POINTS IN PROJECTIVE SPACE
IN VERY UNIFORM POSITION

Abstract. A major problem in the study of smooth curves embedded in projective space is the relationship between the degree and genus of such curves. One general way to study these invariants of curves was extensively exploited by G. Castelnuovo: one intersects the curve with a general hyperplane and studies the resulting finite set of points. A powerful modern tool used to study these points is the Uniform Position Lemma of J. Harris. We introduce, for points in \( \mathbb{P}^n \), a stronger notion of uniformity than that described by Harris in his Lemma. We show that this type of uniformity is also satisfied by the general hyperplane section of an irreducible curve in \( \mathbb{P}^n(\mathbb{C}) \).

A major problem in the study of smooth curves embedded in projective space is the relationship between the degree and genus of such curves. This is equivalent to a description of all possible Hilbert polynomials of such curves.

One general way to study these invariants of curves was extensively exploited by the Italian geometer, G. Castelnuovo. Roughly speaking, in this approach one intersects the curve with a general hyperplane and studies the

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resulting finite set of points. A powerful modern tool used to study these points is the Uniform Position Lemma of J. Harris [H]. The uniform position property (UPP), which this lemma of Harris asserts is true for the points of a general hyperplane section of a curve, severely restricts the Hilbert function of those points. That restriction, in turn, imposes some calculable restrictions on the Hilbert polynomial of the curve. For example, the theorem of Castelnuovo which bounds the genus of a curve in terms of its degree, follows from such considerations (as does more than Castelnuovo ever asserted).

If one considers only arithmetically Cohen-Macaulay curves, then more precise results can be stated. This is so since the Hilbert function of the general hyperplane section not only determines the Hilbert polynomial of the curve it also determines the Hilbert function of the curve.

For the still more special case of arithmetically Cohen-Macaulay curves in \( \mathbb{P}^3 \), Sauer showed [S] that it was possible to say even more. In effect, Sauer describes the possible minimal free resolutions which ideals of a.C.M. curves in \( \mathbb{P}^3 \) can have. Since the Hilbert function (and hence the Hilbert polynomial) can be easily computed from such a resolution, Sauer's results present very detailed numerical information about the curve. (For details of Sauer's work and also of the contributions of others to this type of study of a.C.M. curves in \( \mathbb{P}^3 \) see the article [Ge-Mig]).

In another direction, Green and Lazarsfeld have proved (in some cases) and conjectured (in others) results which imply that the graded Betti numbers in a minimal free resolution of the ideal of a smooth curve in \( \mathbb{P}^3 \) reflect geometric information about the curve. (See [Gr-L]).

It is in this atmosphere that this paper was conceived. We introduce, for points in \( \mathbb{P}^n \), a stronger notion of uniformity than that described by Harris in his Lemma. We show that this type of uniformity is also satisfied by the general hyperplane section of an irreducible curve in \( \mathbb{P}^n(\mathbb{C}) \), a fact which was independently asserted by Ballico in [B].

The uniformity we describe for the general hyperplane section of an irreducible curve follows from some very general principles; principles which are present in Harris' original discussion of the UPP. We take this opportunity to make those principles very explicit.

To this end, the set up for Proposition 2 and Remark 3 are deliberately very general. If \( \varphi \) is any property of finite sets of points in projective space that leads to a finite constructible stratification of the Hilbert schemes of
finite sets of points in projective space, then a general hyperplane section of an irreducible curve, over \( \mathbb{C} \), will have the uniform \( \varphi \) property. One would suspect that just about any natural geometric or algebraic property of finite sets of points in projective space would lead to a finite constructible stratification of the appropriate Hilbert schemes - and one would often be correct! Nevertheless, one must still check this for each property one wants to use.

