic constraints will be set in the realm of
Tulczyjew's triple [44],

\[ T^*(TQ) \leftarrow^\alpha T(T^*Q) \rightarrow^\beta T^*(T^*Q). \]

We will recall that \( \alpha \) is the Tulczyjew's canonical symplectomorphism from \( T(T^*Q) \) (with
its canonical symplectic structure \( \omega_{TQ} \)) to \( T^*(TQ) \) (with its canonical structure \( \omega_{TQ} \)), and
\( \beta \) is the canonical symplectomorphism defined by the former symplectic structure.

The problem of constrained Hamiltonian systems consists in determining the
equations of motion for a system specified by a constraint submanifold \( C \subset T^*Q \) and a
Hamiltonian function \( H: C \rightarrow \mathbb{R} \). The submanifold \( C \) is usually, but not always, determined
by a self-consistency condition of the system under study and it is often the result of a
Dirac's type constraint algorithm [13, 18]. The submanifold

\[ D = \{ v \in T_C(T^*Q) \mid \omega_{TQ}(v, u) = (dH, u), \forall u \in TC \}, \]

defines an implicit differential equation on \( T(T^*Q) \), called a Dirac system, whose analysis
provide the solution of the posed problem. Such analysis was succesfully done in [36] and
a set of necessary and sufficient conditions for its integrability was given in the following
theorem.

**Theorem 1.** The Dirac system \( D \) defined by a constrained Hamiltonian system
is integrable iff \( C \) is coisotropic and \( H \) projects along the characteristic distribution of \( C \).

Despite the similarities between the Hamiltonian constrained problem and the
problem of non-holonomic constraints, the latter has a different nature. In the non-
holonomic Lagrangian problem a Lagrangian \( L \) will be given in all \( TQ \). This Lagrangian
defines the unconstrained or "free" system. In fact, we can assume in what follows that
\( L \) is regular. This means that on \( TQ \) we have a dynamical vector field given by the
Euler–Lagrange vector field \( \Gamma_L \) defined by the equation,

\[ L_{\Gamma_L} \Theta_L = dL, \]

or equivalently \( (L \) regular),

\[ i_{\Gamma_L} \omega_L = dE_L, \]
where $\Theta_L = S^*(dL)$ is the Poincaré-Cartan 1-form defined by $L$, $\omega_L = -d\Theta_L$ is the Cartan 2-form and $E_L = \Delta(L) - L$ is the energy of $L$, with $\Delta$ the Liouville vector field on $TQ$.

The canonical tensor field $S$ on $TQ$ is defined in local natural coordinates $(q^i, \nu^i)$ on $TQ$ by

$$S = \frac{\partial}{\partial \nu^i} \otimes dq^i,$$

and it allows us to characterize some as $S(\Gamma) = \Delta$. The kernel and image of $S$ consists of vertical vector fields. On the other hand $S$ acts by duality on forms and the kernel and image of $S^*$ consists on horizontal 1-forms. A remarkable property of $S$ and $\omega_L$ is given by the formula,

$$i_{\omega_L} = 0,$$

or equivalently,

$$(2.4) \quad S^* \circ \dot{\omega}_L = -\dot{\omega}_L \circ S,$$

where $\dot{\omega}_L$ denotes the map $T(TQ) \to T^*(TQ)$ defined by contraction with $\omega_L$.

Notice that $L$ being regular implies that the 2-form $\omega_L$ is nondegenerate. Consequently we can define its inverse bivector $\Lambda_L$ as,

$$(2.5) \quad \Lambda_L(\alpha, \beta) = \omega_L(\dot{\omega}_L^{-1}(\alpha), \dot{\omega}_L^{-1}(\beta)), \forall \alpha, \beta \in T^*(TQ).$$

Now for reasons that in principle have no relation with the Lagrangian $L$, a submanifold $C \subset TQ$ will be selected, the constraint submanifold of the problem. If the submanifold $C$ is not of the form $TN$ with $N \subset Q$, or more generally, if $C$ is not the total space of an integrable distribution $\mathcal{D}$ defined along a submanifold $N \subset Q$, we will say that the constraints are non-holonomic. In general $\Gamma_L$ will not be tangent to $C$, this is, the Lagrangian system defined by $L$ will evolve in time without keeping within the limits imposed by $C$. If we want the system to remain in $C$ then $\Gamma_L$ must be changed. For that we will assume that there is a set of “forces” $F \subset T^*(TQ)$ that allow us to act upon the dynamical system $\Gamma_L$ and eventually to make it to be confined to $C$.

The modified systems that we can obtain from $\Gamma_L$ by means of the forces $F$ are given by the family of vector fields,

$$(2.6) \quad \Gamma \in \Gamma_L + \Lambda_L(F),$$

where $\Lambda_L$ is the Poisson tensor defined by the Cartan 2-form $\omega_L$, eq. (2.5). Notice that Euler-Lagrange’s equations (2.3) in the presence of external forces $f = f_i(q, v) dq^i$ are modified as

$$L_L \Theta_L = dL + f,$$

thus, spanning the Lie derivative, we will get that if $f \in F$, then $\Gamma$ verifies eq. (2.6).

If we want the resulting system to be a system of “mechanical” type, the system of “forces” will have to be given by horizontal 1-forms, i.e., $S^*(F) = 0$. Then a simple
computation shows that \( \Gamma \) is a SODE because \( S(A_L(F')) = A_L(S^*(F')) = 0 \) where we have used eqs. (2.4) and (2.5).

The system of forces \( F \) will be obtained either by a detailed analysis of the constraint submanifold or they will be given in an independent way. For instance, a large class of non-holonomic constraints are originated by the interaction between the surfaces of different components of the system. In this category fall sliding, rolling and friction constraints, which are supposed to create linear relations between the velocities of the components of the system. However, we can also imagine that our system is subjected to the action of servomechanisms or other devices, that modify its dynamical state in such a way that certain "a priori" given conditions are satisfied, for instance, limitations in the acceleration of the center of mass of the system. Very often the system of forces \( F \) can be postulated from the non-holonomic constraints, i.e., in some cases, they will be supposed to be created by the constraints and they will be supposed to have an explicit relation with them. This is the case of the so called Chetaev's conditions for non-holonomic constraints (see Section 5). Some of these possibilities will be explored later on. In this sense the system of forces will be called reaction or control forces, depending if they are derived from the constraints or are imposed externally.

