J. Berndt – L. Vanhecke

GEODESIC SPRAYS AND $\mathcal{C}$- AND $\mathcal{P}$-SPACES

Abstract. We review some properties of $\mathcal{C}$- and $\mathcal{P}$-spaces and derive some new characterizations for these manifolds by means of the geodesic spray associated to the Sasaki metric on the tangent bundle.

1. Introduction

It is a well-known fact that the curvature of a Riemannian manifold $(M, g)$ is geometrically reflected by the behaviour of (one-parameter) families of adjacent geodesics. Analytically, such families of adjacent geodesics are described by Jacobi fields, which are just the solutions of the second order differential equation

$$\nabla_{\gamma} \nabla_{\gamma} Y + R(Y, \dot{\gamma})\dot{\gamma} = 0$$

along some geodesic $\gamma$ in $M$. (Here, $\nabla$ is the Levi Civita connection and $R$ the Riemannian curvature tensor of $(M, g)$.) Thus, along the geodesic $\gamma$, the curvature of $M$ is controlled by the self-adjoint tensor field

$$R_{\gamma} := R(\cdot, \dot{\gamma})\dot{\gamma}$$

along $\gamma$. $R_{\gamma}$ is called the Jacobi operator of $M$ along $\gamma$. The eigenvalues of $R_{\gamma}$ are real everywhere and, at least in the real analytic situation, $R_{\gamma}$ is diagonalizable by an orthonormal frame field along $\gamma$. In [1] we have shown that a Riemannian manifold is a locally symmetric space if and only if for every geodesic $\gamma$ in $M$ the associated Jacobi operator $R_{\gamma}$ has constant eigenvalues.
and is diagonalizable by a parallel orthonormal frame field along $\gamma$. This suggests to study the following two classes of Riemannian manifolds:

1. the class $\mathfrak{C}$ consisting of all connected Riemannian manifolds $M$ with the property that for every geodesic $\gamma$ in $M$ the Jacobi operator $R_\gamma$ has constant eigenvalues, and
2. the class $\mathfrak{P}$ consisting of all connected Riemannian manifolds $M$ with the property that for every geodesic $\gamma$ in $M$ the Jacobi operator $R_\gamma$ is diagonalizable by a parallel orthonormal frame field along $\gamma$.

We say that $M$ is a $\mathfrak{C}$-space (resp. a $\mathfrak{P}$-space) if $M$ belongs to the class $\mathfrak{C}$ (resp. to the class $\mathfrak{P}$).

We now summarize some basic results on $\mathfrak{C}$- and $\mathfrak{P}$-spaces. For details we refer to [1] and [2]. Henceforth all manifolds are assumed to be connected and of class $C^\infty$.

A. The Ricci tensor $\text{ric}$ of any $\mathfrak{C}$-space is a Killing tensor, that is, the cyclic sum over all entries in $\nabla\text{ric}$ vanishes. Consequently, every $\mathfrak{C}$-space is real analytic in suitable coordinates and has constant scalar curvature.

B. The Riemannian product $M \times N$ of two Riemannian manifolds $M$ and $N$ is a $\mathfrak{C}$-space if and only if $M$ and $N$ are $\mathfrak{C}$-spaces. An analogous statement holds for $\mathfrak{P}$-spaces if $M$ and $N$ are assumed to be real analytic. So the classification of $\mathfrak{C}$-spaces and, in the real analytic case, of $\mathfrak{P}$-spaces can be reduced to irreducible manifolds.

C. Any Riemannian g.o. space and any commutative space (or Gelfand space) is a $\mathfrak{C}$-space. A Riemannian g.o. space is a Riemannian homogeneous space $M$ whose geodesics are orbits of one-parameter subgroups of the group of isometries of $M$. Every naturally reductive Riemannian homogeneous space is a Riemannian g.o. space. Note that generalized Heisenberg groups with two-dimensional center are Riemannian g.o. spaces which are in no way naturally reductive. A commutative space is a Riemannian homogeneous space whose algebra of all invariant (with respect to the connected component of the identity of the full isometry group of the space) differential operators is commutative. Examples of Riemannian g.o. spaces which are not commutative spaces are also provided by certain generalized Heisenberg groups.
All (simply connected) complete $\mathfrak{C}$-spaces know to us are homogeneous Riemannian manifolds. Therefore we want to pose the

**Problem 1.** Is any (simply connected) complete $\mathfrak{C}$-space a homogeneous Riemannian manifold?

**D.** In dimension two, the $\mathfrak{C}$-spaces are precisely the spaces of constant curvature, which are also just the locally symmetric spaces.