Note also that in Proposition 2 and Remark 3 the target scheme \( Z \) does not ever need to be the Hilbert scheme of points in projective space. For instance, one may think of a hyperplane section of a curve as a divisor on that curve. One could then use various symmetric products of that curve as the target \( Z \). In this way one could write down a complete detailed proof of the version of the uniform position theorem given in [ACGH - pp. 112-13].

The moral of this story is that there are many uniform position type theorems out there. If you ever need one, think about the general set-up we have explicitly given here. It might help you!

Finally, we would like to take this opportunity to express our warmest regards to Professor Paolo Salmon on the occasion of his 60th birthday. The second author, in particular, wants to thank Professor Salmon for his many kindnesses over the years and for the introduction which the author received, from several of Salmon's students, to the many splendors of modern Italian geometry. Affettuosi auguri.

1. Generalities

First we quote a definition from [H, pg. 94]. "Constructible Sets. Let \( X \) be a Zariski topological space. A constructible subset of \( X \) is a subset which belongs to the smallest family \( \mathcal{F} \) of subsets such that:

1) every open subset is in \( \mathcal{F} \);
2) a finite intersection of elements of \( \mathcal{F} \) is in \( \mathcal{F} \);
3) the complement of any element of \( \mathcal{F} \) is in \( \mathcal{F} \)."

Let \( \pi : X \rightarrow Y \) be a finite étale morphism of degree \( d \). Assume that \( Y \) is a nonsingular irreducible variety over \( \mathbb{C} \) (the complex numbers), and that \( X \) is a disjoint union of nonsingular irreducible varieties over \( \mathbb{C} \). Let \( Z \) be some scheme over \( \mathbb{C} \). Assume that \( Z \) is expressed as a finite disjoint union: \( Z = \bigcup_{i=1}^{k} W_i \) where each \( W_i \) is a constructible subset of \( Z \). We call this a
stratification of $\mathcal{Z}$ and we say that the $\mathcal{W}_i$ are the strata of this stratification. For each positive integer $\ell$, $1 \leq \ell \leq d$, let $\mathcal{X}(\ell)$ be the $\ell$-fold fibre product of $\mathcal{X}$ with itself over $\mathcal{Y}$. As a set, we may think of $\mathcal{X}(\ell)$ as

$$\{(p_1, \ldots, p_\ell) | p_i \in \mathcal{X} \text{ and } \pi(p_i) = \pi(p_j), \ 1 \leq i, j \leq \ell\}. $$

Let $\mathcal{X}^0(\ell)$ be the union of all connected components of $\mathcal{X}(\ell)$ consisting of points where $p_1, \ldots, p_\ell$ are all distinct, and let $\mathcal{X}^{0,1}(\ell), \ldots, \mathcal{X}^{0,m}(\ell)$ be the connected components of $\mathcal{X}^0(\ell)$. Since $\mathcal{X}(\ell)$ is nonsingular, its connected components are the same as its irreducible components and each connected component is both open and closed in $\mathcal{X}(\ell)$. Finally, we suppose that for some $\ell$ we have a morphism

$$f : \mathcal{X}^0(\ell) \rightarrow \mathcal{Z}. $$

We start by making a very elementary observation. Via $f$ we may pull back the stratification of $\mathcal{Z}$ to get a stratification of $\mathcal{X}^0(\ell)$. From elementary properties of constructible sets ([H, pg. 94]) one can deduce that each $\mathcal{X}^{0,i}(\ell)$ must have a dense stratum and this stratum must contain a Zariski open $U_i$ which is dense in $\mathcal{X}^{0,i}(\ell)$. In other words, each $\mathcal{X}^{0,i}(\ell)$ contains a dense Zariski open $U_i$ such that $f(U_i)$ is entirely contained in one stratum $\mathcal{W}_j$ of $\mathcal{Z}$. The morphism $\pi$ was assumed to be finite and thus the natural projection of $\mathcal{X}(\ell)$ to $\mathcal{Y}$ is finite, and hence proper. Let $A_i$ be the closure of $\left(\mathcal{X}^{0,i}(\ell) \setminus U_i\right)$ in $\mathcal{X}(\ell)$. Notice that $A_i$ has codimension at least 1 in $\mathcal{X}(\ell)$. The image of $A_i$ in $\mathcal{Y}$ is closed, call it $B_i$. Notice that $B_i$ has codimension at least 1 in $\mathcal{Y}$. Set $V_i = (\mathcal{Y} \setminus B_i)$ and $\mathcal{Y}^0 = \cap_{i=1}^m V_i$. Note that $\mathcal{Y}^0$ is Zariski open and dense in $\mathcal{Y}$. For any $y \in \mathcal{Y}^0$ and any $i, 1 \leq i \leq m$, all the preimages of $y$ in $\mathcal{X}^{0,i}(\ell)$ live in $U_i$.