Thus the Lagrangian non-holonomic constraint problem can be stated as follows: given a regular Lagrangian \( L \), a constraint submanifold \( C \) and a system of forces \( F \), determine whether or not there exists a vector field \( \Gamma \) of the form given by eq. (2.6) parallel to \( C \).

This problem admits a simple generalization that is of interest in a variety of situations. We can replace the symplectic manifold \((TQ, \omega_L)\) by a Poisson manifold \((P, \Lambda)\) and the dynamical vector field \( \Gamma_L \) by a Hamiltonian vector field \( \Gamma_H = \Lambda(dH) \). We will consider a constraint submanifold \( C \subset P \) and, contrary to the situation discussed above, we can think that a further restriction on the possible directions of the vector field \( \Gamma_H \) can be imposed, i.e., a field of allowed or permitted directions \( D \) along \( C \) will be introduced\(^1\). The geometrical model for it will consists in a vector subbundle \( p: D \to C \) of \( \tau_P: TP \to P \) (see Fig. 1). The reaction forces now will be modelled by a subbundle \( \eta: F \to C \) of \( \pi_P: T^*P \to P \). Then, the generalized non-holonomic problem can be stated as follows:

**Poisson non-holonomic constraint problem:** Given a Hamiltonian vector field \( \Gamma_H \) on a Poisson manifold \( P \) with Poisson tensor \( \Lambda \), a constraint submanifold \( C \), a field of permitted directions \( D \) and a system of control forces \( F \), determine if there is a vector field \( \Gamma \) of the form

\[
\Gamma \in \Gamma_H + \Lambda(F),
\]

contained in \( D \).

\(^1\)If we are discussing a Lagrangian system, such restriction can be understood as a limitation on the accelerations of the system.
We should point it out that in both versions of the non-holonomic constraint problem the sought vector field can exist or not; it can exist only on a subset of the constraint submanifold, and even if it exists, it is not necessarily unique. It is clear that all kind of possibilities can actually occur and it is not difficult to exhibit examples of them [35]. The discussions in Sections 3 and 4 address the existence and uniqueness of such dynamical vector fields. Particular situations will be also discussed where explicit conditions can be done that will allow to give definite answers to them.

3. Implicit differential equations

The main tool to analyze the general problems discussed above will be the geometrical setting developed in [35] to describe implicit differential equations.

An implicit differential equation on the manifold \( P \) is a submanifold \( E \subset TP \). A solution of \( E \) is any curve \( \gamma: I \rightarrow P, I \subset \mathbb{R} \), such that the tangent curve \( (\gamma(t), \dot{\gamma}(t)) \in E \) for all \( t \in I \). The implicit differential equation will be said to be integrable at a point if there exists a solution \( \gamma \) of \( E \) such that the tangent curve passes through it. The implicit differential equation will be said to be integrable if it is integrable at all its points. Integrability does not imply uniqueness. The integrable part of \( E \) is the subset of all integrable points of \( E \). The integrability problem consists in identifying such subset.

Denoting as before the canonical projection \( TP \rightarrow P \) by \( \tau_P \), a sufficient condition for the integrability of \( E \) is

\[ E \subset TC, \]

where \( C = \tau_P(E) \), provided that the projection \( \tau_P \) restricted to \( E \) is a submersion onto \( C \).

Extracting the integrable part of \( E \): A recursive algorithm was presented in [35] that allows to extract the integrable part of an implicit differential equation \( E \). We shall define the submanifolds

\[ E_0 = E, \ C_0 = C, \]

and recursively for every \( k \geq 1 \),

\[ E_k = E_{k-1} \cap TC_{k-1}, \ C_k = \tau_P(E_k), \]
Non-holonomic constrained systems

then, eventually the recursive construction will stabilize in the sense that \( E_k = E_{k+1} = \cdots = E_\infty \), and \( C_k = C_{k+1} = \cdots = C_\infty \). It is clear by construction that \( E_\infty \subseteq TC_\infty \). Then, provided that adequate regularity conditions are satisfied during the application of the algorithm, the implicit differential equation \( E_\infty \) will be integrable and it will solve the integrability problem.

In what follows we will refer to the non-holonomic problem defined by the Lagrangian \( L \), the constraint submanifold \( C \), the distribution \( D \) and the system of forces \( F \), as the system \( (L, C, D, F) \).

4. Integrability of non-holonomic systems

4.1. The implicit differential equation associated to a system with non-holonomic constraints

The implicit differential equation associated to a non-holonomic system can be described in the Lagrangian and/or the Hamiltonian formalism.

Lagrangian picture: We will be given the data defining a non-holonomic constrained Lagrangian system, i.e., a regular Lagrangian \( L \), a field of permitted directions \( D \subseteq TC(TQ) \) along the constraint submanifold \( C \subseteq TQ \) and a system of forces \( F \subseteq TC(TQ) \). Then, the implicit differential equation associated to \( L, D, F \) will be defined by

\[
E = \alpha^{-1}(dL + F) \subseteq T(T^*Q).
\]

Notice that \( \pi_T \cdot Q(E) = C^* \) is the constraint submanifold considered in the problem and it coincides with \( T\mathcal{F}_L(C) \) where \( \mathcal{F}_L \) denotes the Legendre transformation defined by \( L \) (see below). It is also noticeable that \( dL + F \) defines an affine subbundle of \( T^*(TQ) \) along \( C \). The regularity of \( L \) guarantees the transversality of the submanifold \( E \) with respect to the projection map \( \pi_T \cdot Q \).

Hamiltonian picture: As in the Lagrangian case, a non-holonomic constrained Hamiltonian system is defined by a Hamiltonian function \( H \) on \( T^*Q \), a permitted directions field \( D \subseteq TC^*(T^*Q) \) along the submanifold \( C^* \subseteq T^*Q \) and a system of forces \( F \subseteq T_{C^*}^*(T^*Q) \). The implicit differential equation associated to the data \( (H, C^*, D, F) \) is defined by

\[
E = \beta^{-1}(dH + F),
\]

that projects onto \( C^* \).

