L. Saal

THE AUTOMORPHISM GROUP OF A LIE ALGEBRA OF HEISENBERG TYPE

Abstract. Let \( \eta = \xi \oplus \vartheta \) be a Lie algebra of Heisenberg type, where \( \xi \) denotes the centre of \( \eta \) and \( \vartheta \) its orthogonal complement.

Here, it is shown that the group of automorphisms of \( \eta \) is the semidirect product of an abelian subgroup by the group of automorphisms of \( \eta \) that preserve \( \vartheta \), which, in turn, is described in terms of classical groups.

1. Introduction and some preliminaries

Algebras of type \( H \) were introduced by A. Kaplan in [3] as follows: Let \( \eta \) be a real two-step nilpotent Lie algebra, endowed with an inner product \( \langle , \rangle \). Let \( \xi \) denote the centre of \( \eta \) and let \( \vartheta \) denote its orthogonal complement.

Let \( J : \xi \to \text{End}(\vartheta) \) be the linear mapping defined by

\[
(1.1) \quad (J_z v, v') = \langle z, [v, v'] \rangle, \quad v, v' \in \vartheta, \quad z \in \xi.
\]

We say that \( \eta \) is an algebra of Heisenberg type (or of \( H \)-type) if, for every unit element \( z \in \xi \), \( J_z^2 = -\text{Id} \). Since \( J_z \) is skew-symmetric, this is equivalent to ask \( J_z \) to be an orthogonal transformation of \( \vartheta \).

We denote by \( \text{Aut}(\eta) \) the group of automorphisms of \( \eta \) and by \( \mathcal{A}(\eta) \) the subgroup of those automorphisms that act as orthogonal transformations on \( \eta \).

The structure of \( \mathcal{A}(\eta) \) has been described by Riehm in [5]. The aim of this note is to describe \( \text{Aut}(\eta) \).

Indeed, we will see that \( \text{Aut}(\eta) \) is the semidirect product of an abelian subgroup by the group of automorphisms of \( \eta \) that preserve \( \vartheta \). We denote the last one by \( \text{Aut}_\vartheta(\eta) \).

Now, from (1.1), it follows by linearity and polarization that

\[
(1.2) \quad J_z^2 = -|z|^2 \text{Id}, \quad z \in \xi
\]

and

\[
(1.3) \quad J_z J_{z'} + J_{z'} J_z = -2 \langle z, z' \rangle \text{Id}, \quad z, z' \in \xi
\]
In particular \( J_z J_{z'} + J_{z'} J_z = 0 \) if \( (z, z') = 0 \).

Let \( m = \dim(\zeta) \) and let \( C(m) \) be the Clifford algebra \( C(\zeta, -| \cdot |^2) \). By (1.2), (1.3) and the universal property of Clifford algebras, the action \( J \) of \( \zeta \) on \( \vartheta \) extends to an unitary representation of \( C(m) \) that we also denote by \( J \). So \( \vartheta \) is a \( C(m) \)-module and, via the theory of orthogonal multiplication, it is possible to show that every Clifford module arises in this way [3].

So, the notions "irreducible" and "isotypic", when attributed to \( \vartheta \), refer to its Clifford module structure.

It is known that \( C(m) \) is either simple or a direct sum of two simple algebras, (see, for example, [2]) and an explicit realization of Table 1 is given in Proposition 3 in [6].

| Table 1 |
|---|---|---|---|---|
| \( m \) (mod 8) | 0 | 1 | 2 | 3 |
| \( C(m) \) | \( \mathbb{R}_{2^p} \) | \( \mathbb{C}_{2^p} \) | \( \mathbb{H}_{2^p-1} \) | \( \mathbb{H}_{2^p-1} \otimes \mathbb{H}_{2^p-1} \) |
| dim_{\mathbb{R}} | \( 2^p \) | \( 2^{p+1} \) | \( 2^{p+1} \) | \( 2^{p+1} \) |

<table>
<thead>
<tr>
<th>( m ) (mod 8)</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(m) )</td>
<td>( \mathbb{H}_{2^p-1} )</td>
<td>( \mathbb{C}_{2^p} )</td>
<td>( \mathbb{R}_{2^p} )</td>
<td>( \mathbb{R}<em>{2^p} \oplus \mathbb{R}</em>{2^p} )</td>
</tr>
<tr>
<td>dim_{\mathbb{R}}</td>
<td>( 2^{p+1} )</td>
<td>( 2^{p+1} )</td>
<td>( 2^p )</td>
<td>( 2^p )</td>
</tr>
</tbody>
</table>

We denote by \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) the real and complex numbers and the quaternions respectively, and by \( \mathbb{R}_n, \mathbb{C}_n, \mathbb{H}_n \), the algebra of matrices of order \( n \) with coefficients in \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \). Also \( p \) is such that \( m = 2p \) if \( m \) is even, \( m = 2p + 1 \) otherwise.

We denote by \( C^+(m) \) the even Clifford algebra. Let \( \{z_1, \ldots, z_m\} \) be an orthonormal basis of \( \zeta \). By the universal property of \( C(m) \), the map given by

\[
(1.4) \quad z_j \rightarrow z_j z_m, \text{ for } 1 \leq j \leq m, \text{ and } z_m \rightarrow z_m
\]

extends to an automorphism of \( C(m) \) which takes \( C(m - 1) \) onto \( C^+(m) \). As stated in Remark 1 in [6], this automorphism enables us to extend the realization of \( C^+(m) \) to \( C(m) \) in the same way that Prop. 3 extends it from \( C(m - 1) \) to \( C(m) \).
The automorphism group of a Lie algebra

For \( z \in \zeta \), \( |z| = 1 \), let \( \rho_z : \zeta \to \zeta \) be the reflection through the hiperplane orthogonal to \( z \). It is easy to see that \( J_z : \vartheta \to \vartheta \) is extended to an orthogonal automorphism of \( \eta \), acting on \( \zeta \) by \(-\rho_z\). In this way, we denote by \( \text{Pin}(m) \) the subgroup of \( \mathcal{A}(\eta) \) generated by \( \{(-\rho_z, J_z), \ |z| = 1\} \) and by \( \text{Spin}(m) \) the subgroup of \( \mathcal{A}(\eta) \) generated by \( \{(\rho_z, \rho_{z'}, J_z, J_{z'}), \ |z| = |z'| = 1\} \). Let \( \text{Cliff}(m) \) be the group generated by \( \{(-|z|^2 \rho_z, J_z), \ z \in \zeta - \{0\} \} \).