Fix a point $y_0 \in \mathcal{Y}$ and set $\pi^{-1}(y_0) = \{p_1, \ldots, p_d\}$. Consider the natural monodromy action of the fundamental group $\Pi_1(\mathcal{Y}, y_0)$ on $\{p_1, \ldots, p_d\}$. We know that two ordered $\ell$-tuples $(s_1, \ldots, s_\ell)$ and $(t_1, \ldots, t_\ell)$ of distinct elements of $\{p_1, \ldots, p_d\}$ lie in the same connected component of $\mathcal{X}^0(\ell)$ if and only if there is an element $\sigma$ of $\Pi_1(\mathcal{Y}, y_0)$ with $\sigma(t_1, \ldots, t_\ell) = (s_1, \ldots, s_\ell)$. With this in mind, the following two propositions are immediate.

**Proposition 1.** Assume that $y_0 \in \mathcal{Y}^0$ and for some two ordered $\ell$-tuples $(s_1, \ldots, s_\ell)$ and $(t_1, \ldots, t_\ell)$ of distinct elements of $\pi^{-1}(y_0)$ there is a
\[ \sigma \in \Pi_1(\mathcal{Y}, y_0) \text{ such that } \sigma(s_1, \ldots, s_\ell) = (t_1, \ldots, t_\ell). \] Then \( f(t_1, \ldots, t_\ell) \) and \( f(s_1, \ldots, s_\ell) \) lie in the same stratum of \( \mathcal{Z} \).

**Proposition 2.** Assume that \( y_0 \in \mathcal{Y}^0 \) and that the monodromy action of \( \Pi_1(\mathcal{Y}, y_0) \) on \( \pi^{-1}(y_0) \) is that of the full symmetric group on \( d \) letters. Then for any two ordered \( \ell \)-tuples \((s_1, \ldots, s_\ell)\) and \((t_1, \ldots, t_\ell)\) of distinct elements of \( \pi^{-1}(y_0) \) we have that \( f(s_1, \ldots, s_\ell) \) and \( f(t_1, \ldots, t_\ell) \) lie in the same stratum of \( \mathcal{Z} \).

**Remark 3.** The set \( \mathcal{Y}^0 \) is a dense Zariski open set. The intersection of finitely many dense Zariski open sets is again a dense Zariski open set. Suppose one has a finite number of morphisms \( f_j : \mathcal{X}^0_{(n_j)} \to \mathcal{Z}_j \). Each will lead to a \( \mathcal{Y}^0_j \subset \mathcal{Y} \). The intersection of these \( \mathcal{Y}^0_j \) will then be a dense Zariski open set for which Propositions 1 and 2 apply for all the \( f_j \) simultaneously.

A specific situation to which we want to apply this general observation is when \( \pi : \mathcal{X} \to \mathcal{Y} \) is a family of points in projective space and \( \mathcal{Z} \) is \( \text{Hilb}^\ell_{\mathbb{P}^n} \), the Hilbert scheme of \( \ell \) points in \( \mathbb{P}^n \). Let us elaborate.