In dimension three, a Riemannian manifold is a $\mathfrak{C}$-space if and only if it is locally isometric to a naturally reductive Riemannian homogeneous space. In [3] we derived the following complete list of all three-dimensional, irreducible, simply connected, naturally reductive Riemannian homogeneous spaces:

- the standard spaces of constant curvature $c$ (namely $S^3(c), E^3, H^3(c)$);
- geodesic hyperspheres in $\mathbb{C}P^2(c)$ or $\mathbb{C}H^2(c)$;
- horospheres in $\mathbb{C}H^2(c)$;
- universal covering spaces of tubes about the totally geodesic subspace $\mathbb{C}H^1(c)$ in $\mathbb{C}H^2(c)$.

Here, $\mathbb{C}P^2(c)$ and $\mathbb{C}H^2(c)$ denote the two-dimensional complex projective and hyperbolic space, respectively, endowed with the Fubini-Study metric of constant holomorphic sectional curvature $c$. The original classification in terms of Lie groups is due to F. Tricerri and the second author [17] and to O. Kowalski [13].

There is the obvious

**Problem 2.** Classify the $\mathfrak{C}$-spaces in dimensions greater than three.

**E.** Any two-dimensional Riemannian manifold is a $\mathfrak{P}$-space. A three-dimensional $\mathfrak{P}$-space is almost everywhere locally isometric to one of the following spaces:

- a Riemannian symmetric space;
- a warped product $M^1 \times_f M^2$, where $M^k$ is a $k$-dimensional Riemannian manifold;
- a warped product $M^2 \times_f M^1$, where $M^1$ is a one-dimensional Riemannian manifold, $M^2$ is a Liouville surface with Riemannian metric.
\[ ds^2 = (\varphi(x) + \psi(y))(dx^2 + dy^2), \text{ and } f^2(x, y) = |\varphi(x) \cdot \psi(y)|; \]

- a triply orthogonal system of surfaces with Riemannian metric \( ds^2 \) cyclic sum of \( F_k(x) |x_1 - x_2| |x_1 - x_3| dx^2 \), where \( F_k \) is a positive function depending on \( x_k \) only.

Conversely, every real analytic space in this list is a \( \mathfrak{P} \)-space. The proof of this classification is intimately related to the problem whether the Schrödinger equation of the geodesics of \( M \) can be solved by simple separation of variables in orthogonal coordinates. This question has been studied by H.P. Robertson and L.P. Eisenhart.

Other examples are given in [7] where it is proved that each semi-symmetric manifold of cone type (see [14], [15]) is a non-complete \( \mathfrak{P} \)-space. Further it is proved that a complete semi-symmetric \( \mathfrak{P} \)-space is a local product of symmetric spaces and two-dimensional manifolds. So we mention

**Problem 3.** Find other locally irreducible \( \mathfrak{P} \)-spaces in dimensions greater than three.

A much more difficult problem might be the classification of \( \mathfrak{P} \)-spaces:

**Problem 4.** Classify the \( \mathfrak{P} \)-spaces in dimensions greater than three.

\( \Gamma \). Let \((M, g)\) be a Riemannian manifold and consider two geodesic spheres \( G_1 \) and \( G_2 \) in \( M \) of some sufficiently small radius \( r \in \mathbb{R}_+ \) and being tangent to each other at some point \( m \in M \). The second author and T.J. Willmore [18] have shown that \( M \) is a locally symmetric space if and only if for all such configurations \((G_1, G_2, r, m)\) the shape operators of \( G_1 \) and \( G_2 \) at \( m \) coincide.

Bearing this in mind we proved in [2] that a real analytic Riemannian manifold \( M \) is a \( \mathfrak{P} \)-space if and only if for all such configurations \((G_1, G_2, r, m)\) the shape operators of \( G_1 \) and \( G_2 \) at \( m \) are simultaneously diagonalizable.