Finally, we denote by \( \text{Aut}_\vartheta(\eta)^0 \) the subgroup of \( \text{Aut}_\vartheta(\eta) \) of those automorphisms acting trivially on the center.

In the next section we determine \( \text{Aut}_\vartheta(\eta)^0 \) in terms of classical groups and then, we show that \( \text{Aut}_\vartheta(\eta) \) is the semidirect product of \( \text{Aut}_\vartheta(\eta)^0 \) by \( \text{Cliff}(m) \), or the index \( \left[ \text{Aut}_\vartheta(\eta) : \text{Aut}_\vartheta(\eta)^0 \left< \text{Cliff}(m) \right> \right] = 2 \).

Furthermore \( \text{Cliff}(m) \cap \text{Aut}_\vartheta(\eta)^0 \) has two or four elements. The precise situation is in Proposition 2.2.

2. Determination of \( \text{Aut}_\vartheta(\eta)^0 \)

Let, as in 1, \( \eta = \zeta \oplus \vartheta \) be a Lie algebra of \( H \)-type, with inner product \( \langle \cdot, \cdot \rangle \). Let \( \psi : \vartheta \times \vartheta \to \zeta \) be the antisymmetric form such that \( [z + v, z' + v'] = \psi(v, v') \), for \( z, z' \in \zeta, v, v' \in \vartheta \).

From now on, we fix an orthonormal basis \( \{z_1, \ldots, z_m\} \) of \( \zeta \) and set \( J_i = J_{z_i} \). We can write \( \psi(u, v) = \sum_{i=1}^m \psi_i(u, v) z_i \) with

\[
\psi_i(u, v) = \langle z_i, \psi(u, v) \rangle = \langle z_i, [u, v] \rangle = \langle J_i u, v \rangle \tag{2.1}
\]

Thus

\[
\text{Aut}_\vartheta(\eta) = \{(h, g) \in \text{Gl}(\zeta) \times \text{Gl}(\vartheta) : \psi(gu, gv) = h\psi(u, v) \forall u, v \in \vartheta\}
\]

where \( \text{Gl} \) denotes the general linear group.

Also

\[
\text{Aut}_\vartheta(\eta)^0 = \{g \in \text{Gl}(\vartheta) : \psi(gu, gv) = \psi(u, v) \forall u, v \in \vartheta\}.
\]

From (2.1) it follows that

\[
g \in \text{Aut}_\vartheta(\eta)^0 \text{ if and only if } g^T J_i g = J_i \text{ for all } i = 1, \ldots, m. \tag{2.2}
\]

where \( g^T \) denotes the transpose of \( g \).

It is easy to see that every element of \( \mathcal{A}(\eta) \) that acts on \( \zeta \) by the identity, commutes with the action \( J \) of \( C(m) \) on \( \vartheta \).
For $\text{Aut}_\vartheta(\eta)^0$ we have the following

**Lemma 2.1.** Every element of $\text{Aut}_\vartheta(\eta)^0$ intertwines the representation $J$ of $C^+(m)$ on $\vartheta$.

**Proof.** For $1 \leq i, k \leq m$,

$$\psi_i(u,v) = \langle J_i u, v \rangle = -\langle J_k^2 J_i u, v \rangle = \langle J_k J_i J_k u, v \rangle = \psi_k(J_i J_k u, v).$$

Since $g$ in $\text{Aut}_\vartheta(\eta)^0$ preserves each $\psi_i$, we have

$$\psi_k(g J_i J_k u, g v) = \psi_k(J_i J_k g u, g v).$$

Since $\psi_k$ is not degenerated, $g(J_i J_k) = (J_i J_k)g$.

**Remark 2.1.** We denote by $\text{End}_{C^+(m)}(\vartheta)$ the algebra of linear maps on $\vartheta$ which intertwines the action of $C^+(m)$. By Lemma 2.1,

$$\text{Aut}_\vartheta(\eta)^0 = \left\{ g \in \text{End}_{C^+(m)}(\vartheta) : g^i J_i g = J_i \text{ for a single } J_i \right\}.$$

Indeed, for $k \neq i$, we have

$$g^i J_k g = -g^i J_k J_i J_i g = g^i J_i J_k J_i g = g^i J_i g J_k J_i = J_k$$

and so (2.2) holds.

**Remark 2.2.** Set $K = J_1 \cdots J_m$. Then $K$ commutes with $C^+(m)$. Furthermore if $m \equiv 1,2(\text{mod }4)$, $K^2 = -1$ and since $K$ is orthogonal $K^i = -K$. If $m \equiv 0,3(\text{mod }4)$, $K^2 = 1$ and $K^i = K$.

We are now able to compute $\text{Aut}_\vartheta(\eta)^0$. We remark that Proposition 9 in [6] enable us to make a systematic use of the explicit representations of $C(m)$ as they are given in [6].