4.2. Non-holonomic integrability algorithm

If we apply the integrability algorithm given by eqs. (3.2)-(3.3) to the implicit differential equation associated to a non-holonomic Lagrangian system given by eq. (4.1)

\[2\]

we will obtain,

\[
E_0 = E = \alpha^{-1}(dL + F), \quad C_0 = \pi_T \cdot Q(E) = C^*,
\]

\[
E_1 = E_0 \cap TC_0, \quad C_1 = \pi_T \cdot Q(E_1), \ldots
\]

Notice that a similar algorithm can be applied to non-holonomic Hamiltonian problems, eq. (4.2).
and eventually \( E_\infty \subset TC_\infty \). However we must notice that the stability of the algorithm does not imply automatically that \( E_\infty \) is going to define a vector field lying in \( D \) and the integrable equation will not be in general a solution of the non-holonomic problem.

Then, the integrability algorithm needs a small adaptation to render the required outcome in the context of non-holonomic systems. We need to translate the field of allowed directions to \( T^*Q \). The natural way to do that is by means of the Legendre transformation \( T_L \) defined by \( L \). The derivative of this map \( T_T: T(T^*Q) \rightarrow T(T^*Q) \) allows to define the bundle of permitted directions on \( T^*Q \) as

\[
\hat{D} = T_T(D) \subset T(T^*Q),
\]

then we will construct an adapted integrability algorithm as follows,

\[
\begin{align*}
E_0 &= E, \\
C_0 &= C^*, \\
\hat{D}_0 &= \hat{D}, \\
E_1 &= E_0 \cap \hat{D}_0, \\
C_1 &= \tau_{T^*Q}(E_1), \\
\hat{D}_1 &= \hat{D}_0 \cap C_1, \\
E_k &= E_{k-1} \cap \hat{D}_{k-1}, \\
C_k &= \tau_{T^*Q}(E_k), \\
\hat{D}_k &= \hat{D}_{k-1} \cap C_k,
\end{align*}
\]

and then, eventually when the algorithm stops, we will obtain the stable data \( E_\infty, \hat{D}_\infty, C_\infty \) defining an integrable implicit differential equation (see Fig. 2).

Notice that \( E_\infty \subset \hat{D}_\infty \subset TC_\infty \) which together with the appropriate regularity assumptions guarantees the integrability of \( E_\infty \) and its compatibility with the non-holonomic constraints of the problem.

Obviously all kind of situations for \( E_\infty, \hat{D}_\infty, C_\infty \) can happen. The simplest

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Figure 2: The non-holonomic integrability algorithm.
nontrivial\textsuperscript{3} situation occurs when $E_1 = E_\infty$, $C^* = C_\infty$ and $\dot{D}_\infty = \dot{D}$. Then, the algorithm stops in the first step and,

\begin{equation}
(4.3) \quad E_1 = E \cap \dot{D} = E \cap TF_L(D), \quad C_1 = \tau_{T^*Q}(E_1) = C^*, \quad \dot{D}_1 = \dot{D} \cap TC^* = TF_L(D).
\end{equation}

We will say that the regular Lagrangian $L$ and the non-holonomic constraints $D$, $F$ are compatible if the conditions above, eq. (4.3), are met, i.e., the adapted integrability algorithm for them stops at the first step. In this case, the integrable implicit differential equation defined by them are the submanifold of $T(T^*Q)$ given by,

\begin{equation}
(4.4) \quad E_\infty = \alpha^{-1}(dL + F) \cap TF_L(D),
\end{equation}

and it gives the solution of the non-holonomic Lagrangian problem $(L, C, D, F)$.

We must notice here that the outcome of the integrability algorithm does not necessarily provide us with a SODE. If we require that the solution must be a SODE, then we must modify the algorithm again to incorporate this fact. This problem has been exhaustively discussed in the setting of Lagrangians systems were it is known as the SODE problem for singular Lagrangians.

4.3. Compatibility conditions

It is important to remark again that the previous characterization of compatibility, eq. (4.3), does not imply the uniqueness of solutions passing through its points. This is a general feature of non-holonomic constraints often encountered in practical discussions. It is also important to notice again that even if the solutions were unique, i.e., the implicit differential equation $E_\infty$ would define a bona fide differential equation, they would not be in general necessarily solutions of a SODE\textsuperscript{4}. The following theorem gives sufficient conditions for the non-holonomic Lagrangian problem $(L, C, D, F)$ to be compatible and, as a bonus, it is found that they also guarantee the uniqueness and the second order character of it.

**Theorem 2.** Let $(L, C, D, F)$ be a non-holonomic Lagrangian problem. If

\begin{equation}
(4.5) \quad \text{rank}(F) = \text{corank}(D),
\end{equation}

and

\begin{equation}
(4.6) \quad D^0 \cap F^\perp = 0,
\end{equation}

then the Lagrangian and the non-holonomic constraints are compatible. Moreover, there exists a unique SODE $\Gamma$ such that its graph is the integrable implicit differential equation $\alpha^{-1}(dL + F) \cap TF_L(D)$ associated to it.

\textsuperscript{3}Nontrivial here means that $C$ is actually a submanifold of $TQ$ of codimension at least one and $F$ is a subbundle of rank at least one.

