Abstract. We describe some recent results concerning irrationality measures of values of hypergeometric functions at suitable rational points. Our method yields the best known irrationality measures of the logarithms of several rational numbers and of the constant \( \zeta(2) = \pi^2/6 \). The classical irrationality measures of these constants obtained by Beukers and by Alladi and Robinson through a purely analytic method are considerably improved using an arithmetical method introduced by Siegel in a different context. In the application of Siegel's method outlined here, we make use of the \( p \)-adic valuation of the gamma-factors occurring in the Euler integral representation of Gauss's hypergeometric function. For \( \zeta(2) \), we also use a suitable permutation group.

1. Introduction

For a given irrational number \( \alpha \), let \( \lambda \in \mathbb{R} \) satisfy the following property: for any \( \varepsilon > 0 \) there exists \( q_0 = q_0(\varepsilon) > 0 \) such that

\[
|\alpha - \frac{p}{q}| > q^{-\lambda - \varepsilon}
\]

for all integers \( p \) and \( q \) with \( q > q_0 \). Then \( \lambda \) is said to be an irrationality measure of \( \alpha \), and the least irrationality measure of \( \alpha \) is denoted by \( \mu(\alpha) \). Also, we say that \( \lambda \) is an effective irrationality measure of \( \alpha \) if the above \( q_0(\varepsilon) \) can be effectively computed as a function of \( \varepsilon \).

By well-known elementary arguments, we have \( \mu(\alpha) \geq 2 \) for all irrational \( \alpha \), and \( \mu(\alpha) = 2 \) for almost all irrational \( \alpha \), i.e. the set of \( \alpha \) for which \( \mu(\alpha) > 2 \) has measure zero. On the other hand, for any given real number \( \mu \) satisfying \( 2 \leq \mu \leq +\infty \), it is easy to construct numbers \( \alpha \) for which \( \mu(\alpha) = \mu \). The simplest way of doing this is to define \( \alpha \) by a suitable simple continued fraction expansion. If \( \mu(\alpha) = +\infty \), i.e. if for any arbitrarily large \( \lambda \) the inequality \( |\alpha - p/q| < q^{-\lambda} \) has infinitely many rational solutions \( p/q \), then \( \alpha \) is called a Liouville number.

Since \( \mu(\alpha) = 2 \) for almost all \( \alpha \), for any given irrational \( \alpha \) satisfying no special diophantine properties by definition, one expects that \( \mu(\alpha) = 2 \). For instance, we know
that \( \mu(e) = 2 \) because the simple continued fraction expansion of \( e \) is known to be

\[
e = [2; 1, 2, 1, 4, 1, \ldots, 1, 2n, 1, \ldots],
\]

and we conjecture that \( \mu(\pi) = 2 \), a result which is still unproved. So far, the best known irrationality measure of \( \pi \) is Hata’s upper bound [11]: \( \mu(\pi) < 8.016045 \ldots \). Hata’s result, as well as the above \( \mu(e) = 2 \), is effective.

The study of the irrationality measures of algebraic numbers has a long history. Roth’s celebrated theorem [14] states that for every algebraic irrational number \( \alpha \) (of any degree) we have \( \mu(\alpha) = 2 \). Unfortunately, Roth’s theorem is ineffective. Weaker theorems of the type \( \mu(\alpha) \leq \lambda(d) \), where \( d \geq 2 \) is the degree of the algebraic number \( \alpha \), were previously obtained by Liouville (with \( \lambda(d) = d \)), by Thue (\( \lambda(d) = d/2 + 1 \)), by Siegel (\( \lambda(d) = \min_{r \in \mathbb{N}} (d/(r+1) + r) \)), and by Dyson and Gelfond (\( \lambda(d) = \sqrt{2d} \)) independently. All these theorems, except Liouville’s elementary result \( \mu(\alpha) \leq d \), are also ineffective.

Important progress towards finding effective irrationality measures, improving upon Liouville’s theorem, of algebraic numbers as such, i.e. not viewed as special values of suitable algebraic functions to be approximated by Padé or Padé-type polynomials (see below), was obtained by Bombieri [6] in the early eighties (see also [7] for recent advances). Using a variant of the Thue–Siegel–Dyson method, Bombieri got in [6] effective and uniform irrationality measures for all the generators of some number fields. Bombieri’s results were subsequently extended by Viola [17] to all the irrational numbers (not necessarily generators) in those fields.

Our present knowledge about the irrationality measures of transcendental (or conjecturally transcendental) numbers is highly fragmentary. However, several results are known for the irrationality measures of the values of Gauss’s hypergeometric function and its generalizations at some rational or algebraic points. We recall that Gauss’s hypergeometric function is defined as follows:

\[
\begin{equation}
\begin{split}
_2 F_1(\alpha, \beta; \gamma; z) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n z^n}{(\gamma)_n n!},
\end{split}
\end{equation}
\]

where \( \alpha, \beta \) and \( \gamma \) are complex parameters, \( \gamma \neq 0, -1, -2, \ldots \), \( z \) is a complex variable satisfying \( |z| < 1 \), and the Pochhammer symbols \( (\alpha)_n \), \( (\beta)_n \) and \( (\gamma)_n \) are defined by

\[
(\xi)_0 = 1, \quad (\xi)_n = \xi(\xi + 1) \cdots (\xi + n - 1) \quad (n = 1, 2, \ldots).
\]