- **$m \equiv 1(\text{mod }8).$**

According to Proposition 4 in [6] there exists an isomorphism between $C^+(m)$ and $\mathbb{R}_{2r}$ that can be extended to one between $C(m)$ and $\mathbb{C}_{2r}$ in such a way that $K = J_1 \cdots J_m$ acts by $i \text{Id}$.

We first assume $\vartheta$ irreducible. Thus there exists an isomorphism $\alpha : \vartheta \rightarrow \mathbb{C}^{2r}$ as $C(m)$ modules. Moreover since $\vartheta$ is also an unitary irreducible $Pin(m)$-module we have that $\langle u, v \rangle = c \text{ Re } \left( \alpha(u) \overline{\alpha(v)} \right)$ for some positive constant $c$, where $ab$ denotes the standard hermitian inner product for $a, b \in \mathbb{C}^{2r}$. 
The automorphism group of a Lie algebra

On the other hand, if \( w \) is an irreducible \( \mathbb{C}^+(m) \)-module, so is \( J_iw \) for all \( i = 1, 2, \ldots, m \). Thus \( w \cap J_iw \) is invariant under \( C(m) \) and therefore trivial. In particular \( \vartheta = w \oplus J_m w \) and also \( J_m w = Kw \).

We now fix \( w = \alpha^{-1}(\mathbb{R}_{2\pi}) \). Then for \( w, w' \in w \), \( \langle w, Kw' \rangle = c \text{Re} \left( \alpha(w) \overline{\alpha(w')} \right) = 0 \) since \( \alpha K = i\alpha \).

So \( \langle w, J_m w' \rangle = 0 \) \( \forall w, w' \in w \). We fix an orthonormal basis of \( \vartheta \), \( \{ w_1, \ldots, w_{2^p}, J_m w_1, \ldots, J_m w_{2^p} \} \). Since \( K \) intertwines \( w \) with \( J_m w \) as \( \mathbb{C}^+(m) \)-modules we have that the matrix of \( K \) in this basis is \( \begin{pmatrix} 0 & -Q' \\ Q & 0 \end{pmatrix} \). The properties \( K = -K^t \), \( K^2 = -1 \) and \( KJ_m = J_m K \) imply that \( Q^t Q = QQ^t = 1 \) and \( Q = Q' \) (i.e., \( Q^2 = 1 \)). So if \( g \in \text{End}_{\mathbb{C}^+(m)}(\vartheta) \), \( g \) has a matrix of the form \( \begin{pmatrix} a_1 & -cQ \\ bQ & d1 \end{pmatrix} \),

where \( 1 \) denotes the \( 2^p \times 2^p \) identity. Since the matrix of \( J_m \) is \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) we have that

\[ g'J_m g = J_m \quad \text{if and only if} \quad \begin{pmatrix} a & -c \\ b & d \end{pmatrix} \text{ is Sp}(1, \mathbb{R}). \]

Thus \( \text{Aut}_0(\eta)^0 \simeq \text{Sp}(1, \mathbb{R}) \).

If \( \vartheta = (\vartheta_0)^t \) with \( \vartheta_0 \) \( C(m) \)-irreducible the above argument shows that \( g \in \text{End}_{\mathbb{C}^+(m)}(\vartheta) \) has a matrix

\[
\begin{pmatrix}
  a_{11}1 & \ldots & a_{11}1 & \ldots & -a_{1,1+1}Q & \ldots & -a_{1,2}Q \\
  a_{2,1}1 & \ldots & a_{2,1}1 & \ldots & -a_{2,1+1}Q & \ldots & -a_{2,2}Q \\
  a_{3,1}Q & \ldots & a_{3,1}Q & \ldots & a_{3,1+1}Q & \ldots & a_{3,2}Q \\
  a_{4,1}Q & \ldots & a_{4,1}Q & \ldots & a_{4,1+1}Q & \ldots & a_{4,2}Q \\
  a_{5,1}Q & \ldots & a_{5,1}Q & \ldots & a_{5,1+1}Q & \ldots & a_{5,2}Q \\
  a_{6,1}Q & \ldots & a_{6,1}Q & \ldots & a_{6,1+1}Q & \ldots & a_{6,2}Q \\
  a_{7,1}Q & \ldots & a_{7,1}Q & \ldots & a_{7,1+1}Q & \ldots & a_{7,2}Q \\
  a_{8,1}Q & \ldots & a_{8,1}Q & \ldots & a_{8,1+1}Q & \ldots & a_{8,2}Q \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}
\]

and thus \( \text{Aut}_0(\eta)^0 \simeq \text{Sp}(l, \mathbb{R}) \).

**Remark.** We can look at \( Q : w \to w \) such that \( K = J_m Q \). Moreover \( Q = J_1 \ldots J_{m-1} \). It is easy to see that the eigenspaces of \( Q \) associated to the eigenvalues \( \pm 1 \) have the same dimension and one can take a basis of eigenvectors for \( w \).

\( m \equiv 2 \pmod{8} \).

By proposition 3 and 4 in [6], we can extend an isomorphism between \( \mathbb{C}^+(m) \) and \( \mathbb{C}^{2^p-1} \) to one between \( C(m) \) and \( \mathbb{H}^{2^p-1} \) by sending \( z_m \) to \( j \text{Id} \). (Here we denote by \( i, j, k \) the units of \( \mathbb{H} \).)