\textsuperscript{4}It can be shown that for regular Lagrangians and permitted directions field $D$ such that $\in S(D)$ the solution is a SODE.
Proof. The symbol \( D^0 \) denotes the annihilator of \( D \), i.e.,
\[
D^0 = \{ a \in T^*_C(TQ) | \langle a, D \rangle = 0 \},
\]
and \( F^\perp \) denotes the symplectic orthogonal to \( F \) with respect to the Poisson tensor \( \Lambda_L \), that is,
\[
F^\perp = \{ f \in T^*_C(TQ) | \Lambda_L(f, F) = 0 \}.
\]
The vector fields constructed from \( \Gamma_L \) to solve the Lagrangian non-holonomic problem have the form
\[
\Gamma = \Gamma_L + \Lambda_L(f), \; f \in F.
\]
If \( \Gamma \in D \), then pairing the previous equation with elements in \( D^0 \), we will get
\[
\Lambda_L(a, f) = \langle \Gamma_L, a \rangle, \; \forall a \in D^0.
\]
Consider the map \( \phi: F \rightarrow (D^0)^* \) defined by \( \phi(f)(a) = \Lambda_L(a, f) \), for all \( a \in D^0 \), whose kernel is \( F \cap (D^0)^\perp \). Notice that \( (F \cap (D^0)^\perp)^\perp = F^\perp + D^0 \). But \( \dim(F^\perp + D^0) = \dim F^\perp + \dim D^0 \) because of eq. (4.6), then,
\[
\dim(F \cap (D^0)^\perp) = 2n - (\dim F^\perp + \dim D^0) = 2n - (2n - \text{rank} F) - \text{corank} D = 0.
\]
Then we conclude that \( F \cap (D^0)^\perp = 0 \) and thus conditions (4.5)-(4.6) imply the injectivity of the map \( \phi \). Moreover, again because of \( \text{rank}(F) = \text{corank}(D) \), the previous map is surjective. Hence the map \( \phi \) is a bundle isomorphism and there exists a unique element \( f \in F \) such that \( \phi(f) = (\Gamma_L, \_ \_ \_) \). Then there exists a unique vector field that gives a solution of the non-holonomic Lagrangian problem, i.e., the submanifold \( E \) cuts \( T\mathcal{F}_L(D) \) along a submanifold which is the graph of the Legendre transform of the vector field \( \Gamma \). Moreover, because the forces \( f \) a horizontal 1–forms, the vector field \( \Lambda_L(f) \) is vertical and \( \Gamma \) is a sode. 

**Remark.** It follows from the proof of the previous theorem that the conditions for compatibility can also be written as
\[
\text{rank}(F^\perp) = \text{corank}(D^0), \; F \cap (D^0)^\perp = 0,
\]
or they can be simultaneously encoded as
\[
T^*_C(TQ) = F^\perp \oplus D^0.
\]

4.4. Local expressions

Because of its practical interest we shall write the conditions in Thm. 2 in local coordinates.

We will suppose that the submanifold \( C \) is defined locally by the set of functions
\[
\phi^a(q, v) = 0, \; a = 1, \ldots, r.
\]
Then \( \dim C = 2n - r \). The annihilator of the subbundle \( D \subset TC \) will verify \((TC)^0 \subset D^0\). Thus, we will decompose \( D^0 = (TC)^0 \oplus K \). We will suppose that \( K \) is a subbundle with a local basis in the chart \( q^i, v^i \) given by the set of 1-forms

\[
\beta^b = \gamma^b_i dq^i + \beta^b_i dv^i, \quad b = 1, \ldots, s.
\]

A system of local 1-forms generating \((TC)^0\) is given by \( d\phi^a, \ a = 1, \ldots, r \). Thus, \( D^0 \) will be locally generated by the set of 1-forms \( d\phi^a, \beta^b \), that will be collectively denoted by

\[
\beta^C = \gamma^C_i dq^i + \beta^C_i dv^i, \quad C = 1, \ldots, r + s.
\]

The corank of \( D \) will then be \((r + s)\):

We will now assume that the system of forces \( F \) will be defined by a subbundle of rank \( r + s \), satisfying thus eq. (4.5), and it will have a system of local generating horizontal 1-forms \( f^C = f^C_i dq^i, \ C = 1, \ldots, r + s \). The local expression of the Poisson tensor \( \Lambda_L \) is given by

\[
\Lambda_L = W^{ij} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j} + M^{ij} \frac{\partial}{\partial v^i} \wedge \frac{\partial}{\partial v^j}.
\]

The intersection condition (4.6) is equivalent to the statement that \( \Lambda_L \) defines a nondegenerate pairing between \( F \) and \( D^0 \), i.e., that the matrix

\[
M^{BC} = W^{ij}(q, v)f^B_i(q, v)\beta^C_j(q, v),
\]

is nondegenerate. In the particular case that \( D = TC \), i.e., there are no limitation to the permitted directions for the vector field, then the set of 1-forms spanning \( D^0 \) are given by \( d\phi^a, a = 1, \ldots, r \). Then, the compatibility condition becomes,

\[
0 \neq \det M^{ab} = \det W^{ij}(q, v)f^a_i(q, v)\frac{\partial \phi^b}{\partial v^i}.
\]

**Linear non-holonomic constraints:**

The situation that usually is dealt with is when the submanifold \( C \) is an affine subbundle \( A \) of \( TQ \), i.e., locally, \( C \) is defined by local functions,

\[
\phi^a(q, v) = \mu^a_i(q)v^i + \mu^a_0(q) = 0, \quad a = 1, \ldots, r.
\]

The tangent submanifold \( TA \subset T_A(TQ) \) is defined by the set of 1-forms \( d\phi^a \), i.e., \((TA)^0 = \text{span}\{d\phi^a | a = 1, \ldots, r \}\). Hence, because

\[
d\phi^a = \mu^a_i(q)dv^i + \left(v^j \frac{\partial \mu^a_i}{\partial q^j} + \frac{\partial \mu^a_0}{\partial q^j} \right) dq^i,
\]

namely,

\[
\frac{\partial \phi^a}{\partial v^i} = \mu^a_i(q),
\]

and the compatibility condition becomes,

\[
(4.7) \quad \det M^{ab} = \det W^{ij}(q, v)f^a_i(q, v)\mu^a_j(q) \neq 0.
\]
If, in addition we accept Chetaev's forces (see next Section), i.e., we define,

\( F = S^*(TA)^0 \),

the subbundle \( F \) will be generated by the 1-forms \( \mu^a = \mu^a_i(q)dq^i \) because \( S^*(d\phi^a) = \mu^a_i(q)dq^i \), and the compatibility condition will be simply given by

\[ \det M^{ab} = \det W^{ij}(q)\mu^a_i(q)\mu^b_j(q) \neq 0, \]

reproducing results in [29]. Notice that if \( L \) is a mechanical Lagrangian \( L = T - V \), where \( T \) is the kinetic energy corresponding to a Riemannian metric \( g \), then,

\[ M^{ab} = g^{ij}\mu^a_i(q)\mu^b_j(q), \]

which is obviously invertible.