Let \( \mathfrak{T} \mathfrak{C} \) be the class of all Riemannian manifolds with the property that for all such configurations \((G_1, G_2, r, m)\) the shape operators of \( G_1 \) and \( G_2 \) at \( m \) have the same eigenvalues (counted with multiplicities). Then \( \mathfrak{T} \mathfrak{C} = \mathfrak{C} \) in dimensions two and three, and \( \mathfrak{T} \mathfrak{C} \subset \mathfrak{C} \) in dimensions greater than three. Therefore we state

**Problem 5.** Do the classes \( \mathfrak{T} \mathfrak{C} \) and \( \mathfrak{C} \) coincide in general?
It is worth to mention that every $\mathcal{C}$-space is a space with volume-preserving geodesic symmetries.

G. In order to treat $\mathcal{C}$- and $\mathcal{P}$-spaces analytically we have the following useful characterizations. For any tangent vector $v$ to a Riemannian manifold $M$ we define the covariant derivative $R'_v$ of $R_v := R(.,v)v$ in direction of $v$ by

$$R'_v := (\nabla_v R)(.,v)v.$$

Then $M$ is a $\mathcal{C}$-space if and only if for every $p \in M$ and $v \in T_pM$ there exists a skew-symmetric endomorphism $T_v$ of $T_pM$ such that $R'_v = R_v \circ T_v - T_v \circ R_v$. If $M$ is real analytic then $M$ is a $\mathcal{P}$-space if and only if $R'_v \circ R_v = R_v \circ R'_v$ for all $v \in TM$.

H. For more details and applications (in particular to the theory of curvature-adapted submanifolds) we refer to [1], [2] and [4].

We shall now continue our work on $\mathcal{C}$- and $\mathcal{P}$-spaces by studying the Jacobi operator $\bar{R}_S$ of the geodesic spray $S$ on the tangent bundle $TM$ of $M$, where $TM$ is endowed with the Sasaki metric $\bar{g}$. The motivation for this arises from the following consideration. Endowing $TM$ with a Riemannian metric $\bar{g}$ we can define the Jacobi operator $\bar{R}_S$ of $(TM,\bar{g})$ with respect to the geodesic spray $S$. Via the canonical projection from $TM$ onto $M$ there is a natural one-to-one correspondence between the geodesics in $M$ and the integral curves of $S$. So, instead of studying the family $\{R_\gamma\}$ of Jacobi operators on $M$ with respect to geodesics $\gamma$ one is tempted to study the single operator $\bar{R}_S$. We shall do this here for the case where the Riemannian metric on $TM$ is chosen to be the Sasaki metric, although there are other possible choices.

In Section 2 we summarize basic facts about the Riemannian geometry of the tangent bundle of a Riemannian manifold arising from the Sasaki metric on $TM$. Then, in Section 3, we shall characterize locally symmetric spaces, $\mathcal{C}$-spaces and $\mathcal{P}$-spaces by properties of the Jacobi operator $\bar{R}_S$.

2. On the tangent bundle of a Riemannian manifold

In this section we recall some basic facts concerning the Riemannian geometry of the tangent bundle of a Riemannian manifold. For details we refer to [8] and [12].
Let $M$ be an $n$-dimensional Riemannian manifold with Riemannian metric $g$, Levi Civita connection $\nabla$ and Riemannian curvature tensor $R$ with the convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. We denote by $TM$ the tangent bundle of $M$ and by $\pi : TM \to M$ the canonical projection. The connection map $K : TTM \to TM$ is defined by

$$Ku := \nabla_{\pi_*u} I$$

for all $u \in TTM$, where $I := id_{TM}$ is regarded as a vector field along $\pi$. The connection map relates two different methods of differentiating the tangent vector field $\dot{c}$ of any curve $c$ in $M$, namely we have

$$K\dot{c} = \nabla_{\dot{c}} \dot{c}.$$

We denote by

$$\mathcal{H} := \text{kernel of } K$$

and

$$\mathcal{V} := \text{kernel of the differential } \pi_* \text{ of } \pi$$

the horizontal and vertical subbundle of $TTM$, respectively. For each $v \in T_p M, p \in M$, the map