Assume \( \vartheta \) \( C(m) \)-irreducible. Once again we can assume that \( \langle u, v \rangle = \text{Re} \langle uv \rangle \) where \( - \) denotes conjugation in \( \mathbb{H} \). By Remark 2.2 \( K = J_1 \ldots J_m \) commutes with \( \mathbb{C}^+(m) \) and \( K^2 = -1 \). So we look at \( (\vartheta, K) \) as a complex representation of \( \mathbb{C}^+(m) \) of (complex) dimension \( 2^p \). We can decompose it with an argument similar to the above case \( \vartheta = w \oplus J_m w \), where \( w \) and \( J_m w \) are irreducible, equivalent, \( \mathbb{C}^+(m) \)-modules. We
have that $J_m$ acts on $\mathfrak{g}$ by $j\text{Id}$ and thus $\langle w, J_m w' \rangle = 0 \forall w, w' \in \mathfrak{g}$. We take a real basis of $\mathfrak{g}$ of the form
\[
\{ w_1, \ldots, w_{2^r-1}, Kw_1, \ldots, Kw_{2^r-1}, J_m w_1, \ldots, J_m w_{2^r-1}, J_m Kw_1, \ldots, J_m Kw_{2^r-1} \}.
\]

The operator "multiplication by $j$ on the right" intertwines $w$ and $J_m w$ as $C^+(m)$ modules. We denote it by $\phi$. We know that $\phi$ commutes with $K$ and that $\phi$ is skew-symmetric and $\phi^2 = -1$.

Moreover the matrix of $\phi$ in the above basis is
\[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\phi_1 \\
\phi_1 & 0
\end{pmatrix}.
\]

Also the matrix of $K$ is
\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
K_1 & 0 \\
0 & -K_1
\end{pmatrix}.
\]

Thus $g \in \text{End}_{C^+(m)}(\mathfrak{g})$ if and only if the matrix of $g$ is
\[
\begin{pmatrix}
a_{11} + b_{11}K_1 & -\phi_1(a_{12} - b_{12}K_1) \\
\phi_1(a_{21} + b_{21}K_1) & a_{22} - b_{22}K_1
\end{pmatrix}.
\]

Then $g^t J_m g = J_m$ if and only if $a_{11}a_{22} - b_{11}b_{22} + a_{12}a_{21} - b_{21}b_{12} = 1$ and $a_{21}b_{12} + a_{12}b_{21} + a_{11}b_{22} + b_{11}a_{22} = 0$ if and only if
\[
\begin{pmatrix}
a_{11} + b_{11}i & -a_{12} - b_{12}i \\
a_{21} + b_{21}i & a_{22} + b_{22}i
\end{pmatrix}
\]
is in $\text{Sp}(1, \mathbb{C})$.

If $\mathfrak{g} = (\mathfrak{g}_0)^t$ with $\mathfrak{g}_0 C(m)$ -irreducible, a similar computation shows that $\text{Aut}_C(\mathfrak{g})^0 \simeq \text{Sp}(1, \mathbb{C})$.

- $m \equiv 4 \pmod{8}$.

In this case, $C^+(m) \simeq \mathbb{H}_{2^{r-2}} \oplus \mathbb{H}_{2^{r-2}}$ and it has two irreducible, inequivalent modules. (we consider $\mathbb{H}$ as a right $\mathbb{E}$-module, $\mathbb{E}$ acting on it on the left).
Let us consider \( \vartheta \) irreducible. Since \( K^2 = 1 \), \( \vartheta \) splits as \( \vartheta = \vartheta^+ \oplus \vartheta^- \) where \( \vartheta^\pm \) are the eigenspaces of \( K \) associated to the eigenvalues \( \pm 1 \). We have that \( J_i K = -K J_i \) for all \( i \), so \( J_i : \vartheta^\pm \to \vartheta^\mp \) and \( \dim \vartheta^+ = \dim \vartheta^- \). Since \( K \) is in the center of \( C^+(m) \), \( \vartheta^\pm \) are inequivalent \( C^+(m) \)-modules and, by dimension, they must be irreducible. Thus we identify \( \text{End}_{C^+(m)}(\vartheta) \) with \( \mathbb{H} \times \mathbb{H} \) (acting by right multiplication).

If \( \vartheta = (\vartheta_0)^l \) with \( \vartheta_0 \) irreducible, we have \( \vartheta^\pm \simeq \left( \mathbb{H}^{2^{r-2}} \right)^l \), that is, the \( 2^{r-2} \times 1 \) matrix over \( \mathbb{H} \) and so \( \text{End}_{C^+(m)}(\vartheta) \simeq \mathbb{H}_l \times \mathbb{H}_l \). Since \( \vartheta = \vartheta^+ \oplus J_i \vartheta^+ \), there exists an orthonormal basis of \( \vartheta \) such that \( J_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). If we write \( g \in \text{End}_{C^+(m)}(\vartheta) \) as a pair \( g = (g_1, g_2) \), the condition \( g^i J_i g = J_i \) implies that \( g_1^2 = g_2^{-1} \).

Thus \( \text{Aut}_\vartheta(\eta)^0 \simeq \text{Gl}_l(\mathbb{H}) \).

- \( m \equiv 0 \) (mod 8).

This case is completely analogous to the above: \( C^+(m) \simeq \mathbb{R}_{2^{r-1}} \oplus \mathbb{R}_{2^{r-1}}, \ K^2 = 1 \), and the eigenspaces of \( K \) are the two inequivalent, irreducibles modules of \( C^+(m) \).

For \( \vartheta = \vartheta_0^l \) with \( \vartheta_0 \) irreducible, we have \( \text{End}_{C^+(m)}(\vartheta) \simeq \mathbb{R}_l \times \mathbb{R}_l \) and \( \text{Aut}_\vartheta(\eta)^0 \simeq \text{Gl}_l(\mathbb{R}) \).

- \( m \equiv 3, 7 \) (mod 8).

We remark that in these cases \( C(m) \) has two inequivalent, irreducible modules.