5. Comparison with previous results

We have already seen in the previous section, that particular choices of the set of reaction forces \( F \), for instance those given in eq. (4.8), give special expressions for the compatibility conditions discussed in Section 4 before, Thm. 2. A special choice for the reaction forces is given by Chetaev's conditions. In intrinsic terms the system of forces corresponding to a non-holonomic constraint submanifold \( C \subset TQ \) under Chetaev's conditions is given by the subbundle (see also [34]),

\( F = S^*((TC)^0). \)

More generally, if we consider a bundle of permitted directions \( D \rightarrow C \), then Chetaev's bundle of reaction forces will be given by (see also [29]),

\( F = S^*(D^0). \)

Following [16] we will say that \( D \) is admissible if \( S^* \) is injective when restricted to \( D^0 \), i.e., \( D^0 \) does not contain horizontal 1-forms. This implies that the rank of the Chetaev bundle coincides with the rank of \( D^0 \), i.e., \( \text{rank} F = \text{corank} D \). Hence, Faddeev's admissibility condition is a particular instance of the condition (4.5) in Thm. 2.

The Hessian of the Lagrangian \( L \) can be defined as the symmetric \((0,2)\)-tensor,

\[ H_L(\alpha, \beta) = \Lambda_L(S^*(\alpha), \beta), \quad \alpha, \beta \in T^*(TQ). \]

Then, we will say that \( L \) is definite if the Hessian \( H_L \) is definite as a symmetric tensor. Then, if \( L \) is definite, \( H_L \) is nondegenerate when restricted to the subspace \( D^0 \), but \( H_L(D^0, D^0) = \Lambda_L(S^*(D^0), D^0) = \Lambda_L(F, D^0) \), and \( \Lambda_L \) is nondegenerate on the pair \( F, D^0 \), i.e., \( F^\perp \cap D^0 = 0 \). Then if \( L \) is definite ("normal" according to the terminology in [33]) this implies condition (4.6). Then Thm. 2 implies the main result in [16], [29].

\[^5\text{Notice that } H_L \text{ is symmetric because of eq. (2.4)}\]
Theorem 3. Let \( L \) be a definite Lagrangian and a non-holonomic constraint defined by the permitted field of directions \( D \rightarrow C \). If \( D \) is admissible and the reaction forces are given by the Chetaev's bundle \( F = S^*(D^0) \), then there exists a unique SODE solution of the non-holonomic Lagrangian problem defined by \( L \) and \( D \rightarrow C \).

A different condition is used in [3] and [29] to find a solution of the Lagrangian non-holonomic problem. Now we will assume that a submanifold \( C \subset TQ \) is given and \( D = TC \). Let \( S_C \) be the distribution

\[
S_C = S(TC^\perp).
\]

A Lagrangian system \( L \) with a non-holonomic constraint is said to be regular if

\[
S_C \cap TC = 0.
\]

The following theorem is again a particular instance of Thm. 2.

Theorem 4. If \((L, C)\) is a regular non-holonomic Lagrangian system, then it has a unique solution. In addition this solution is a SODE.

Proof The analysis of the regularity condition \( S_C \cap TC = 0 \) leads to,

\[
0 = \dot{\omega}_L(S_C \cap TC) = \dot{\omega}_L(S_C) \cap \dot{\omega}_L(TC) = \dot{\omega}_L(S(TC^\perp)) \cap \dot{\omega}_L(((TC)^0)^0),
\]

where as before \( \dot{\omega}_L \) is the natural bundle map \( T(TQ) \rightarrow T^*(TQ) \) defined by contraction with the 2-form \( \omega_L \). Then it is obvious that \( \dot{\omega}_L(TC^\perp) = TC^0 \). In fact, \( \alpha \in \dot{\omega}_L(TC^\perp) \) iff there is \( u \in TC^\perp \) such that \( \dot{\omega}_L(u) = \alpha \), but \( \omega_L(u,TC) = 0 \), then \( \alpha \in TC^0 \) and conversely. Then, continuing with the computation in eq. (5.5) we get,

\[
\dot{\omega}_L(S(TC^\perp)) \cap \dot{\omega}_L(((TC)^0)^0) = S^*(\dot{\omega}_L(TC^\perp)) \cap (TC^0)^\perp
\]

\[
= S^*(TC^0) \cap (TC^0)^\perp = F \cap (D^0)^\perp,
\]

with \( F = S^*(TC^0) \) and \( D = TC \), and again we are in the conditions of Thm. 2.

As it was pointed before, a similar approach to this paper was adopted by S. Benenti in [5]. There, the problem of linear non-holonomic Lagrangian or Hamiltonian systems was addressed and the compatibility conditions exhibited there (Props. 2 and 3) are equivalent to the regularity of \( L \) and eq. (4.9). The nonlinear case is also discussed and the conditions in Prop. 4 of [5] are equivalent respectively to the regularity of the Lagrangian, the rank condition in Thm. 2, and the regularity of the matrix \( M^{ab} \) in eq. (4.7) plus Chetaev's conditions; hence the conclusions there follow from Thm. 2.

To end this section we will comment on the characterization given by C.M. Marle in [33]. The subbundles \( W \) (called the projection bundle) and \( TD \), the tangent bundle of the Hamiltonian constraint submanifold, are \( W = TFE(S_C) \) and \( D = C^* \), using the notation in this paper. Then, \( W \oplus TD \) is the translation to \( T^*Q \) of the regularity condition eq. (5.4). Then, the existence and uniqueness of solutions stated in [33] (Prop. 2.15 and Thm. 2.16) are again particular instances of the previous results.
6. Examples and applications

6.1. Disk rolling on a surface

As a simple example we will discuss first a disk rolling vertically on a rough surface. The disk of mass $M$ and radius $R$ will be described by the coordinates $x, y$ of the contact point with the surface, the angle $\theta$ defined by the plane containing the disk and a fixed plane normal to the surface, and the angle $\phi$ parametrizing the position of the disk with respect to its center (see also Fig. 3). The configuration space of the system will be $Q = \mathbb{R}^2 \times \mathbb{T}^2$. The Lagrangian describing the free system is,

\begin{equation}
L = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 \dot{\phi}^2,
\end{equation}

with $I_1, I_2$ the corresponding moments of inertia.