$$T_v TM \to T_p M \oplus T_p M, \quad u \mapsto (\pi_*u, Ku)$$

is an isomorphism. This gives rise to define for each $v \in TM$ the horizontal lift $v^\mathcal{H} \in \mathcal{H}_v$, and the vertical lift $v^\mathcal{V} \in \mathcal{V}_v$, of $v$ by

$$\pi_* v^\mathcal{H} = v, \quad Kv^\mathcal{H} = 0$$

and

$$\pi_* v^\mathcal{V} = 0, \quad Kv^\mathcal{V} = v,$$

respectively.

We now define a Riemannian metric $\tilde{g}$ on $TM$ by

$$\tilde{g}(A, B) := g(\pi_* A, \pi_* B) + g(KA, KB)$$

for all vector fields $A, B$ on $TM$. With respect to this metric, which is also known as the Sasaki metric on $TM$, the projection $\pi$ becomes a Riemannian
submersion. We denote by $\nabla$ the Levi Civita connection and by $\bar{R}$ the Riemannian curvature tensor of $(TM, \bar{g})$. For vector fields $A, B, C$ on $TM$ we have

$$\pi_* \nabla_A B = \nabla_{\pi_* A} \pi_* B + \frac{1}{2} R(I, KA) \pi_* B + \frac{1}{2} R(I, KB) \pi_* A,$$

$$K \nabla_A B = \nabla_{\pi_* A} KB - \frac{1}{2} R(\pi_* A, \pi_* B) I,$$

$$\pi_* \bar{R}(A, B) C = R(\pi_* A, \pi_* B) \pi_* C$$

$$- \frac{1}{2} R(KB, KC) \pi_* A + \frac{1}{2} R(KA, KC) \pi_* B + R(KA, KB) \pi_* C$$

$$+ \frac{1}{2} (\nabla_{\pi_* B} R)(KC, I) \pi_* A - \frac{1}{2} (\nabla_{\pi_* A} R)(KC, I) \pi_* B$$

$$+ \frac{1}{2} (\nabla_{\pi_* B} R)(KA, I) \pi_* C - \frac{1}{2} (\nabla_{\pi_* A} R)(KB, I) \pi_* C$$

$$- \frac{1}{4} R(KB, I) R(KC, I) \pi_* A + \frac{1}{4} R(KA, I) R(KC, I) \pi_* B$$

$$- \frac{1}{4} R(KB, I) R(KA, I) \pi_* C + \frac{1}{4} R(KA, I) R(KB, I) \pi_* C$$

$$- \frac{1}{4} R(R(\pi_* A, \pi_* C) I, I) \pi_* B + \frac{1}{4} R(R(\pi_* B, \pi_* C) I, I) \pi_* A$$

$$- \frac{1}{2} R(R(\pi_* A, \pi_* C) I, I) \pi_* C$$

$$K \bar{R}(A, B) C = R(\pi_* A, \pi_* B) KC - \frac{1}{2} R(\pi_* B, \pi_* C) KA + \frac{1}{2} R(\pi_* A, \pi_* C) KB$$

$$+ \frac{1}{2} (\nabla_{\pi_* C} R)(\pi_* A, \pi_* B) I$$

$$- \frac{1}{4} R(R(KB, I) \pi_* C, \pi_* A) I + \frac{1}{4} R(R(KA, I) \pi_* C, \pi_* B) I$$

$$- \frac{1}{4} R(R(KC, I) \pi_* B, \pi_* A) I + \frac{1}{4} R(R(KC, I) \pi_* A, \pi_* B) I.$$

These formulae describe completely the Levi Civita connection and the Riemannian curvature tensor on the tangent bundle with respect to the Sasaki metric.