Natural generalizations of \( _2 F_1 \) are the functions

\[
pFq(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n z^n}{(\gamma_1)_n \cdots (\gamma_q)_n n!}
\]

(see [2]), the most important case being \( p = q + 1 \).
On Siegel's method in diophantine approximation to transcendental numbers

A typical feature of the results obtained for the hypergeometric functions is that they provide effective irrationality measures. It is also worth noting that for suitable special values of the parameters, the hypergeometric functions are indeed algebraic functions, so that general methods applying to values of hypergeometric functions at algebraic points yield, in some cases, effective irrationality measures of special algebraic numbers. The most important case is the binomial series

\[(1 + z)^x = \sum_{n=0}^{\infty} \binom{x}{n} z^n = \text{$_2$F$_1$}(-\alpha, \beta; \beta; -z)\]

(here $\beta$ is of course arbitrary), which takes on algebraic values for $\alpha \in \mathbb{Q}$, $z \in \overline{\mathbb{Q}}$. Along these lines Baker proved, in his pioneering paper [3], that $\mu(\sqrt{2}) < 2.955$, thus giving the first effective improvement upon Liouville's bound $\mu(\sqrt{2}) \leq 3$. Baker's method was refined and extended to several powers of rational numbers with rational exponents by Chudnovsky [8], who proved in particular the effective result $\mu(\sqrt{2}) < 2.42971$. Recent results of this type for simultaneous rational approximations can be found in [4] and [13].

The search for irrationality measures of values of hypergeometric functions is based on several methods, and relies in particular on the hypergeometric differential equation, or on the various integral representations of the hypergeometric functions (see [2] and [10]). Such methods are applied to construct Padé or Padé-type approximations to the hypergeometric functions involved, yielding good rational approximations at suitable rational or algebraic values of the variable. The appearance of Beukers' paper [5] aroused new interest in the Euler (or Euler-Pochhammer) integral representation of the hypergeometric functions $_{q+1}F_q$. The Euler representation for $_2F_1$, valid for $\Re \gamma > \Re \alpha > 0$:

\[
_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \int_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1} \frac{1}{(1-xz)\beta} \, dx,
\]

yields the analytic continuation of $_2F_1$ outside the disk $|z| < 1$ (see [10], p. 59), and was extended by Pochhammer to the functions $_{q+1}F_q$ in the following form (see [9]):

\[
_0^q F_q(\alpha_1, \ldots, \alpha_{q+1}; \gamma_1, \ldots, \gamma_q; z) = \frac{\Gamma(\gamma_1) \cdots \Gamma(\gamma_q) \Gamma(\gamma_1 - \alpha_1) \cdots \Gamma(\alpha_q) \Gamma(\gamma_q - \alpha_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_q) \Gamma(\gamma_1 - \alpha_1) \cdots \Gamma(\gamma_q - \alpha_q) \Gamma(\gamma_q)} \prod_{h=1}^{q} \int_0^1 \cdots \int_0^1 x_1^{\alpha_1-1} \cdots x_{q+1}^{\alpha_{q+1}-1} (1-x_1)^{\gamma_1-\alpha_1} \cdots (1-x_q)^{\gamma_q-\alpha_q} \frac{1}{(1-x_1 \cdots x_q z)^{\gamma_0}} \, dx_1 \cdots dx_q,
\]

for $\Re \gamma_h > \Re \alpha_h > 0$ ($h = 1, \ldots, q$).

In [5] Beukers considered integrals of Euler's type related to the constants

\[
\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 / 6 = _3F_2(1, 1, 1; 2, 2; 1)
\]
and

\[ \zeta(3) = \sum_{n=1}^{\infty} n^{-3} = {}_4F_0(1,1,1,1;2,2,2;1), \]

and gave a new proof of Apéry's theorem on the irrationality of such constants. Also, Beukers' method easily yields the irrationality measures \( \mu(\zeta(2)) = \mu(\pi^2) < 11.85078 \ldots \) and \( \mu(\zeta(3)) < 13.41782 \ldots \). The method introduced in [5] was subsequently applied by Alladi and Robinson [1] to one-dimensional integrals of Euler's type, thus obtaining irrationality measures of logarithms of rational numbers.

The irrationality results for the values of hypergeometric functions obtained through the construction of Padé or Padé-type approximating polynomials can be often improved by eliminating common prime factors of the values of such polynomials at the points considered. This powerful elimination method was introduced by Siegel [16] in a different context, i.e. in the study of E-functions, and systematically applied by Baker [3], Chudnovsky [8] and others, and recently by Hata in a series of papers (including [11]).