The constraint submanifold $C \subset TQ$ is given by the rolling conditions,

\begin{equation}
\Psi_1 = \dot{x} - R \cos \theta \dot{\phi} = 0; \quad \Psi_2 = \dot{y} - R \sin \theta \dot{\phi} = 0.
\end{equation}

In the cotangent bundle $T^*Q = T^*\mathbb{R}^2 \times T^*\mathbb{T}^2$ we introduce coordinates $(x, y, \theta, \phi, p_x, p_y, p_\theta, p_\phi)$, and finally in $T(T^*Q)$ we will have coordinates $(x, y, \theta, \phi, p_x, p_y, p_\theta, p_\phi; \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}, \dot{p}_x, \dot{p}_y, \dot{p}_\theta, \dot{p}_\phi)$. The annihilator of $TC$ on $T^*_C(TQ)$ will be spanned by $d\Psi_1, d\Psi_2$. Using Chetaev's forces, we will have,

\begin{equation*}
F = \{ \lambda_1 (dx - R \cos \theta d\phi) + \lambda_2 (dy - R \sin \theta d\phi) \mid \lambda_1, \lambda_2 \in \mathbb{R} \}.
\end{equation*}

Thus, the affine subbundle $dL + F \subset T^*_C(TQ)$ is given by,

\begin{equation*}
dL + F = \{ M \dot{x} dx + M \dot{y} dy + I_1 \dot{\phi} d\phi + I_2 \dot{\phi} d\phi + \\
\quad \quad + \lambda_1 (dx - R \cos \theta d\phi) + \lambda_2 (dy - R \sin \theta d\phi) \mid \lambda_1, \lambda_2 \in \mathbb{R} \}.
\end{equation*}

The map $\alpha: T(T^*Q) \to T^*(TQ)$ is defined in coordinates $(q^i, p_i; q^i, p_i)$ for $T(T^*Q)$ and $(q^i, v^i; r_i, s_i)$ by

\begin{align*}
s_i \circ \alpha &= p_i, & r_i \circ \alpha &= \dot{p}_i, & v^i \circ \alpha &= \dot{q}^i, & q^i \circ \alpha &= q^i,
\end{align*}

or equivalently, $\alpha(q^i, p_i, q^i, p_i) = (q^i, \dot{q}^i, \dot{p}_i, p_i)$. Then, the submanifold $E = \alpha^{-1}(dL + F)$ is given by the equations,

\begin{equation*}
E = \{ (x, y, \theta, \phi, p_x, p_y, p_\theta, p_\phi; \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}, \dot{p}_x, \dot{p}_y, \dot{p}_\theta, \dot{p}_\phi) \in T(T^*Q) \mid \\
\quad \quad \dot{p}_x = \lambda_1, \quad \dot{p}_y = \lambda_2, \quad \dot{p}_\theta = -\lambda_1 R \cos \theta - \lambda_2 R \sin \theta, \quad \dot{p}_\phi = 0, \quad p_x = M \dot{x}, \quad p_y = M \dot{y}, \\
\quad \quad p_\theta = I_1 \phi, \quad p_\phi = I_2 \phi, \quad \Psi_1 = 0, \quad \Psi_2 = 0 \}.
\end{equation*}

The Legendre transformation $T_L: TQ \to T^*Q$ is given by

\begin{equation*}
p_x = M \dot{x}, \quad p_y = M \dot{y}, \quad p_\theta = I_1 \phi, \quad p_\phi = I_2 \phi,
\end{equation*}

and maps the submanifold $C$ into the submanifold $C^* \subset T^*Q$ given by

\begin{equation*}
C^* = \{ (x, y, \theta, \phi; p_x, p_y, p_\theta, p_\phi) \in T^*Q \mid I_1 p_x = M R \cos \theta p_\phi, I_1 p_y = M R \sin \theta p_\phi \}.
\end{equation*}
It is clear that $\tau^*Q(E) = C^*$, and thus $E \cap TC^* = E_1$ and the integrability algorithm stops. Computing the intersection of $TC^*$ with $E$ we obtain,

$$\lambda_1 = -\frac{MR\sin \theta}{I_1 I_2} p_\phi \rho, \quad \lambda_2 = -\frac{MR\cos \theta}{I_1 I_2} p_\phi \rho, \quad \dot{\rho} = 0, \quad \dot{\rho} = 0.$$  

In this form we will obtain the following vector field $\Gamma$ defined on $C^*$,

$$\Gamma = \frac{R \cos \theta}{I_1} \frac{\partial}{\partial x} + \frac{R \sin \theta}{I_1} \frac{\partial}{\partial y} + \frac{p_\phi}{I_1} \frac{\partial}{\partial \phi} + \frac{p_\rho}{I_2} \frac{\partial}{\partial \rho},$$  

which is mapped by $\mathcal{F}_L^{-1}$ into a SODE, or equivalently the following set of equations of motion,

$$\dot{x} = \frac{R \cos \theta}{I_1} p_\phi, \quad \dot{y} = \frac{R \sin \theta}{I_1} p_\phi, \quad \dot{\phi} = \frac{p_\phi}{I_1}, \quad \dot{\rho} = \frac{p_\rho}{I_2},$$  

whose solutions are found immediately.

The results thus obtained are in full agreement with the solutions obtained in any elementary course in Analytical Mechanics and illustrate the predictions of Thm. 3.