Suppose that we have a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{X} & \to & \mathcal{Y} \times \mathbb{P}^n \\
\downarrow{\pi} & & \downarrow{\text{projection}} \\
\mathcal{Y} & \leftarrow & \\
\end{array}
\]

This is a family of \( d \) points in \( \mathbb{P}^n \). From this we may construct another commutative diagram:

\[
\begin{array}{ccc}
\mathcal{X}^0_{(\ell)} \times_\mathcal{Y} \mathcal{X} & \to & \mathcal{X}^0_{(\ell)} \times \mathbb{P}^n \\
\downarrow{\text{projection}} & & \downarrow{\text{projection}} \\
\mathcal{X}^0_{(\ell)} & \to & \\
\end{array}
\]
so that each of the projection maps has $\ell$ sections $\sigma_1, \ldots, \sigma_\ell$ defined by $\sigma_i(p_1, \ldots, p_\ell) = p_i$. We may think of the union of the images of these $\ell$ sections as a family of $\ell$ points in $\mathbb{P}^n$ and thereby obtain a morphism

$$f : \mathcal{X}_\ell^0 \longrightarrow \text{Hilb}_\mathbb{P}^n_\ell.$$  

To make practical use of these generalities we must interesting stratifications of $\text{Hilb}_\mathbb{P}^n_\ell$.

2. An Interesting Stratification

In what follows we wish to think of a point $I \in \text{Hilb}_\mathbb{P}^n_\ell$ as a perfect homogeneous ideal of height $n$ in $R = \mathbb{C}[x_0, \ldots, x_n]$. Consider the minimal free $R$-resolution of $I$:

$$(4) \quad 0 \longrightarrow F_{n-1} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \xrightarrow{\varepsilon} I \longrightarrow 0$$

Here $F_i = \bigoplus_{j=1}^{r_i} (R(-\gamma_{ij}))^{a_{ij}}$. The numbers $r_i, \gamma_{ij}$ and $\alpha_{ij}$ are determined by $I$, up to ordering the direct sum decomposition of each $F_i$. The numbers $\alpha_{ij}$ are called the graded Betti numbers of the ideal $I$. Given two points $I, I'$ in $\text{Hilb}_\mathbb{P}^n_\ell$ whose minimal free $R$-resolutions lead to numbers $r_i, \gamma_{ij}, \alpha_{ij}$ and $r'_i, \gamma'_{ij}, \alpha'_{ij}$ respectively, we say that $I$ and $I'$ have numerically equivalent minimal free resolutions when $r_i = r'_i$, $\gamma_{ij} = \gamma'_{ij}$, and $\alpha_{ij} = \alpha'_{ij}$. In this way one may define an equivalence relation on the closed points of $\text{Hilb}_\mathbb{P}^n_\ell$. Two points are equivalent if and only if they have numerically equivalent resolutions.

**Proposition 5.** Under this equivalence relation on $\text{Hilb}_\mathbb{P}^n_\ell$ there are only finitely many equivalence classes and each equivalence class is a constructible set in the Zariski topology. In other words, this equivalence relation gives rise to a finite constructible stratification of $\text{Hilb}_\mathbb{P}^n_\ell$.

The proof will follow from a Lemma.

**Lemma 6.** Let $\mathcal{X}$ be any irreducible closed subset of $\text{Hilb}_\mathbb{P}^n_\ell$. Then there exists a dense Zariski open set $\mathcal{U} \subset \mathcal{X}$ such that all points in $\mathcal{U}$ have numerically equivalent minimal free resolutions.
Proof: Without loss of generality we may replace $\mathcal{X}$ by any dense affine open subset of $\mathcal{X}$. Thus we may assume that $\mathcal{X}$ is affine. Let $\mathbb{C}[\mathcal{X}]$ be the coordinate ring of $\mathcal{X}$ and let $\mathcal{K} = \mathbb{C}(\mathcal{X})$ the function field of $\mathcal{X}$. Consider the pull-back of the universal family of points over $\text{Hilb}^d_{\mathbb{P}^n}$ to $\mathcal{X}$:

$$\mathcal{P} \longrightarrow \mathcal{X} \times \mathbb{P}^n \longrightarrow \mathcal{X}$$