Assume, first, \( \eta \) isotypical, that is, \( \vartheta = (\vartheta_0)^l \) with \( \vartheta_0 \) an irreducible \( C(m) \)-module. Since \( K \) commutes with \( C(m) \) and \( K^2 = 1 \), \( K \) acts on \( \vartheta_0 \) by \( K = 1 \) or by \( K = -1 \). Therefore \( J_1 = J_2 \cdots J_m \) or \( J_1 = -J_2 \cdots J_m \). Since \( J_2 \cdots J_m \) is in \( C^+(m) \), every element of \( \text{Aut}_\vartheta(\eta)^0 \) commutes with \( J_1 \). Thus, every element of \( \text{Aut}_\vartheta(\eta)^0 \) is an orthogonal automorphism (see [5]).

We assume \( m \equiv 3(8) \) (resp. \( m \equiv 7(8) \)) and we now compute \( \text{Aut}_\vartheta(\eta)^0 \) in the case that \( \eta \) is not isotypical.

According to proposition 3 in [6] we can extend the isomorphism between \( C(m-1) \) and \( \mathbb{H}_{2^{r-1}} \) (resp. \( \mathbb{R}_{2^r} \)) which sends \( z_i \to j_{z_i}, \) for \( 1 \leq i \leq m - 1, \) to an isomorphism between \( C(m) \) and \( \mathbb{H}_{2^{r-1}} \oplus \mathbb{H}_{2^{r-1}} \) (resp. \( \mathbb{R}_{2^r} \oplus \mathbb{R}_{2^r} \)) by the rule

\[
\begin{align*}
    z_i & \to J_i = (j_{z_i}, j_{z_i}), \quad 1 \leq i \leq m - 1 \\
    z_m & \to J_m = (j_{z_1} \cdots j_{z_{m-1}}, -j_{z_1} \cdots j_{z_{m-1}})
\end{align*}
\]

Furthermore, as is stated in Remark 1 in [5], the automorphism from \( C(m-1) \) to \( C^+(m) \) which sends \( z_i \to z_i z_m, \) \( 1 \leq i \leq m - 1, \) enables us to apply Prop. 3 to \( C(m) \).
extending the representation from $C^+(m)$ to $C(m)$ in the same way that Prop. 3 extends it from $C(m-1)$ to $C(m)$.

So the two irreducible -up to isomorphism- inequivalent representations of $C(m)$, say $\vartheta_0^+$ and $\vartheta_0^-$, are equivalent as $C^+(m)$ -modules, being the identity an intertwining operator.

Assume, now, $\vartheta = \vartheta^+ \oplus \vartheta^-$, where $K = J_1 \ldots J_m$ acts by $\pm 1$ on $\vartheta^\pm$ respectively. Thus $\vartheta^+ = \vartheta_0^+ \oplus \ldots \oplus \vartheta_0^+$, $p$ times, and $\vartheta^- = \vartheta_0^- \oplus \ldots \oplus \vartheta_0^-$, $q$ times.

By the above, $\text{End}_{C^+(m)}(\vartheta) \cong \text{Gl}(p + q, \mathbb{H})$ for $m \equiv 3(8)$ and $\text{End}_{C^+(m)}(\vartheta) \cong \text{Gl}(p + q, \mathbb{R})$ for $m \equiv 7(8)$.

We remember that $\text{Aut}_\vartheta(\eta)^0 = \{ g \in \text{End}_{C^+(m)}(\vartheta) : g^t J_m g = J_m \}$. If $g \in \text{Aut}_\vartheta(\eta)^0$, $g$ commutes with $J_1 \ldots J_{m-1}$ and so $g^t J_m g = J_m$ if and only if $g^t K g = K$.

We set $I_{p,q} = \begin{pmatrix} \mathbb{I} & \mathbb{0} \\ \mathbb{0} & -\mathbb{I} \end{pmatrix}$, $1 p \times p$ matrix, $-1 q \times q$ matrix.

So, for $m \equiv 3(8)$, $\text{Aut}_\vartheta(\eta)^0 \cong \{ g \in \text{Gl}(p + q, \mathbb{H}) : g^t I_{p,q} g = I_{p,q} \}$ $\cong U(p, q, \mathbb{H})$ where $-$ denotes conjugation in $\mathbb{H}$ and for $m \equiv 7(8)$, $\text{Aut}_\vartheta(\eta)^0 \cong O(p, q, \mathbb{R})$.

Assume, first, $\vartheta$ irreducible. Set $K_1 = J_1 \ldots J_{m-1}$. Then $K_1^2 = 1$ and $\vartheta = \vartheta^+ \oplus \vartheta^-$, where $\vartheta^\pm$ are the eigenspaces of $K_1$ associated to $\pm 1$, respectively. Furthermore $J_m = \pm K$ on $\vartheta^\pm$. We also observe that $\text{End}_{C^+(m)}(\vartheta)$ commutes with $K_1$ and so it preserves $\vartheta^\pm$.

By dimension, $\vartheta$ is $C^+(m)$-irreducible and we can identify $\text{End}_{C^+(m)}(\vartheta)$ with $\mathbb{H}$ (by right multiplication). Now $K$ commutes with $C^+(m)$ and, by Remark 2.2, $K^2 = -K$ and $K^2 = -1$. Thus $K$ acts on $\vartheta$ by a quaternion, say $q_1$, which is imaginary pure and it has norm one. Moreover we can assume that $q_1 = b_1 i + c_1 j + d_1 k$ with $b_1 = c_1 = 0$, $d_1 = 1$.

Associating to each $g$ in $\text{End}_{C^+(m)}(\vartheta)$ a quaternion $q(g)$, we have that the condition $g^t J_m g = J_m$ is equivalent to $\overline{q(g)} q_1 q(g) = q_1$, when $\overline{q(g)}$ denotes the conjugate of $q(g)$. This implies that the norm of $q(g)$ must be one and so $q(g)$ commutes with $q_1$. Thus $\text{Aut}_\vartheta(\eta)^0 \cong U(1)$.