---

This was known in advance because $L$ is definite and the constraints are admissible.
6.2. Appell's machine

We will discuss now the well-known Appell's machine (see Figure 3). Let \( \rho = 0 \): It consists in a disk of radius \( R \) with a small cylinder rigidly attached to it of radius \( r \) with total mass \( M \). A rope passes through the small cylinder and hangs a mass \( m \) on the other extreme of a vertical frame. The frame is of negligible mass and the disk rolls over the surface. The coordinates of the contact point of the main wheel will be \((x, y)\) and the coordinates of the small mass \( m \) will be thereby \((x, y, z)\). The angle defined by the horizontal axis of the wheel with the \( Ox \) axis will be denoted by \( \theta \) and the angular position of the wheel will be denoted by \( \phi \) as in the example before. The configuration space being \( Q = \mathbb{R}^3 \times T^2 \) and the tangent bundle becomes the 10-dimensional manifold \( TQ = T\mathbb{R}^3 \times T^2 \). The Lagrangian of the system will be given by

\[
L = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_2 \dot{\theta}^2 - mgz.
\]

The rolling conditions between the disk and the floor and the nonsliding condition between the rope and the cylinder define the constraint submanifold \( C \subset TQ \). The 7-dimensional submanifold \( C \) is then characterized by the equations,

\[
\Psi_1 = \dot{x} - R \cos \theta \phi = 0, \quad \Psi_2 = \dot{y} - R \sin \theta \phi = 0, \quad \Psi_3 = \dot{z} = 0.
\]

The non-holonomic constraints are linear and the system can be considered to be of Chetaev's type with \( F = S^* (TC) \). The Legendre transformation of \( C \) gives the submanifold \( C^* \),

\[
C^* = \{ (x, y, z, \theta, \phi, p_x, p_y, p_z, p_\phi) \in T^* Q \mid I_1 p_x = (M + m) R \cos \theta p_\phi, \]

\[
I_1 p_y = (M + m) R \sin \theta p_\phi, \quad I_1 p_z = m R p_\phi \}.
\]

Repeating the computations of the previous section we will arrive to the following equations of motion on the submanifold \( C^* \subset T^* Q \),

\[
\dot{p}_x = \lambda_1, \quad \dot{p}_y = \lambda_2, \quad \dot{p}_z = \lambda_3 - mg, \tag{6.3}
\]

\[
\dot{p}_\phi = -\lambda_1 R \cos \theta - \lambda_2 R \sin \theta - \lambda_3 r, \quad \dot{p}_\theta = 0. \tag{6.4}
\]

Thus, solving the Lagrange's multipliers, \( \lambda_1, \lambda_2, \lambda_3 \), i.e., computing \( E \cap TC^* \), we get,

\[
\dot{\phi} = -\frac{mgr}{I_1 + (M + m) R^2 + mr^2}, \quad \dot{\theta} = 0,
\]

which is integrated immediately. Two comments are pertinent here. Notice that the last equation is simply the conservation of the \( z \) component of the angular momentum, which is obvious from the analysis of the forces acting upon the system. Secondly, we know in advance that the system is compatible and has a unique and well defined solution without having to consider it as a limit case \( \rho \to 0 \) of a "well posed" system. In fact we will see immediately that the case \( \rho \neq 0 \) (see Fig. 4) has extra difficulties because of the nonconservation of angular momentum.
$\rho \neq 0$: The system now consists as before on a disk an a small cylinder rigidly attached to it. A rope passes through the small cylinder and hangs a mass $m$ on the other extreme of a frame of length $\rho$. The frame is of negligible mass and slides without friction. Now the coordinates of the contact point of the main wheel will be $(x_D, y_D)$ and the coordinates of the mass $m$ will be $(x, y, z)$. The angle defined by the horizontal axis of the wheel with the $Ox$ axis will be denoted again by $\theta$ and the angular position of the wheel will be denoted by $\phi$. The Lagrangian of the system will be given now by

$$L = \frac{1}{2} M (\dot{x}_D^2 + \dot{y}_D^2) + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_2 \dot{\phi}^2 - mgz.$$ 

The geometry of the system imposes that $x - x_D = \rho \cos \theta$ and $y - y_D = \rho \sin \theta$, (that can be understood as holonomic constraints). We can eliminate them from the Lagrangian $L$ and we get a system on $Q = \mathbb{R}^3 \times T^2$ as in the situation before. The constraints now are given by,

$$\Psi_1 = \dot{x}_D - \rho \cos \theta \dot{\phi} = 0, \quad \Psi_2 = \dot{y}_D - \rho \sin \theta \dot{\phi} = 0, \quad \Psi_3 = r \dot{\phi} - \dot{z} = 0.$$ 

Repeating the computations and using the Legendre transform to write now the equations of motion directly on $C \subset TQ$ we will obtain,

$$(M + m) \ddot{x} + M \rho \dot{\theta}^2 \cos \theta + M \rho \dot{\theta} \sin \theta = \lambda_1,$$

$$(M + m) \ddot{y} + M \rho \dot{\theta}^2 \sin \theta - M \rho \dot{\theta} \cos \theta = \lambda_2,$$

$$m \ddot{z} + mg = -\lambda_3,$$

$$I_1 \ddot{\phi} = -\lambda_1 R \cos \theta - \lambda_2 R \sin \theta + \lambda_3 \dot{\theta},$$

$$I_2 - M \rho^2) \ddot{\theta} - M \rho R \dot{\phi} \dot{\phi} = \lambda_1 \rho \sin \theta - \lambda_2 \rho \cos \theta.$$ 

Fig. 4: Appell’s machine with $\rho \neq 0$. 

Repeating the computations and using the Legendre transform to write now the equations of motion directly on $C \subset TQ$ we will obtain,
Eliminating the Lagrange's multipliers, we obtain \( \phi \) and \( \theta \) variables,

\begin{align}
(I_2 + (M + m)\rho^2)\ddot{\theta} + \rho R m \dot{\phi} &= 0, \quad (6.5) \\
(I_1 + (m + M)R^2 + mr^2)\ddot{\phi} - M R \rho^2 &= -mrg. \quad (6.6)
\end{align}

Eq. (6.5) shows that there is no conservation of the angular momentum, the reason being that for the mass \( m \) to move rigidly with the frame, this has to act on it with a horizontal force. The previous equations, (6.6), (6.5), can be solved and we obtain \( \phi(t) \) implicitely from the integral,

\[
\frac{1}{\sqrt{2}} \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{E - mgr\phi + \frac{I_1 + (M + m)\rho^2}{I_1 + (M + m)R^2 + m\rho^2} \exp \left(-\frac{2MRe}{I_2 - (M + m)\rho^2}\right)} (\phi - \phi_0)} = t - t_0,
\]

and

\[
\theta(t) = \theta_0 + \int_{t_0}^{t} e^{-\frac{2MRe}{I_2 - (M + m)\rho^2} (\phi(t) - \phi_0)} dt.
\]

6.3. A variation of Benenti's problem

Finally we will consider some variants of an example proposed by Benenti [5] (see Fig. 5).