Siegel's method is usually applied through the analysis of the \( p \)-adic valuation of suitable binomial coefficients. A new idea was introduced in the recent paper [12], where the \( p \)-adic valuation of binomial coefficients is replaced by the \( p \)-adic valuation of the gamma-factors occurring in the above mentioned Euler-Pochhammer integral representation of the hypergeometric functions \( {}_{g+1}F_g \). This approach looks more intrinsic, and was combined in [12] with the symmetry properties of the hypergeometric functions with respect to the parameters, and with the study of a suitable permutation group, thus obtaining symmetric statements on the \( p \)-adic valuation of rational approximations to \( \zeta(2) \), and hence the best known irrationality measure \( \mu(\zeta(2)) = \mu(\pi^2) < 5.441242 \ldots \). For one-dimensional integrals, the method of [12] has been applied in [18] to obtain the best known irrationality measures of logarithms of several rational numbers.

The rest of this paper is a survey of the methods developed in [18] and [12].

We conclude this introduction by stating a well-known fundamental lemma, which shows that obtaining a good irrationality measure of a number \( \alpha \) depends on the determination of a sequence \( p_n/q_n \) of sufficiently good rational approximations to \( \alpha \).

**Lemma.** Let \( \alpha \in \mathbb{R} \), and let \( (p_n), (q_n) \) be sequences of integers satisfying

\[ \lim_{n \to \infty} \frac{1}{n} \log |p_n - q_n \alpha| = -R \]

and

\[ \limsup_{n \to \infty} \frac{1}{n} \log |q_n| \leq S \]
for some positive numbers $R$ and $S$. Then

$$\mu(\alpha) \leq \frac{S}{R} + 1.$$  

For a proof, see e.g. [8], Lemma 3.2. Note that if $p_n$ and $q_n$ have a common divisor $\Delta_n$ such that $\lim \frac{1}{n} \log \Delta_n = U > 0$, replacing in the Lemma $p_n$ with $p_n/\Delta_n$ and $q_n$ with $q_n/\Delta_n$ yields the improvement $\mu(\alpha) \leq (S - U)/(R + U) + 1$. This justifies the use of Siegel's method in our context.

2. One-dimensional Euler integrals

For integers $h, j, l$ satisfying $h \geq \max\{0, -l\}$, $j \geq \max\{0, l\}$, $\max\{h, j\} \leq \max\{j - l, h + l\}$, and for real $z > -1$, $z \neq 0$, let

$$I(h, j, l; z) = \frac{1}{1 + xz} \left(1 + x^h(1 - x)^j \right)^{\max\{0, -l\}} \int_0^1 \frac{x^h(1 - x)^j}{(1 + xz)^{j - l}} \frac{dx}{1 + xz}.$$ 

The change of variable $1 + xz = t$ yields

$$I(h, j, l; z) = (1 + z)^{\max\{0, -l\}} \int_1^{1 + z} \frac{(t - 1)^h(1 + z - t)^j}{t^{j - l}} \frac{dt}{t}.$$ 

This implies (see [18] for details)

$$I(h, j, l; z) = a(1 + z) + b(1 + z) \log(1 + z)$$

with $a(1 + z) \in \mathbb{Q}[1 + z]$ and $b(1 + z) \in \mathbb{Z}[1 + z]$. For any positive integer $n$ let

$$d_n = \text{l.c.m.}\{1, 2, \ldots, n\}.$$ 

Defining

$$M = \max\{j - l, h + l\},$$

we easily obtain $d_M a(1 + z) \in \mathbb{Z}[1 + z]$. By the above assumption $\max\{h, j\} \leq M$ we also have $\deg a(1 + z) \leq M$, $\deg b(1 + z) \leq M$. Moreover, a straightforward application of Cauchy’s integral formula allows one to express $b(1 + z)$ as a contour integral, in the form

$$b(1 + z) = (1 + z)^{\max\{0, -l\}} \frac{1}{2\pi i} \int_{|t| = \rho} \frac{(t - 1)^h(1 + z - t)^j}{t^{j - l}} \frac{dt}{t}$$

for any $\rho > 0$.

Since we are interested in applying the Lemma in Section 1 to get irrationality measures of logarithms of rational numbers, we choose $z = r/s$, with integers $r$ and $s$ satisfying $r \neq 0$, $s \geq 1$, $r > -s$, $(r, s) = 1$. Then

$$s^M d_M a(1 + r/s) \in \mathbb{Z}, \quad s^M b(1 + r/s) \in \mathbb{Z}.$$
The simplest choice for the parameters $h, j, l$, and in fact the best from a purely analytic viewpoint, is $h = j, l = 0$. Let, for $n = 1, 2, \ldots$

$$I_n = I(n, n, 0; r/s) = (r/s)^{2n+1} \int_0^1 \left( \frac{x(1-x)}{1+(r/s)x} \right)^n \frac{dx}{1+(r/s)x}$$

$$= a_n + b_n \log(1 + r/s).$$

Then, by (2.3),

$$s^n d_n I_n = s^n d_n a_n + s^n d_n b_n \log(1 + r/s) \in \mathbb{Z} + \mathbb{Z} \log(1 + r/s).$$