Now $\mathcal{X} \times \mathbb{P}^n = \text{Proj}\mathbb{C}[\mathcal{X}][x_0, \ldots, x_n]$ and $\mathcal{P}$ corresponds to a homogeneous ideal of $\mathbb{C}[\mathcal{X}][x_0, \ldots, x_n]$. This ideal extends to a perfect homogeneous ideal of height $n$, $J \subset \mathcal{K}[x_0, \ldots, x_n]$. $J$ is the ideal of the generic point over $\mathcal{X}$.

Set $S = \mathbb{K}[x_0, \ldots, x_n]$. Consider the minimal free $S$-resolution of $J$:

$$0 \longrightarrow F_{n-1} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \xrightarrow{\varepsilon} J \longrightarrow 0. \quad (8)$$

Here $F_1 = \bigoplus_{j=1}^{r_i} S(-\gamma_{ij})^{a_{ij}}$. The strategy of the proof is to investigate under what conditions one can specialize this minimal free $S$-resolution of the generic point $J$ to a minimal free $\mathbb{R}$-resolution of a particular closed point $I \in \mathcal{X}$. The maps $d_i$ and $\varepsilon$ are given by matrices of elements of $S$, but one may change the maps by multiplying by non-zero elements of $\mathcal{K}$ without changing the fact that we have a minimal free $S$-resolution. Thus we may assume that all the entries of the matrices are in $\mathbb{C}[\mathcal{X}][x_0, \ldots, x_n]$. Then for any closed point $I \in \mathcal{X}$ one can specialize $(8)$ to a (perhaps non-exact) sequence:

$$0 \longrightarrow \bigoplus_{i=1}^{r_{n-1}} \mathbb{R}(-\gamma_{i,n-1})^{a_{i,n-1}} d_{n-2} \longrightarrow \cdots \longrightarrow d_1 \longrightarrow d_0 \xrightarrow{\varepsilon} I \longrightarrow 0. \quad (9)$$

by replacing any element of $\mathbb{C}[\mathcal{X}]$ occurring in the matrices given the maps $d_i$ and $\varepsilon$ by its value at $I$. The question is; When will $(9)$ be a minimal free $\mathbb{R}$-resolution for $I$?

Let $I_e$ be the degree $e$ component of $I$, and similarly let $J_e$ be the degree $e$ component of $J$. 
CLAIM. There is a dense Zariski open subset $U \subset X$ such that for all $I \in U$ and all integers $e$, $\dim_C I_e = \dim_K J_e$.

Proof of Claim. From the diagram

$$
\begin{array}{ccc}
\mathcal{P} & \longrightarrow & \mathcal{X} \times \mathbb{P}^n \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
\mathcal{X} & & \mathbb{P}^n
\end{array}
$$

we obtain the following map of vector bundles on $X$:

$$
\varphi_e : (\pi_1)_*(\pi_2)^* \mathcal{O}_{\mathbb{P}^n}(e) \longrightarrow (\pi_1)_* (((\pi_2)^* \mathcal{O}_{\mathbb{P}^n}(e)) \otimes \mathcal{O}_\mathcal{P}) .
$$

For any closed point $I \in \mathcal{X}$, $I_e = \text{Ker}(\varphi_e)|_I$. Similarly for the generic point $J$.

Recall that the rank of a vector bundle map is a lower semi-continuous function. Therefore, there is a nonempty open subset $U$ of $\mathcal{X}$ on which the rank of $\varphi_e$ is maximal. Since any non-empty open subset of $\mathcal{X}$ contains the generic point $J$, we have that on $U$, $\dim_C I_e = \dim_K J_e$. End of proof of claim.