First we will solve the problem of two point masses forced to move on a plane with parallel velocities. The configuration space will be \( Q = \mathbb{R}^2 \times \mathbb{R}^2 \) and we will denote by \((x_1, y_1)\) the position of the particle of mass \( m_1 \) and by \((x_2, y_2)\) the position for the particle of mass \( m_2 \). The constraint on the velocities is given by the function on the tangent bundle \( TQ \),

\[
\Psi = \dot{x}_1 \dot{y}_2 - \dot{x}_2 \dot{y}_1. 
\]

The Lagrangian of the system is,

\[
L = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2). 
\]

The non-holonomic constraint \( \Psi \) is a genuine non-linear non-holonomic constraint. If we solve the system with Chetaev's forces (5.1) we will obtain the following system of equations on \( TQ \),

\[
m_1 \ddot{x}_1 = \lambda \dot{y}_2, \quad m_1 \ddot{y}_1 = -\lambda \dot{x}_2, \quad m_2 \ddot{x}_2 = -\lambda \dot{y}_1, \quad m_2 \ddot{y}_2 = \lambda \dot{x}_1,
\]

thus derivating eq. (6.7), and substituting on it, we get,

\[
\left( \frac{\dot{x}_2^2 + \dot{y}_2^2}{m_1} + \frac{\dot{x}_1^2 + \dot{y}_1^2}{m_2} \right) \lambda = 0.
\]

Hence, \( \lambda = 0 \) and Chetaev's forces vanish. The solution of the system is simply free motion of the two particles with initial parallel velocities. Notice that Chetaev's forces have the form \( f_1 = \lambda (\dot{y}_2 dx_1 - \dot{x}_2 dy_1) \), \( f_2 = \lambda (\dot{y}_1 dx_2 + \dot{x}_1 dy_2) \) where \( f_i \) is the force.
acting on the particle $i$, thus the only solution leaving the submanifold $\Psi = 0$ invariant is obtained with $\lambda = 0$. However if we choose the system of forces

$$F = \{ \lambda (m_1 f_1 dx_1 + m_1 f_2 dy_1 + m_2 f_1 dx_2 + m_2 f_2 dy_2) \mid \lambda \in \mathbb{R} \},$$

then we will have the equations,

$$\ddot{x}_i = \lambda f_1, \quad \dot{y}_i = \lambda f_2, \quad \ddot{x}_2 = \lambda f_1, \quad \dot{y}_2 = \lambda f_2,$$

thus if the initial velocities were parallel, the corresponding motions will be a solution for arbitrary $\lambda$. Thus, in this particular situation there are non unique solution of the problem.

Fig. 5.

The problem can be modified substituting the point masses by disks rolling without sliding on the plane. Then we will be considering two copies of the system discussed in Section 6.1., eqs. (6.1), (6.2), with the additional constraint,

(6.8) \quad \Psi_5 = v_1 \land v_2,

where $v_1$ denotes the velocity of the center of mass of the $i$th disk. Replacing the constraints given by the rolling conditions, eq. (6.2), on eq. (6.8), we will obtain,

$$R_1 R_2 \sin(\theta_2 - \theta_1) \dot{\theta}_1 \dot{\theta}_2 = 0.$$ 

Then, either one of the disks is still ($\dot{\theta}_i = 0$) and the angles $\theta_i$ are arbitrary, or if the disks are rolling ($\dot{\theta}_i \neq 0$), then $\theta_1 = \theta_2 = \theta$ and the nonlinear constraint $\Psi_5$ is redundant. Then the two disks roll independently and freely keeping parallel directions.

If we remove the rolling conditions and we keep only the parallelism condition $\Psi_5 = 0$, then we will obtain again the solutions of the point masses problem before, i.e.,
free motion with initial parallel velocities. Under this condition, the disks slide without rolling. It is obvious that using non-Chetaev forces, other solutions can be found.

7. Conclusions and outlook

We have reviewed the theory of Lagrangian systems with non-holonomic constraints from the viewpoint of implicit differential equations in the realm of symplectic and tangent bundle geometry. From the beginning we have deliberately separated the non-holonomic constraints from the forces or controls that we can use to force the system to satisfy them, emphasizing in this way that non-Chetaev's systems can be included in this picture. In fact, we have found general conditions that guarantee the existence and uniqueness of solutions for this generalized non-holonomic problem. We have specialized this general compatibility conditions to different particular situations already discussed in the literature.

The discussion has been made in the autonomous case but it is obvious that it can be extended to the time-dependent setting easily (see for instance [40], [30], [19]). Some of the beautiful aspects of Tulczyjew's triple are lost in a time-dependent setting and this justifies to keep the discussion at the autonomous level. The usual extended phase space trick can be used to include time as an additional variable and proceed as it has been done here. An alternative path would be to extend Tulczyjew's triple to the setting of cosymplectic/contact geometry to recover the same geometrical setting without introducing spurious variables as suggested by [28].

Subtler is the problem of extending the theory of non-holonomic constraints to singular Lagrangians. This is a significant extension of the theory because physical Lagrangians are very often singular. Singular lagrangians introduce their own constraints, that arise as integrability conditions of an implicit differential equation. A simultaneous analysis of Lagrangian constraints and non-holonomic constraints is needed to obtain the equations of motion of the theory. This problem has been addressed in [31] and [22].

An important aspect of the theory of non-holonomic constraints is its relation with Quantum Mechanics. Apart from a paper by R.J. Eden [15], there has been little attention to the quantization of systems with non-holonomic constraints (see also [39]). One reason for this is that fundamental theories do not include non-holonomic constraints because non-holonomic constraints are phenomenological models that offer a reasonable description for the (often unknown) true fundamental theory that describes the interaction between the surfaces of the systems in contact, or other interactions present in the system which cannot be described at the same fundamental level. Some aspects concerning the Hamiltonian structure of non-holonomic systems and its quantization are being discussed in [22].

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Non-holonomic constrained systems


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