Clearly $\log d_n = \sum_{p \leq n} \log p = \psi(n)$, the Čebyšev $\psi$-function. The Prime Number Theorem yields $\log d_n = \psi(n) \sim n$. Therefore

$$\lim_{n \to \infty} \frac{1}{n} \log(s^n d_n) = \log s + 1.$$

Denoting by $x_0$ and $x_1$ the stationary points $\neq 0, 1$ of the function

$$f(x) = \frac{x(1-x)}{1+(r/s)x},$$

i.e. the solutions of $rx^2 + 2sx - s = 0$ with $0 < x_0 < 1$ and $1 + (r/s)x_1 < 0$, we plainly have

$$\lim_{n \to \infty} \frac{1}{n} \log |a_n + b_n \log(1 + r/s)| = \lim_{n \to \infty} \frac{1}{n} \log |I_n| = 2 \log |r/s| + \log f(x_0),$$

and from (2.2) we obtain

$$\frac{1}{n} \log |b_n| \leq \min_{\varrho > 0} \log \frac{(1+\varrho)(1 + r/s + \varrho)}{\varrho}.$$

With the change of variable $\varrho = -1 - (r/s)x$ we get

$$\min_{\varrho > 0} \log \frac{(1+\varrho)(1 + r/s + \varrho)}{\varrho} = \log((r/s)^2 f(x_1)).$$

Hence

$$\limsup_{n \to \infty} \frac{1}{n} \log |b_n| \leq 2 \log |r/s| + \log f(x_1).$$

It follows that

$$\lim_{n \to \infty} \frac{1}{n} \log |s^n d_n a_n + s^n d_n b_n \log(1 + r/s)| = \log f(x_0) + 2 \log |r/s| + 1 + \log s,$$

$$\limsup_{n \to \infty} \frac{1}{n} \log |s^n d_n b_n| \leq \log f(x_1) + 2 \log |r/s| + 1 + \log s,$$

whence, by the Lemma,

$$\mu(\log(1 + r/s)) \leq \frac{\log f(x_1) - \log f(x_0)}{-\log f(x_0) - 2 \log |r/s| - 1 - \log s},$$

for any $r$ and $s$ such that $-\log f(x_0) > 2 \log |r/s| + 1 + \log s$ (see [1]).
The above natural choice $h = j, l = 0$, however, does not permit the use of Siegel's method. For any integers $h, j, l$ satisfying the inequalities stated at the beginning of this section, we now define

$$I_n = I(hn, jn, ln; r/s) = a_n + b_n \log(1 + r/s) \quad (n = 1, 2, \ldots),$$

whence, by (2.3),

$$s^{Mn}d_{Mn}I_n = s^{Mn}d_{Mn}a_n + s^{Mn}d_{Mn}b_n \log(1 + r/s) \in \mathbb{Z} + \mathbb{Z}\log(1 + r/s).$$

For suitable choices of $h, j, l$ with $l \neq 0$, the integers $s^{Mn}d_{Mn}a_n$ and $s^{Mn}d_{Mn}b_n$ have a large common divisor $\Delta_n$ which must be eliminated to get an improvement on (2.4).

By the definition (1.1) we have

$$2F_1(\alpha, \beta; \gamma; z) = 2F_1(\beta, \alpha; \gamma; z).$$

Hence, for $\Re \gamma > \max\{\Re \alpha, \Re \beta\}$ and $\min\{\Re \alpha, \Re \beta\} > 0$, the Euler integral representation (1.2) yields

$$\int_0^1 \frac{x^{\alpha-1}(1-x)^{\gamma-\alpha-1}}{(1-xz)^\beta} \, dx = \frac{\Gamma(\alpha) \Gamma(\gamma - \alpha)}{\Gamma(\beta) \Gamma(\gamma - \beta)} \int_0^1 \frac{x^{\beta-1}(1-x)^{\gamma-\beta-1}}{(1-xz)^\alpha} \, dx.$$

Choosing $\alpha = h+1$, $\beta = j - l + 1$, $\gamma = h + j + 2$, and changing $z$ into $-z$, we obtain by (2.1)

$$I(h, j, l; z) = \frac{h! \, j!}{(j - l)! \, (h + l)!} I(j - l, h + l, l; z).$$

Let

$$I_n^* = I((j - l)n, (h + l)n, ln; r/s) = a_n^* + b_n^* \log(1 + r/s) \quad (n = 1, 2, \ldots).$$

By (2.5), (2.7), (2.8) and by the irrationality of $\log(1 + r/s)$ we have

$$((j - l)n)! \, ((h + l)n)! \, a_n = (hn)! \, (jn)! \, a_n^*.$$

Denote $M^* = \max\{h, j\}$. We have assumed $M^* \leq M$. Multiplying (2.9) by $s^{Mn}d_{Mn}$ we get

$$((j - l)n)! \, ((h + l)n)! \, A_n = K(hn)! \, (jn)! \, A_n^*,$$

where $A_n = s^{Mn}d_{Mn}a_n$, $A_n^* = s^{Mn}d_{M^*n}a_n^*$ and $K = d_{Mn}/d_{M^*n}$ are integers since $M^* \leq M$.