We may assume the $U$ of the claim is affine and replace $X$ by $U$. Notice that, now, on $\mathcal{X}$ we have that $\text{Ker}(\varphi_e)$ is a vector bundle.

Now let us check (9) for exactness. All maps are graded homomorphisms so we may check exactness by checking it on each graded component. By [L1, Thm. 2.2] and [M, Lecture 14] we see that we only need to check exactness in finitely many degrees. Fixing a degree $e$, but letting $I$ vary over $\mathcal{X}$, the degree $e$ component of (9) becomes a sequence of vector bundles on $\mathcal{X}$. This sequence is, by assumption, exact at the generic point, $J$. Again, since the rank of a vector bundle map is lower semi-continuous, there is an open dense $\mathcal{U} \subset \mathcal{X}$ such that for any $I \in \mathcal{U}$ the ranks of all the $d_i$'s and $e$ in degree $e$ are the same as for $J$. We claim that for any $I \in \mathcal{U}$, (9) is exact in degree $e$. Indeed, for any vector bundle with two subbundles of the same rank, the set of points of the base over which the two subbundles are equal is a closed subset of the base. In our case, at each step of the sequence the two subbundles are kernel and image. By assumption, these are equal at the generic point $J$ and the only closed subset of $\mathcal{U}$ containing $J$ is all of $\mathcal{U}$. Intersecting the finitely
many $U$'s we get from the finitely many degrees $e$ at which we must check exactness, we get a $U$ on which (9) is exact for all $I$.

For all $I$ in this $U$, (9) is a free $R$-resolution of $I$. To conclude the proof we must check that (9) is minimal. Again, assume $U$ is affine and replace $X$ by $U$.

The definition of "minimal", as stated for instance in [L1], it as follows: let $m$ be the unique homogeneous maximal ideal of $R$, $m = \bigoplus_{d>0} R_d$, and set

$$N_i = \begin{cases} \ker d_{i-1} & \text{for } i \geq 1 \\ \ker \varepsilon & \text{for } i = 0 \end{cases}$$

then (4), (8) or (9) is minimal if and only if $N_i \subseteq mF_i$ for all $i \geq 0$. Again we check this one graded piece at a time. In a fixed degree $e$, each of the $F_i$ becomes a vector bundle on $X$. The degree $e$ component of $F_i$ has two subbundles: $(N_i)_e$ the degree $e$ component of $N_i$, and $(mF_i)_e$ the degree $e$ component of $mF_i$.

It is a general "vector bundle fact" that the set of all points $I$ and $X$ for which we have the containment $[(N_i)_e]_I \subseteq [(mF_i)_e]_I$ in the fibre $[(F_i)_e]_I$ of $(F_i)_e$ over $I$ is a closed subset of $X$. By assumption, this set contains the generic point $J$. Therefore,

$$[(N_i)_e]_I \subseteq [(mF_i)_e]_I$$

holds for all points of $X$. 

Proof (of Proposition (5)) This is an easy Noetherian induction argument. Set $X_0 = \text{Hilb}^d_{\mathbb{P}^n}$. From lemma (6) we see that each irreducible component of $\text{Hilb}^d_{\mathbb{P}^n}$ has an open dense subset on which numerical equivalence of ideals is constant. (We are not saying it is constant between these open sets). Call these open sets $U_{0,1}, \ldots, U_{0,j_0}$. Having constructed $X_i$ and $U_{i,1}, \ldots, U_{i,j_i}$, we construct $X_{i+1}$ and $U_{i+1,1}, \ldots, U_{i+1,j_{i+1}}$ as follows. Set $X_{i+1} = X_i \setminus \bigcup_{j=1}^{j_i} U_{ij}$. Then $X_{i+1}$ is a finite union of closed subvarieties of $\text{Hilb}^d_{\mathbb{P}^n}$. Call them, $X_{i+1,1}, \ldots, X_{i+1,j_{i+1}}$. Let $U_{i+1,j}$, $1 \leq j \leq j_{i+1}$, be the open dense subset of $X_{i+1,j}$, obtained from lemma (6), on which the numerical equivalence class of a minimal free resolution is constant. Since the dimension of $X_{i+1}$ is strictly less than the dimension of $X_i$, only finitely many $X_i$ are
nonempty. Therefore there are only finitely many \( U_{h,k} \). Each \( U_{h,k} \) is an open subset of a closed set. Each equivalence class is a union of \( U_{h,k} \)'s, and therefore there are only finitely many equivalence classes and each of them is constructible. \( \blacksquare \)