Assume $\vartheta = (\vartheta_0)^l$ with $\vartheta_0 C(m)$ -irreducible. Then $\text{End}_{C^+(m)}(\vartheta) \cong \mathbb{H}_l$ and we need to determine $\{ q \in \mathbb{H}_l : \overline{q}^t(q_1 I) q = q_1 I \}$.

Now, let $\varphi : \mathbb{H}_l \to \mathbb{C}_{2l}$ be the canonical homomorphism given by $q \mapsto \begin{pmatrix} \alpha & -\beta \\ \beta & \overline{\alpha} \end{pmatrix}$, with $\alpha = A + B i$, $\beta = D + C i$ if $q = A + B i + C j + D k$. Then $\varphi(q_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $\varphi(q_1) \varphi(q) \varphi(q_1)^{-1} = \overline{\varphi(q)}$, we have that $\overline{q}^t q_1 q = q_1$ if and only if $\overline{\varphi(q)} \varphi(q_1) \varphi(q) = \varphi(q_1)$. 


if and only if $\varphi(q) = 1$. Thus $\text{Aut}_0(\eta) = \text{Gl}(l, \mathbb{H}) \cap O(2l, \mathbb{C}) = SO^*(2l)$.

- $m \equiv 6 \pmod{8}$.

Here $K^i = -K$, $K$ anticommutes with $J_i$ and $K^2 = -1$. Assume, first, $\vartheta$ irreducible. Thus $(\vartheta, K)$ is a complex vector space and, by dimension, it must be irreducible as a $C^+(m)$-module. Thus $g \in \text{End}_{C^+(m)}(\vartheta)$ if and only if $g = a + bK$, $a, b \in \mathbb{R}$.

Now, $g^tJ_ig = J_i$ if and only if $(a + bK^i)J_i(a + bK) = J_i$ if and only if $a^2J_i + abK^iJ_i + abJ_iK + b^2K^iJ_iK = J_i$ iff $(a^2 - b^2)J_i + 2abJ_iK = J_i$. Since $J_i$ and $K$ are linearly independent, $a^2 - b^2 = 1$, and $ab = 0$ which implies $\text{Aut}_0(\eta)^0 = \{\pm 1\}$.

We assume $\vartheta = (\vartheta_0)^l$ with $\vartheta_0$ irreducible. We identify $\mathbb{C}$ with $\text{End}_{C^+(m)}(\vartheta)$ via the rule that maps the matrix $A + iB = [a_{ij} + ib_{ij}]_{1 \leq i,j \leq l}$ to the element $g = (a_{ij} + b_{ij}K_0)_{1 \leq i,j \leq l}$ where $K_0$ is the restriction of $K$ on $\vartheta_0$. Also let $J_m^0$ the restriction of $J_m$ on $\vartheta_0$.

Now, $g^tJ_m^0 = J_m^0$ if and only if

$$
\sum_j (a_{ji} + b_{ji}K_0^i) J_m^0 (a_{jr} + b_{jr}K_0) = \delta_{ir}J_m^0.
$$

But this implies

$$
\sum_j (a_{ji}a_{jr} - b_{ji}b_{jr}) J_m^0 + (a_{jr}b_{ji} + b_{jr}a_{ji}) J_m^0 K_0 = \delta_{ir}J_m^0
$$

and so $A^tA - B^tB = 1$, $A^tB + B^tA = 0$, i.e. $A + iB \in O(l, \mathbb{C})$. Thus $\text{Aut}_0(\eta)^0 \cong O(l, \mathbb{C})$.

We have, just, proved the following

**Proposition 2.1.** Let $\eta = \zeta \oplus \vartheta$ be an algebra of $H$-type.

Let $\dim_{\mathbb{R}} \vartheta = n$. Then $\text{Aut}_0(\eta)^0$ is isomorphic to

- $\text{Sp} \left(n2^{-\left(\frac{m+1}{2}\right)}, \mathbb{R}\right)$ if $m \equiv 1 \pmod{8}$
- $\text{Sp} \left(n2^{-\left(\frac{m+2}{2}\right)}, \mathbb{C}\right)$ if $m \equiv 2 \pmod{8}$
- $U \left(n_12^{-\left(\frac{m+1}{2}\right)}, n_{-1}2^{-\left(\frac{m+1}{2}\right)}, \mathbb{H}\right)$ if $m \equiv 3 \pmod{8}$
- $\text{Gl} \left(n2^{-\left(\frac{m+2}{2}\right)}, \mathbb{H}\right)$ if $m \equiv 4 \pmod{8}$
$$SO^* \left( 2n2^{-\left(\frac{m+1}{2}\right)} \right) \quad \text{if } m \equiv 5(\text{mod } 8)$$

$$O \left( n2^{-\left(\frac{m}{2}\right)}, \mathbb{C} \right) \quad \text{if } m \equiv 6(\text{mod } 8)$$

$$O \left( n_12^{-\left(\frac{m-1}{2}\right)}, n_{-1}2^{-\left(\frac{m+1}{2}\right)}, \mathbb{R} \right) \quad \text{if } m \equiv 7(\text{mod } 8)$$

$$Gl \left( n2^{-\left(\frac{m}{2}\right)}, \mathbb{R} \right) \quad \text{if } m \equiv 0(\text{mod } 8)$$

Here we kept the notation of Theorem 6 in [5] and $n_1$ and $n_{-1} = n - n_1$ are the dimensions of the eigenspaces of $\vartheta$ with respect to $K = J_1 \cdots J_m$.

Our purpose, now, is to describe $\text{Aut}_\vartheta(\eta)$.