For any prime $p$ and any integer $H > 0$, let $v_p(H)$ be the $p$-adic valuation of $H$, i.e. the exponent of $p$ in the factorization of $H$ into prime powers. Clearly

$$v_p(H!) = \sum_{m \geq 1} \left[ \frac{H}{p^m} \right].$$
whence \( v_p(H!) = [H/p] \) if \( p > \sqrt{H} \). Therefore, denoting

\[
\alpha_p = v_p(((j - l)n)!(h + l)n)!), \quad \beta_p = v_p((hn)!(jn)!),
\]

for any prime \( p > \sqrt{Mn} \) we have

\[
\alpha_p = \left[ \frac{(j - l)n}{p} \right] + \left[ \frac{(h + l)n}{p} \right]
\]

and

\[
\beta_p = \left[ \frac{hn}{p} \right] + \left[ \frac{jn}{p} \right].
\]

Let

\[
\omega = \{ n/p \} = n/p - \lfloor n/p \rfloor
\]

be the fractional part of \( n/p \). For any integer \( H \) we have

\[
\frac{Hn}{p} = H \left[ \frac{n}{p} \right] + H\omega,
\]

whence

\[
\left[ \frac{Hn}{p} \right] = H \left[ \frac{n}{p} \right] + [H\omega].
\]

Thus

\[
\alpha_p - \beta_p = \left[ (j - l)\omega \right] + \left[ (h + l)\omega \right] - [h\omega] - [j\omega].
\]

The integer \( \alpha_p - \beta_p \) satisfies \(-1 \leq \alpha_p - \beta_p \leq 1 \) (see [12], Lemma 4.1). Hence, removing from (2.10) the primes \( p > \sqrt{Mn} \) dividing the factorials on both sides, we obtain

\[
(2.11) \quad (p_1 \cdots p_\lambda) PA_n = K(p'_1 \cdots p'_{\lambda'}) P'A_n^*,
\]

where \( p_1, \ldots, p_\lambda \) are the distinct primes \( > \sqrt{Mn} \) for which \( \alpha_p - \beta_p = 1 \), \( p'_1, \ldots, p'_{\lambda'} \) are the distinct primes \( > \sqrt{Mn} \) for which \( \alpha_p - \beta_p = -1 \), i.e.

\[
(2.12) \quad [(j - l)\omega] + [(h + l)\omega] < [h\omega] + [j\omega],
\]

and \( P, P' \) are products of primes \( \leq \sqrt{Mn} \). By (2.11), \( p'_1, \ldots, p'_{\lambda'} \) divide \( A_n \). Thus any prime \( p > \sqrt{Mn} \) satisfying (2.12) divides \( A_n \).

The set \( \Omega \) of \( \omega \in [0, 1) \) satisfying (2.12) is clearly the union of finitely many intervals \([\ell_q, \sigma_q]\), where each \( \ell_q \) or each \( \sigma_q \) is a rational number whose denominator divides at least one of the integers \( h, j, j - l, h + l \). Let

\[
\Delta_n = \prod_{p > \sqrt{Mn}} p, \quad D_n = \frac{d_{Mn}}{\Delta_n}.
\]

By the above argument,

\[
\delta^{Mn} D_n a_n = \frac{\Lambda_n}{\Delta_n} \in \mathbb{Z}.
\]
Since \( 1/M \leq 1/M^* = \min \{1/h, 1/j\} \), for \( \omega < 1/M \) we have \([h\omega] = [j\omega] = 0\), whence \( \omega \not\in \Omega \). Thus \( p | \Delta_n \) yields \( \omega = \{n/p\} \in \Omega \), \( \omega \geq 1/M \), \( n/p \geq \omega \geq 1/M \), \( p \leq Mn \), \( p \nmid d_{Mn} \). Hence \( D_n \in \mathbb{Z} \). Also \( s^{Mn} b_n \in \mathbb{Z} \) by (2.3). Therefore we can apply the Lemma to

\[
s^{Mn} D_n l_n = s^{Mn} D_n a_n + s^{Mn} D_n b_n \log(1 + r/s) \in \mathbb{Z} + \mathbb{Z} \log(1 + r/s).
\]

Note that for \( l = 0 \) the set \( \Omega \) is empty and \( \Delta_n = 1 \), so that no arithmetical correction to the constant \( \lim \frac{1}{n} \log(s^{Mn} d_{Mn}) = M(\log s + 1) \) is possible, i.e. Siegel's method is not applicable for \( l = 0 \).

Since

\[
\lim_{n \to \infty} \frac{1}{n} \log(s^{Mn} D_n) = \lim_{n \to \infty} \frac{1}{n} \log(s^{Mn} d_{Mn}) - \lim_{n \to \infty} \frac{1}{n} \log \Delta_n
\]

we must evaluate \( \lim \frac{1}{n} \log \Delta_n \). Let

\[
\tilde{\Delta}_n = \prod_{\{n/p\} \in \Omega} p, \quad \delta_n = \prod_{p \leq \sqrt{Mn}} p,
\]

whence \( \tilde{\Delta}_n = \delta_n \Delta_n \). Denoting by \( \vartheta(x) = \sum_{p \leq x} \log p \) the Čebyšev \( \vartheta \)-function, we have

\[
\log \delta_n \leq \vartheta(\sqrt{Mn}) = O(\sqrt{n}),
\]

whence \( \lim \frac{1}{n} \log \delta_n = 0 \). Therefore

\[
(2.13) \quad \lim_{n \to \infty} \frac{1}{n} \log \tilde{\Delta}_n = \lim_{n \to \infty} \frac{1}{n} \log \Delta_n.
\]