**Remark 10.** That there are only finitely many different possible numerical equivalence classes of minimal free resolutions is shown in [L1, Thm. 2.2] and [M, Lecture 14]. What is new here is the constructibility of the resulting equivalence classes in \( \text{Hilb}_{P^n} \).

Next we wish to apply these results to the family of hyperplane sections of a curve in \( \mathbb{P}^n \). Let \( C \subseteq \mathbb{P}^n \) be a reduced, irreducible, nondegenerate (possibly singular) curve of degree \( d \). Let \( H \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^* \) be the universal family of hyperplanes and set \( \mathcal{P} = H \cap (C \times (\mathbb{P}^n)^*) \). Let \( U \subseteq (\mathbb{P}^n)^* \) be the dense Zariski open set over which \( \mathcal{P} \) is étale and \( \mathcal{P}^0 \) be those points of \( \mathcal{P} \) lying over \( U \). The diagram

\[
\begin{array}{ccc}
\mathcal{P}^0 & \longrightarrow & U \times \mathbb{P}^n \\
\downarrow & & \downarrow \\
U & \leftarrow & \\
\end{array}
\]

then gives a family of \( d \) points in \( \mathbb{P}^n \) as described after remark (3). From [ACGH, pg. 111] we see that the monodromy action for the cover \( \mathcal{P}^0 \longrightarrow U \) is the full symmetric group on \( d \) letters. One may then combine Proposition (2), Remark (3) and Proposition (5) to obtain the following result.

**Theorem 11.** There is a dense Zariski open set \( U' \subseteq U \), such that for any hyperplane \( H \in U' \) the following holds:

if \( \Gamma \) and \( \Gamma' \) are any two subsets of \( H \cap C \), such that \( \Gamma \) and \( \Gamma' \) have the same cardinality (call it \( \ell \) - so we can think of \( \Gamma \) and \( \Gamma' \) as points of \( \text{Hilb}_{P^n} \)), then \( \Gamma \) and \( \Gamma' \) have numerically equivalent minimal free \( R \)-resolutions.

**Remark 11A.** Technically, what we have shown is if \( C \subseteq \mathbb{P}^n \) and \( \Gamma = C \cap H (H \in U') \), then \( \Gamma \) has the uniform resolution property as a subset of \( \mathbb{P}^n \). Since the resolution of a set of points in \( \mathbb{P}^m (m < n) \), considered in \( \mathbb{P}^n \), is the tensor product of the resolution in \( \mathbb{P}^m \) with the Koszul resolution of the field \( k \) over the polynomial ring in the new variables, the numerical features
of minimal free resolutions of point sets in $\mathbb{H}$ (considering their ideals in a polynomial ring in one fewer variable) are completely determined by their resolutions as subsets of $\mathbb{P}^n$. It follows that $\Gamma$ has the uniform resolution property as a subset of $\mathbb{P}^{n-1}$.

3 - An Example and Some Questions

**Definition 12.** a) [Harris] A finite set $\Gamma$ of distinct points in $\mathbb{P}^n$ is said to have the **Uniform Position Property** (UPP) if and only if every two subsets of $\Gamma$ with the same cardinality have the same Hilbert function.

b) A finite set $\Gamma$ of distinct points in $\mathbb{P}^n$ is said to have the **Uniform Resolution Property** (URP) if and only if every two subsets of $\Gamma$ with the same cardinality have numerically equivalent minimal free resolutions. (In [B], Ballico calls this same property "syzygetic uniform position").