We denote by $\delta_d$, $d > 0$, the automorphism given by

$$\delta_d(z, v) = (d^2z, dv), \ z \in \zeta, \ v \in \vartheta.$$  

We recall that $\{z_1, \ldots, z_m\}$ denotes an orthonormal basis of $\zeta$. Thus, for $h \in \text{Gl}(\zeta)$, there exist orthogonal matrices $k_1$ and $k_2$ such that

$$(2.3) \quad h = k_1h_0k_2$$

where $h_0$ is diagonal with positive entries $d_1, \ldots, d_m$.

Set $\dim \vartheta = n$.

We first assume $\dim \zeta = m$ is even.

In this case $\text{Pin}(m)$ acts on $\zeta$ as the full non-connected $O(m)$ since $\det(-\rho_z) = -1$. Thus, by (2.3), we can write each element $(h, g) \in \text{Aut}_\vartheta(\eta)$ as

$$(h, g) = (k_1, g_1)(h_0, g_0)(k_2, g_2), \quad \text{with } (k_i, g_i) \in \text{Pin}(m) \text{ for } i = 1, 2.$$  

Since $(h_0, g_0) \in \text{Aut}_\vartheta(\eta)$, we have that $\psi_i(g_0u, g_0v) = d_i\psi_i(u, v)$ for $u, v \in \vartheta$, $i = 1, \ldots, m$; that is $g_0^{\dagger}J_i g_0 = d_iJ_i$. This implies $d_i^{\dagger} = (\det g_0)^2$ so that $h_0 = d1$ and $(h_0, g_0) = \delta_{d_1/2}(1, g^\prime)$. Thus we conclude that

$$(h, g) = \delta_{d_1/2}(k_1, g_1)(1, g^\prime)(k_2, g_2)$$

So

$$\quad (2.4) \quad h, g) = \delta_{d_1/2}(k_3, g_3)(1, g^{\prime\prime})$$

with $(k_3, g_3) \in \text{Pin}(m)$, $(1, g^{\prime\prime}) \in \text{Aut}_\vartheta(\eta)^0$.

We now assume $m = \dim \zeta$ is odd.
The automorphism group of a Lie algebra

In this case, \( \text{Pin}(m) \) acts on \( \zeta \) as \( \text{SO}(m) \).

Case \( m \equiv 3, 7 \mod 8 \). If \( (-1, g) \) is in \( \text{Aut}_\phi(\eta) \), we have \( \psi_i(gu, gv) = -\psi_i(u, v) \) so that \( \psi_i(gJ_kJ_lu, gv) = -\psi_i(J_kJ_lu, v) = -\psi_k(u, v) = \psi_k(gu, gv) = \psi_i(J_kJ_lgu, gv) \).

So \( (-1, g) \in \text{Aut}_\phi(\eta) \) if and only if

\[
(2.5) \quad g \in \text{End}_{C^+(m)}(\vartheta) \quad \text{and} \quad g^tJ_mg = -J_m
\]

As before, we decompose \( \vartheta = \vartheta^+ \oplus \vartheta^- \), where \( K = J_1 \ldots J_m \) acts by \( \pm 1 \) on \( \vartheta^\pm \) respectively and \( \vartheta^+ = \vartheta^1_+ \oplus \ldots \oplus \vartheta^n_+ \), \( p \) times, \( \vartheta^- = \vartheta^1_- \oplus \ldots \oplus \vartheta^n_- \), \( q \) times.

We have that \( g \) satisfies (2.5) if and only if \( g \in \text{End}_{C^+(m)}(\vartheta) \) and \( g^tKg = -K \).

So if \( p = q \), there exists such \( g \): indeed, for \( m \equiv 3(8) \), \( g \in \text{Gl}(p + q, \mathbb{H}) \) and 
\[
\bar{g}^tI_p,qg = -I_p,q. \quad \text{For example } g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \text{The same holds for } m \equiv 7(8).
\]

On the other hand, we have that if \( \dim \vartheta^+ = \dim \vartheta^- \) and \( \phi(z) = -z \quad \forall z \in \zeta \) and we set \( J_z = J_{\phi(z)} \) then \( J_z \mid_{\vartheta^+} \) is equivalent to \( J_{\phi(z)} \mid_{\vartheta^-} \) as \( C(m) \)-modules. So there exists an orthonormal \( g : \vartheta \to \vartheta \) such that \( gJ_z = J_{\phi(z)}g \). So \( g \in \text{End}_{C^+(m)}(\vartheta) \) and \( gK = -Kg \).

Now if \( p \neq q \), there is not such \( g \) since if \( B \) is a non degenerated symmetric bilinear form on a real vector space \( V \), \( \dim V = n \), then there exists a basis of \( V \) \( \{\beta_j\} \) such that \( B(\beta_j, \beta_j) = 1 \) for \( 1 \leq j \leq r \) and \( B(\beta_j, \beta_j) = -1 \) for \( r + 1 \leq j \leq n \) and the number \( r \) is independent of the choice of the basis (see, for example, [1]).

Thus for \( (h, g) \in \text{Aut}_\phi(\eta) \), either \( \det h > 0 \) and we can write \( (h, g) \) as in (2.4), or \( (h, g) = (-1, g_0)(h', g') \) with \( \det h' > 0 \).

Case \( m \equiv 1, 5 \mod 8 \). Here, there exists \(-1, g_0) \in A(\eta) \) (see [41]).

Then, we have the following

**PROPOSITION 2.2.** Let \( N = \text{Aut}_\phi(\eta)^0 \cap \text{Cliff}(m) \) and \( G = \text{Aut}_\phi(\eta)^0 \rtimes \text{Cliff}(m) \).