The geodesic spray $S$ of $\nabla$ is the vector field on $TM$ characterized by

$$\pi_* S = I \quad \text{and} \quad KS = 0.$$

Thus $S$ is a horizontal vector field on $TM$. If $\alpha$ is an integral curve of $S$ with $\alpha(0) = v \in TM$ then $\gamma := \pi \circ \alpha$ is a geodesic in $M$ with $\dot{\gamma}(0) = v$. Conversely,
if $\gamma$ is a geodesic in $M$ with $\dot{\gamma}(0) = v \in TM$ then $\gamma$ is of the form $\pi \circ \alpha$, where $\alpha$ is an integral curve of $S$ with $\alpha(0) = v$. So the geodesics in $M$ are in a one-to-one correspondence with the integral curves of $S$. Note that the integral curves of $S$ are geodesics in $TM$.

We now turn our attention to the Jacobi operator $\bar{R}_S := \bar{R}(.,S)S$. By the above formulae we get

$$\pi_* \bar{R}_S A = \left( R_I - \frac{3}{4} R^2_I \right) \pi_* A + \frac{1}{2} R'_I \pi_* K A,$$

$$K \bar{R}_S A = \frac{1}{4} R^2_I K A + \frac{1}{2} R'_I \pi_* A,$$

where $R'_I := (\nabla I R)(.,I)I$. In particular, for horizontal vectors $X \in \mathcal{H}_v$ and vertical vectors $V \in \mathcal{V}_v$, $v \in TM$, we get

1. $\pi_* \bar{R}_S X = \left( R_v - \frac{3}{4} R^2_v \right) \pi_* X,$
2. $K \bar{R}_S X = \frac{1}{2} R'_v \pi_* X,$
3. $\pi_* \bar{R}_S V = \frac{1}{2} R'_v K V,$
4. $K \bar{R}_S V = \frac{1}{4} R^2_v K V.$

Finally, let $J$ be the almost Hermitian structure on $(TM, \tilde{g})$ characterized by

$$\pi_* \circ J = -K \quad \text{and} \quad K \circ J = \pi_*.$$

As is well-known, $J$ is integrable, that is, $J$ is a complex structure on $TM$, if and only if $R = 0$.

To each non-zero tangent vector $v \in T_p M$, $p \in M$, we associate a two-dimensional $J$-invariant subspace $L(v)$ of $T_v TM$ by

$$L(v) := \mathbb{R}v^\mathcal{V} \oplus \mathbb{R}v^\mathcal{H}.$$  

Note that $Jv^\mathcal{H} = v^\mathcal{V}$ and $Jv^\mathcal{V} = -v^\mathcal{H}$.  

3. The Jacobi operator with respect to the geodesic spray

We start off by characterizing locally symmetric spaces by properties of the Jacobi operator $\tilde{R}_S$.

**Theorem 1.** For a Riemannian manifold $M$ the following statements are equivalent:

(a) $M$ is a locally symmetric space;
(b) $\mathcal{H}$ (or equivalently, $\mathcal{V}$) is invariant by $\tilde{R}_S$;
(c) the horizontal (or equivalently, vertical) lift of any eigenvector of $R_v$ is an eigenvector of $\tilde{R}_{S(v)}$ for all $v \in TM$;
(d) for every $p \in M$ and $v, w \in T_p M$ the vectors $\tilde{R}_{S(v)} w^\mathcal{H}$ and $w^\mathcal{V}$ are orthogonal with respect to the Sasaki metric on $TM$.

**Proof.** Firstly, we assume that $M$ is locally symmetric. Then $\nabla R = 0$ and hence $R'_v = 0$ for all $v \in TM$. Then (2) implies the invariance of $\mathcal{H}$ by $\tilde{R}_S$. (The invariance of $\mathcal{V}$ by $\tilde{R}_S$ follows from (3).)

Next, we assume that $\tilde{R}_S \mathcal{H} \subset \mathcal{H}$. Let $w$ be an eigenvector of $R_v$ for some $v \in TM$, say $R_v w = \kappa w$. Then $K \tilde{R}_S w^\mathcal{H} = 0$ by the assumption and $\pi_* \tilde{R}_S w^\mathcal{H} = \left(\kappa - \frac{1}{4}\kappa^2\right) \pi_* w^\mathcal{H}$ by means of (1). This implies $\tilde{R}_S w^\mathcal{H} = \left(\kappa - \frac{1}{4}\kappa^2\right) w^\mathcal{H}$. The proof for vertical vectors is analogous and involves (4) instead of (1).