Let now

\[
\Omega = \bigcup_{q = 1}^{Q} [q, \sigma_q),
\]

where \( [q, \sigma_q) \) and \( [q', \sigma_{q'}) \) are disjoint for \( q \neq q' \), and denote \( [n/p] = \nu \). Then

\[
\log \tilde{\Delta}_n = \sum_{q = 1}^{Q} \sum_{e \leq \{n/p\} < \sigma_q} \log p = \sum_{q = 1}^{Q} \sum_{\nu = 0}^{\infty} \sum_{e \leq \{n/p\} < \sigma_q} \log p,
\]

where the dash means that the sum ranges over the primes \( p \) satisfying

\[
\frac{n}{\nu + \sigma_q} < p \leq \frac{n}{\nu + q_q}.
\]

It follows that

\[
(2.14) \quad \frac{1}{n} \log \tilde{\Delta}_n = \sum_{q = 1}^{Q} \sum_{\nu = 0}^{\infty} \vartheta(q) \Phi_{\nu}(n),
\]
where
\[ \Theta^{(q)}(n) = \frac{1}{n} \sum \log p = \frac{1}{n} \left( \vartheta \left( \frac{n}{\nu + q} \right) - \vartheta \left( \frac{n}{\nu + \sigma_q} \right) \right). \]

For brevity, let
\[ \Lambda^{(q)}(n) = \frac{1}{\nu + q} - \frac{1}{\nu + \sigma_q} = \frac{\sigma_q - q}{(\nu + q)(\nu + \sigma_q)}. \]

For any \( q \) and \( \nu \) we get, by the Prime Number Theorem,
\[ (2.15) \quad \lim_{n \to \infty} \Theta^{(q)}(n) = \Lambda^{(q)}. \]

Since the series \( \sum_{\nu=0}^{\infty} \Lambda^{(q)}(q) \) converges, we have, for any fixed \( q \),
\[ \left| \sum_{\nu=0}^{\infty} \Theta^{(q)}(n) - \sum_{\nu=0}^{\infty} \Lambda^{(q)}(n) \right| \leq \left| \sum_{\nu \leq N} \Theta^{(q)}(n) - \sum_{\nu \leq N} \Lambda^{(q)}(n) \right| + \sum_{\nu > N} \Theta^{(q)}(n) + \sum_{\nu > N} \Lambda^{(q)}(n), \]
where the last two sums are small for any sufficiently large \( N \) and any \( n \) (note that from \( \sigma_q \leq 1 \leq 1 + q \) we have \( n/(\nu + 1 + q) \leq n/(\nu + \sigma_q) \), whence
\[ \sum_{\nu > N} \Theta^{(q)}(n) = \frac{1}{n} \sum_{\nu > N} \sum \log p \leq \frac{1}{n} \vartheta \left( \frac{n}{N + q} \right) = O(1/N), \]
where the constant implicit in \( O \) is absolute. Once \( N \) is fixed, the difference
\[ \left| \sum_{\nu \leq N} \Theta^{(q)}(n) - \sum_{\nu \leq N} \Lambda^{(q)}(n) \right|, \]
by (2.15), is small for any sufficiently large \( n \). Hence
\[ \lim_{n \to \infty} \sum_{\nu=0}^{\infty} \Theta^{(q)}(n) = \sum_{\nu=0}^{\infty} \Lambda^{(q)}(n). \]

Moreover
\[ \sum_{\nu=0}^{\infty} \Lambda^{(q)}(n) = \Psi(\sigma_q) - \Psi(q), \]
where
\[ \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \]
is the logarithmic derivative of the Euler gamma–function (see [10], formula (3) p. 15).
From (2.13) and (2.14) we obtain
\[ \lim_{n \to \infty} \frac{1}{n} \log \Delta_n = \sum_{q=1}^{Q} (\Psi(\sigma_q) - \Psi(q)) = \int_{\Omega} d\Psi(x). \]
Therefore
\[ \lim_{n \to \infty} \frac{1}{n} \log \left( s^M D_n \right) = M(1 + \log s) - \int_{\Omega} d\Psi(x). \]
Combining the arithmetical correction $\int_{\Omega} d\mathcal{P}(x)$ with the asymptotics for $I_n$ and $b_n$, one obtains good irrationality measures of $\log(1 + r/s)$ (see [18]). The only (theoretically unsolved) problem that remains is to find the optimal values for $h, j, l$ as functions of $r$ and $s$.

In the special case $r = s = 1$, computer tests indicate that the optimal $h, j, l$ are likely to be $h = j = 7, l = 1$. These choices yield the best known irrationality measure of $\log 2$, namely Rukhadze's bound [15]: $\mu(\log 2) < 3.891399\ldots$. Note that in this case Alladi and Robinson's bound [1], obtained without arithmetical correction i.e. for $l = 0$, is $\mu(\log 2) < 4.6221\ldots$.