- If \( m \equiv 0, 2 \mod 4 \) \( \text{Aut}_\phi(\eta) \simeq G/N \) and \( N = \{ \pm 1 \} \).
- If \( m \equiv 1 \mod 4 \), the index \( [\text{Aut}_\phi(\eta) : G/N] = 2 \) and \( N = \{ \pm 1, \pm K \} \).
- If \( m \equiv 3 \mod 4 \) \( \text{Aut}_\phi(\eta) \simeq G/N \) for \( \dim \vartheta^+ \neq \dim \vartheta^- \) and \([\text{Aut}_\phi(\eta) : G/N] = 2 \) for \( \dim \vartheta^+ = \dim \vartheta^- \), where \( N = \{ \pm 1, \pm K \} \) if \( v \) is not isotypical and \( N = \{ \pm 1 \} \) if \( v \) is isotypical.

**Proof.** We only need to compute \( N \).

We have that \( \{ u \in C(m) : u \text{ commutes with } C^+(m) \} = \mathbb{R}1 \oplus \mathbb{R}K \). Furthermore, if \( g \in \text{Aut}_\phi(\eta)^0 \), \( g \) must commute with \( C^+(m) \) and \( (\det g)^2 = 1 \). Thus \( N \subset \{ \pm 1, \pm K \} \).

But now, Remark 2.2 implies that for \( m \equiv 1, 3 \mod 4 \), \( K^tJ_iK = J_i \) and for
Proposition 2.3. $\text{Aut}(\eta)$ is the semidirect product of a normal, abelian subgroup by $\text{Aut}_\eta(\eta)$.

Proof. We fix $\{v_1, \ldots, v_n\}$ an orthonormal basis of $\vartheta$. We take $T \in \text{Aut}(\eta)$ and we write

$$Tz_i = \sum_{j=1}^{m} h_{ji} z_j$$

$$Tv_i = \sum_{j=1}^{m} \alpha_{ji} z_j + \sum_{j=1}^{n} g_{ji} v_j.$$ 

We take $\alpha \in \mathbb{R}_{m \times n}$, $h \in \mathbb{R}_m$ and $g \in \mathbb{R}_n$ given by

$$\alpha = [\alpha_{ij}] \quad h = [h_{ij}] \quad \text{and} \quad g = [g_{ij}].$$

Now, $T$ is an automorphism if and only if

$$(z_j, T[v_i, v_k]) = (z_j, [Tv_i, Tv_k]), \quad 1 \leq j \leq m; \quad 1 \leq i, k \leq n.$$ 

Since $[v_i, v_k] = \sum_{l=1}^{m} \psi_l(v_i, v_k) z_l$, $T[v_i, v_k] = \sum_{1 \leq s, l \leq m} \psi_l(v_i, v_k) h_{sl} z_s$ and $[Tv_i, Tv_k] = \sum_{1 \leq r, t \leq n} g_{rt} g_{tk} [v_r, v_t] = \sum_{1 \leq r, t \leq n} \sum_{s=1}^{m} g_{rt} g_{tk} \psi_s(v_r, v_t) z_s.$

Therefore $T$ is an automorphism if and only if $\sum_{i=1}^{m} \psi_l(v_i, v_k) h_{jl} = \sum_{1 \leq r, t \leq n} g_{rt} g_{tk} \psi_j(v_k, v_t)$ i.e. if and only if $(h, g) \in \text{Aut}_\eta(\eta)$.

For $\beta = [\beta_{ij}] \in \mathbb{R}_{m \times n}$, we define $T_\beta(z_i) = z_i$, $T_\beta(v_i) = v_i + \sum_{j=1}^{m} \beta_{ji} z_j$.

Then $B = \{T_\beta : \beta \in \mathbb{R}_{m \times n}\}$ is a normal abelian subgroup of $\text{Aut}(\eta)$ and by the above $\text{Aut}(\eta) \simeq B \rtimes \text{Aut}_\eta(\eta)$.

Remark 2.3. It is known that every automorphism of a classical Heisenberg algebra (dim$\zeta = 1$) can be obtained as the product of an inner automorphism by an element in $\text{Aut}_\eta(\eta)$.

We see here that, in general, this is not the case. Indeed, we take $x = z + v$, $z \in \zeta$, $v \in \vartheta$. Since $\eta$ is two step nilpotent, $\text{Ad}(\exp x) = I + a dv$ and the matrix of
The automorphism group of a Lie algebra

113

Ad (exp x) with respect to the basis \{z_1, \ldots, z_m, v_1, \ldots, v_n\} is

\[
\begin{pmatrix}
1 & \alpha \\
0 & 1
\end{pmatrix}
\]

with \(\alpha = \begin{pmatrix}
\psi_1(v, v_1) & \cdots & \psi_1(v, v_n) \\
\vdots & \ddots & \vdots \\
\psi_m(v, v_1) & \cdots & \psi_m(v, v_n)
\end{pmatrix} \in \mathbb{R}^{m \times m}.
\]

Furthermore, the rows of \(\alpha\) are orthogonal since

\[
J_i(v) = (\psi_i(v, v_1), \ldots, \psi_i(v, v_n)) \quad i = 1, \ldots, m.
\]

If \(m = 1\) the above assertion holds since for each \(b \in \mathbb{R}^n\) there exists a unique \(v \in \mathfrak{g}\) such that \(J_1(v) = b\).

If \(m \neq 1\), take \(T_\alpha\) with \(\text{rank}(\alpha) < m\). This \(T_\alpha\) cannot be written in this form.

Acknowledgements. I am indebted to Aroldo Kaplan with whom I had the pleasure to work on this problem. I am also grateful to Fulvio Ricci for several invaluable conversations. Research partially supported by CONICET, CONICOR and SecytUNC.

REFERENCES


Linda SAAL

Facultad de Matematica, Astronomia y Fisica

Universidad Nacional de Cordoba, Ciudad Universitaria, 5000 Cordoba, Argentina.

Lavoro pervenuto in redazione il 15.9.1995 e, in forma definitiva, il 20.4.1996.