Now we assume that statement (c) holds. We choose $p \in M$ and $v, w \in T_p M$. Let $e_1, \ldots, e_n$ ($n = \dim M$) be an orthonormal basis of $T_p M$ consisting of eigenvectors of $R_v$. We write $w$ as a linear combination of $e_1, \ldots, e_n$, say $w = \Sigma w_k e_k$. Then $w^\mathcal{H} = \Sigma w_k e_k^\mathcal{H}$, and from the assumption we get that $\tilde{R}_{S(v)} w^\mathcal{H} = \Sigma w_k \tilde{R}_{S(v)} e_k^\mathcal{H}$ is a horizontal vector, whence it must be orthogonal to $w^\mathcal{V}$.

Finally, we assume that (d) is valid. Then the assumption, the definition of $\bar{g}$, and (2) yield

$$0 = \bar{g}(\tilde{R}_{S(v)} w^\mathcal{H}, w^\mathcal{V}) = g(K \tilde{R}_{S(v)} w^\mathcal{H}, w) = \frac{1}{2} g(R'_v w, w)$$

for all $v, w \in T_p M, p \in M$. Hence, $(\nabla_v R)(.,v)w = R'_v = 0$ for all $v \in TM$. This condition is known (see e.g. [11], [18]) to be necessary and sufficient for $R$ to be parallel, by which $M$ is shown to be locally symmetric. $\blacksquare$
The invariance of a space $L(w)$ with respect to $\tilde{R}_{S(v)}$ has a nice geometrical meaning.

**Lemma 1.** Let $v, w$ be non-zero tangent vectors to $M$ at some point $p \in M$. Then $L(w)$ is invariant by $\tilde{R}_{S(v)}$ if and only if $w$ is an eigenvector of both $R_v$ and $R'_v$.

**Proof.** Assume that $\tilde{R}_{S(v)}L(w) \subset L(w)$. Then $\tilde{R}_{S(v)}w^\mathcal{H} = \alpha w^\mathcal{H} + \beta w^\mathcal{V}$ for some $\alpha, \beta \in \mathbb{R}$. From (2) we get

$$R'_v w = R'_v \pi_* w^\mathcal{H} = 2K \tilde{R}_{S(v)}w^\mathcal{H} = 2\beta w,$$

which proves that $w$ is an eigenvector of $R'_v$. We can also write $\tilde{R}_{S(v)}w^\mathcal{V} = aw^\mathcal{H} + bw^\mathcal{V}$ with some $a, b \in \mathbb{R}$. Then (4) implies

$$R'^2_v w = R'^2_v K w^\mathcal{V} = 4K \tilde{R}_{S(v)}w^\mathcal{V} = 4bw.$$

Eventually, (1) and the preceding equation yield

$$R_v w = \pi_* \tilde{R}_{S(v)}w^\mathcal{H} + \frac{3}{4} R'^2_v w = (\alpha + 3b)w.$$

Thus $w$ is also an eigenvector of $R_v$.

Conversely, assume that $R_v w = \kappa w$ and $R'_v w = \bar{\kappa} w$ for some $\kappa, \bar{\kappa} \in \mathbb{R}$. Then the equations (1)-(4) give

$$\pi_* \tilde{R}_{S(v)}w^\mathcal{H} = \left(\kappa - \frac{3}{4}\kappa^2\right) w, \quad K \tilde{R}_{S(v)}w^\mathcal{V} = \frac{1}{4}\kappa^2 w,$$

$$K \tilde{R}_{S(v)}w^\mathcal{H} = \frac{1}{2} kw = \pi_* \tilde{R}_{S(v)}w^\mathcal{V}.$$

This implies $\pi \tilde{R}_{S(v)}L(w) \subset \mathbb{R}w$ and $K \tilde{R}_{S(v)}L(w) \subset \mathbb{R}w$, and hence $\tilde{R}_{S(v)}L(w) \subset \mathbb{R}w^\mathcal{H} \oplus \mathbb{R}w^\mathcal{V} = L(w)$. 

We shall now use Lemma 1 to obtain the following characterization of $\mathcal{P}$-spaces.