3. Two-dimensional Euler integrals

The analogue of the preceding theory for two-dimensional integrals has a far richer structure, and has been thoroughly worked out (in [12]) only for the constant $\zeta(2)$, although it can be also applied, with additional complications, to the values of the dilogarithm at rational points. The dilogarithm is defined by

$$L_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_{0}^{1} \frac{\log(1-t)}{t} \, dt,$$

whence $\zeta(2) = L_2(1)$. Note that $z = 1$ is a branch point of $L_2(z)$, so that one may expect the arithmetical properties of $L_2(1)$ to be in some sense peculiar in comparison with those of $L_2(r/s)$, while no similar singularity case occurs for the function $\log(1 + z)$ in the range $z > -1$.

We sketch here the main results of the paper [12]. The relevant integrals for $\zeta(2)$ are

$$I(h, i, j, k, l) = \int_{0}^{1} \int_{0}^{1} \frac{a^{h}(1-x)^{i}y^{k}(1-y)^{j}}{(1-xy)^{i+j+l}} \, dx \, dy,$$

where the parameters $h, i, j, k, l$ are any non-negative integers. It can be shown that $I(h, i, j, k, l) = a + b\zeta(2)$ with $a \in \mathbb{Q}$ and $b \in \mathbb{Z}$. Moreover, the value of $I(h, i, j, k, l)$ is unchanged under a cyclic permutation of $h, i, j, k, l$. Also, interchanging the variables $x, y$ in the integral yields $I(h, i, j, k, l) = I(k, j, i, h, l)$. Thus, if we call $\tau$ and $\sigma$ the permutations

$$\tau = (h \, i \, j \, k \, l), \quad \sigma = (h \, k)(i \, j),$$

the value of $I(h, i, j, k, l)$ is invariant under the action on the parameters of the permutation group $T = \langle \tau, \sigma \rangle$ generated by $\tau$ and $\sigma$. The group $T$ is clearly isomorphic to the dihedral group $D_5$ of order 10.

Using the permutation $\tau$, one can suitably define two integers $M$ and $N$ with $M \geq N \geq 0$, invariant under the action of the group $T$, such that if $I(h, i, j, k, l) = a + b\zeta(2)$
then \( d_M d_N a \in \mathbb{Z} \) (we define \( d_0 = 1 \)). Also, the integer \( \delta \) can be expressed as a double contour integral (see [12], Section 2, for details).

Assume now that the non-negative integers

\[(3.1) \quad h, i, j, k, l \]

are such that

\[(3.2) \quad j + k - h, k + l - i, l + h - j, h + i - k, i + j - l \]

are also non-negative. Choosing in (2.6) \( \alpha = h + 1 \), \( \beta = i + j - l + 1 \), \( \gamma = h + i + 2 \), \( z = y \), we have

\[
\int_0^1 \frac{x^h(1-x)^i}{(1-xy)^{i+j-l+1}} \, dx = \frac{h! \, i!}{(i+j-l)!(l+h-j)!} \int_0^1 \frac{x^{i+j-l}(1-x)^{i+h-j}}{(1-xy)^{h+1}} \, dx.
\]

Multiplying by \( y^k(1-y)^j \) and integrating in \( 0 \leq y \leq 1 \) we get the transformation formula

\[(3.3) \quad I(h, i, j, k, l) = \frac{h! \, i!}{(i+j-l)!(l+h-j)!} \, I(i+j-l, l+h-j, j, k, l), \]

similar to (2.7). Hence

\[(3.4) \quad \frac{I(h, i, j, k, l)}{h! \, i! \, j! \, k! \, l!} = \frac{I(i+j-l, l+h-j, j, k, l)}{(i+j-l)!(l+h-j)! \, j! \, k! \, l!}. \]

Let \( \varphi \) be the permutation of the ten integers (3.1) and (3.2) which can be naturally associated with (3.4), i.e. the permutation defined by

\[
\varphi = (h \ i \ j \ k \ l)(i \ l+h-j)(j+k-h \ k+l-i),
\]

and let \( \tau \) and \( \sigma \) act on the integers (3.2) in the obvious way, so that

\[
\tau = (h \ i \ j \ k \ l)(j+k-h \ k+l-i \ l+h-j \ h+i-k \ i+j-l),
\]

\[
\sigma = (h \ k)(i \ j)(j+k-h \ h+i-k)(k+l-i \ l+h-j).
\]

By (3.4), the value of

\[(3.5) \quad \frac{I(h, i, j, k, l)}{h! \, i! \, j! \, k! \, l!} \]

is invariant under the action of the permutation group \( \Phi = \{ \varphi, \tau, \sigma \} \).

Applying any product of permutations \( \varphi, \tau \) and \( \sigma \), the invariance of the value of (3.5) yields several transformation formulae for \( I(h, i, j, k, l) \) of the type (3.3). As in Section 2, Siegel's method is applied using the \( p \)-adic valuation of the quotients of factorials occurring in such transformation formulae, and these quotients of factorials are related with the structure of the permutation group \( \Phi \).

The group \( \Phi \) turns out to be isomorphic to the symmetric group \( S_5 \) of the \( 5! = 120 \) permutations of five elements (see [12], Section 3). Since \( |\Phi| = 120 \) and \( |T| = 10 \),
there are 12 left cosets of $T$ in $\Phi$. The ten permutations belonging to any such coset yield transformation formulae having the same quotient of factorials, whereas any two permutations belonging to different cosets yield transformation formulae with different quotients of factorials. Thus the left cosets of $T$ in $\Phi$ can be characterized by all the twelve quotients of factorials satisfying the following properties:

(i) The numerator and the denominator are products of the same number of factorials.