**Theorem 2.** For an $n$-dimensional real analytic Riemannian manifold $M$ the following statements are equivalent:

(a) $M$ is a $\mathcal{P}$-space;

(b) for every $v \in TM$ the space $T_v TM$ can be decomposed orthogonally into two-dimensional $\tilde{R}_{S(v)}$-invariant subspaces $L(e_1), \ldots, L(e_n)$. 
Proof. In part G of the Introduction we mentioned that $M$ is a $\mathcal{V}$-space if and only if $R_v$ and $R'_v$ commute for all $v \in TM$. As $R_v$ and $R'_v$ commute precisely if they are simultaneously diagonalizable the assertion follows by virtue of Lemma 1 (for $v = 0$ statement (b) is always valid).

We now continue with a characterization of $\mathcal{C}$-spaces.

Theorem 3. For a Riemannian manifold $M$ the following statements are equivalent:

(a) $M$ is a $\mathcal{C}$-space;

(b) for every $p \in M, v \in T_p M$ and eigenvector $w$ of $R_v$ the vectors $\bar{R}_S(v)w^H$ and $w^V$ are orthogonal with respect to the Sasaki metric on $TM$.

Proof. Assume that $M$ is a $\mathcal{C}$-space. Let $p \in M, v \in T_p M$ and $w$ be an eigenvector of $R_v$, say $R_v w = \kappa w$. From part G in the Introduction we know that there exists a skew-symmetric endomorphism $T_v$ of $T_p M$ such that $R'_v = R_v \circ T_v - T_v \circ R_v$. Then, using the definition of $\bar{g}$, equation (2) and the above expression for $R'_v$ we get

$$
\bar{g}(\bar{R}_S(v)w^H, w^V) = g(K \bar{R}_S(v)w^H, w) = \frac{1}{2} g(R'_v w, w) = g(R_v w, T_v w) = \kappa g(w, T_v w) = 0
$$

as $T_v$ is skew-symmetric.

Conversely, assume that (b) is valid. Let $\gamma : I \to M$ be a geodesic in $M$ defined on an open interval $I$. From perturbation theory of symmetric operators it is known (see e.g. [1, Lemma 2]) that on an open and dense subset $I_0$ of $I$ the self-adjoint operator $R_\gamma$ can be diagonalized by an orthonormal frame field $E_1, \ldots, E_n$ $(n = \dim M)$ along $\gamma|I_0$, say $R_\gamma E_k = \kappa_k E_k$. By virtue of the assumption we get analogously to the preceding computation

$$
0 = 2\bar{g}(\bar{R}_S(\gamma)E_k^H, E_k^V) = 2g(K \bar{R}_S(\gamma)E_k^H, E_k) = g(R'_\gamma E_k, E_k) = \partial \kappa_k / \partial t
$$
on $I_0$. Thus the eigenvalues of $R_\gamma$ are locally constant on $I_0$. As $I_0$ is open and dense in $I$ the eigenvalues of $R_\gamma$ must be constant on the entire interval $I$. From this we conclude that $M$ is a $\mathcal{C}$-space.
Next, we shall use the Jacobi operator of the geodesic spray to obtain characterizations of spaces of constant sectional curvature and of constant holomorphic sectional curvature which are analogous to well-known results by A. Fialkow [9], E. Cartan [5] and S. Tanno [16].

**Theorem 4.** For a Riemannian manifold $M$ with $\dim M > 2$ the following statements are equivalent:

(a) $M$ is a space of constant sectional curvature;

(b) for every $p \in M$ and every orthogonal triple $u, v, w \in T_p M$ the vectors $\tilde{R}_S(v)^{u \mathcal{H}}$ and $w^{\mathcal{H}}$ are orthogonal with respect to the Sasaki metric on $TM$.

**Proof.** We have

$$g(\tilde{R}_S(v)^{u \mathcal{H}}, w^{\mathcal{H}}) = g(\pi_* \tilde{R}_S(v)^u, w) = g(R_v u, w) - \frac{3}{4} g(R_v^2 u, w).$$

First, assume that $M$ is a space of constant curvature $\kappa$. Then

$$R_v u = \kappa ||v||^2 u.$$

Putting this into (5) gives

$$g(\tilde{R}_S(v)^{u \mathcal{H}}, w^{\mathcal{H}}) = 0,$$

which proves (b). Conversely, suppose (6) is satisfied for any orthogonal triple $u, v, w$. Then (5) yields

$$g(R_v u, w) - \frac{3}{4} g(R_v^2 u, w) = 0$$

for all such triples $u, v, w$. Since this holds also for multiples of $v$ we must have

$$g(R_v u, w) = 0.$$

The result now follows from Fialkow's result [9, Theorem 3.2] (see also Chen [6, p. 130] who refers to Cartan [5] for this result). \qed
THEOREM 5. For a nearly Kähler manifold \((M, J)\) with \(\dim M > 2\) the following statements are equivalent:

(a) \(M\) is a space of constant holomorphic sectional curvature;
(b) for every \(p \in M, v, w \in T_p M\) with \(g(Jv, w) = 0\) the vectors \(\bar{R}S(v)(Jv)^H\) and \(w^H\) are orthogonal with respect to the Sasaki metric on \(TM\).

Proof. We have

\[
(7) \quad \bar{g}(\bar{R}S(v)(Jv)^H, w^H) = g(\pi_* \bar{R}S(v)(Jv)^H, w) = g(R_v Jv, w) - \frac{3}{4} g(R^2_v Jv, w).
\]

First, assume that \(M\) is a space of constant holomorphic sectional curvature c. Due to S. Tanno [16] we have

\[
R_v Jv = c||v||^2 Jv,
\]

and hence

\[
\bar{g}(\bar{R}S(v)(Jv)^H, w^H) = 0,
\]

by which (b) is proved. Conversely, assume that (b) is satisfied. Then (7) implies

\[
g(R_v Jv, w) - \frac{3}{4} g(R^2_v Jv, w) = 0
\]

for all \(v, w\) with \(g(Jv, w) = 0\). Following the same reasoning as in the proof of Theorem 4 we get

\[
g(R_v Jv, w) = 0
\]

for all \(v, w\) with \(g(Jv, w) = 0\). Then the result follows from [16].

Note that in both theorems we could replace the horizontal lifts by the vertical lifts.

We shall finish with two results in almost Hermitian geometry. Therefore, we mention
LEMMA 2. [10] We have

(a) a Kähler manifold $(M, J)$ is locally symmetric if and only if

$$g(R^l_v Jv, Jv) = 0$$

for all tangent vectors $v \in T_p M$ and all $p \in M$;

(b) an analytic nearly Kähler manifold $(M, J)$ is a locally 3-symmetric space with canonical almost complex structure $J$ if and only if

$$g(R^l_v Jv, Jv) = 0$$

for all tangent vectors $v \in T_p M$ and all $p \in M$.

From this lemma we obtain easily

THEOREM 6. For a Kähler manifold $(M, J)$ the following statements are equivalent:

(a) $M$ is a Hermitian locally symmetric space;

(b) for every $p \in M$ and $v \in T_p M$ the vectors $\bar{R}_{S(v)}(Jv)^H$ and $(Jv)^V$ are orthogonal with respect to the Sasaki metric on $TM$.

Proof. Using (2) we have

$$\bar{g}(\bar{R}_{S(v)}(Jv)^H, (Jv)^V) = g(K \bar{R}_{S(v)}(Jv)^H, Jv) = \frac{1}{2} g(R^l_v Jv, Jv).$$

The assertion then follows from this formula and (a) in Lemma 2. 

In a similar way we obtain

THEOREM 7. Let $(M, J)$ be an analytic nearly Kähler manifold. Then the following statements are equivalent:

(a) $M$ is a locally 3-symmetric space with canonical almost complex structure $J$;

(b) for every $p \in M$ and $v \in T_p M$ the vectors $\bar{R}_{S(v)}(Jv)^H$ and $(Jv)^V$ are orthogonal with respect to the Sasaki metric on $TM$. 
REFERENCES


Jürgen BERNDT  
Mathematisches Institut Universität zu Köln  
Weyertal 86-90, 50923 Köln, Germany.

Lieven VANHECKE  
Department of Mathematics Katholieke Universiteit Leuven  
Celestijnenlaan 200 B, 3001 Leuven, Belgium.

*Lavoro pervenuto in redazione il 10.9.1992.*