(ii) The integers occurring in the numerator belong to the set $\{h, i, j, k, l\}$ and the integers in the denominator belong to $\{j+k-h, k+l-i, l+h-j, h+i-k, i+j-l\}$.

(iii) The sum of the integers in the numerator equals the sum of the integers in the denominator.

Let $I_n = I(h_n, i_n, j_n, k_n, l_n) = a_n + b_n \zeta(2) \quad (n = 1, 2, \ldots)$, whence

$$d_{Mn}d_{Nn}I_n = d_{Mn}d_{Nn}a_n + d_{Mn}d_{Nn}b_n \zeta(2) \in \mathbb{Z} + \mathbb{Z}\zeta(2).$$

Applying the arithmetical method described in Section 2 to the quotients of factorials occurring in the transformation formulae for $I(h_n, i_n, j_n, k_n, l_n)$, we can define two unions $\Omega$ and $\Omega'$ of finitely many intervals contained in $[0, 1)$, with $\Omega' \subset \Omega$, such that any prime $p > \sqrt{Mn}$ satisfying $\{n/p\} \in \Omega$ divides $A_n = d_{Mn}d_{Nn}a_n$, and for any prime $p > \sqrt{Mn}$ satisfying $\{n/p\} \in \Omega'$ we have $p^2 \mid A_n$. Therefore, defining

$$\Delta_n = \prod_{p > \sqrt{Mn}} p, \quad \Delta'_n = \prod_{p > \sqrt{Mn}} p \quad (n = 1, 2, \ldots),$$

we get $\Delta_n \Delta'_n \mid A_n$. Also $\Delta_n \mid d_{Mn}$, $\Delta'_n \mid d_{Nn}$. Hence the integer

$$D_n = \frac{d_{Mn}d_{Nn}}{\Delta_n \Delta'_n}$$

is such that

$$D_nI_n = D_na_n + D_nb_n \zeta(2) \in \mathbb{Z} + \mathbb{Z}\zeta(2),$$

and by the method of Section 2 we obtain

$$\lim_{n \to \infty} \frac{1}{n} \log D_n = M + N - \left( \int_{\Omega} d\Psi(x) + \int_{\Omega'} d\Psi(x) \right).$$

Combining this with the asymptotics for $I_n$ and $b_n$, under the further assumption that the integers (3.1) and (3.2) are all strictly positive, one gets irrationality measures of $\zeta(2)$. The choice $h = i = 12$, $j = k = 14$, $l = 13$ yields $\mu(\zeta(2)) < 5.441242 \ldots$ ([12], Section 5).

We conclude this brief outline of the two–dimensional case by mentioning an open problem. We have obtained (3.3), and hence all the other transformation formulae
for $I(h, i, j, k, l)$ corresponding to the permutations in the group $\Phi$, by means of (2.6), i.e. by the symmetry of $2 F_1(\alpha, \beta; \gamma; z)$ with respect to $\alpha$ and $\beta$, or, in other words, using one-dimensional Euler integrals. At first sight, the use of the symmetry of $3 F_2(\alpha_1, \alpha_2, \alpha_3; \gamma_1, \gamma_2; 1)$ with respect to $\alpha_1, \alpha_2, \alpha_3$ or $\gamma_1, \gamma_2$ looks more appropriate, since the Euler--Pochhammer integral representation of $3 F_2$ directly translates this symmetry into transformation formulae for two-dimensional integrals of the desired type. Also, many relations, involving gamma--factors, among functions $3 F_2$ with unit argument are known, and can be conveniently expressed by making use of the Thomae--Whipple functions $F_p$ and $F_n$ (see [2], Chapter III). Surprisingly enough, all the transformation formulae for $I(h, i, j, k, l)$ obtainable in this way correspond to permutations belonging to $\Phi$. This indicates that the structure of the transformation formulae of the type considered ought to be rigid, and leads us to the following conjecture ([12], Conjecture 3.2).

**Conjecture.** Let $h, i, j, k, l$ and $h', i', j', k', l'$ be any non-negative integers such that

\[
j + k - h, \quad k + l - i, \quad l + h - j, \quad h + i - k, \quad i + j - l
\]

and

\[
j' + k' - h', \quad k' + l' - i', \quad l' + h' - j', \quad h' + i' - k', \quad i' + j' - l'
\]

are also non-negative. If

\[
\frac{I(h, i, j, k, l)}{I(h', i', j', k', l')} \in \mathbb{Q},
\]

then there exists a permutation $\chi \in \Phi = \langle \varphi, \tau, \sigma \rangle$ such that

\[
h' = \chi(h), \quad i' = \chi(i), \quad j' = \chi(j), \quad k' = \chi(k), \quad l' = \chi(l).
\]

If this conjecture is true, no permutation group larger than $\Phi$ can be found to which the method outlined above can be applied, so that for each choice of the parameters $h, i, j, k, l$ our method applied to $\Phi$ must yield the best irrationality measure of $\zeta(2)$ obtainable with those parameters.

**References**


On Siegel's method in diophantine approximation to transcendental numbers


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