Since the Hilbert function of a set of points is uniquely determined by the numerical equivalence class of its minimal free resolution, we see that any set of points with URP has UPP.

We first want to show that the converse is not true, i.e. there are points in $\mathbb{P}^n$ with the UPP which do not have the URP.

To see that this is so, we recall some well-known facts. A set $\Gamma$ of $\ell$ points in $\mathbb{P}^n$ is said to have **generic Hilbert function** if and only if $\Gamma$ imposes independent conditions on forms of all degrees. If we use $H_\Gamma(-)$ to denote the Hilbert function of the set $\Gamma$ then

$$H_\Gamma(t) = \begin{cases} \binom{t+n}{n} & \text{if } \binom{t+n}{n} \leq \ell \\ \ell & \text{if } \binom{t+n}{n} > \ell. \end{cases}$$

It is easy to prove that a general set of $\ell$ distinct points has generic Hilbert function.

For the reader's convenience, we state (and prove) some elementary (and probably well-known) facts.

**Lemma 13.** If a finite set of distinct points $\Gamma \subset \mathbb{P}^n$ has generic Hilbert function then there is a dense Zariski open subset $U \subset \mathbb{P}^n$ such that: if $p \in U$ then $\Gamma \cup \{p\}$ has generic Hilbert function.
Proof. Let $I$ be the ideal of $\Gamma$ and $I_d$ its degree $d$ graded component. $\Gamma \cup \{p\}$ will have generic Hilbert function if and only if $p$ is chosen so that for each $d$ for which $I_d \neq (0)$, $p$ is not in the base locus of $I_d$. Those base loci are all proper closed subsets of $\mathbb{P}^n$. Moreover, for $d > 0$ they are all equal to $\Gamma$.

Thus, there are only a finite number of proper closed subsets of $\mathbb{P}^n$ which $p$ must avoid. $$

\text{Lemma 14.} \quad \text{If a finite set of distinct points } \Gamma \subset \mathbb{P}^n \text{ has the property that all of its subsets have generic Hilbert function, then there is a dense Zariski open subset } U \subset (\mathbb{P}^n)^d (d \text{ any integer } > 0), \text{ such that if } (p_1, \cdots, p_d) \in U \text{ then } \Gamma \cup \{p_1, \cdots, p_d\} \text{ has the property that all of its subsets have generic Hilbert function.}

\text{Proof.} \text{ Add the points one at a time using Lemma 13.}$$

\text{Lemma 15.} \quad \text{If a finite set of distinct points } \Gamma \subset \mathbb{P}^n \text{ has generic Hilbert function and UPP then all subsets of } \Gamma \text{ have generic Hilbert function.}

\text{Proof.} \text{ See [GMR, Lemma 2.3c)]}

\text{Example 16.} \text{ In this example we show that there are always sets of points in } \mathbb{P}^n \text{ which have UPP but do not have URP.}

\text{Let } \ell = 2n + 1 \text{ and let } \Gamma \text{ be a set of } \ell \text{ distinct points on the rational normal curve in } \mathbb{P}^n. \text{ It is easy to see that any such set } \Gamma \text{ has UPP and generic Hilbert function. Also, the quadrics through such a set are all the quadrics which vanish on the rational normal curve itself. Consequently, the ideal of } \Gamma \text{ is minimally generated by } \binom{n+2}{2} - \ell \text{ quadrics and } n \text{ cubics. That information is enough for us to describe the first term in a minimal free resolution of the ideal of } \Gamma.

\text{On the other hand, it follows from [GGR - Thm. 5.8] that for } n \geq 4, \text{ the general set of } \ell \text{ points in } \mathbb{P}^n \text{ has its ideal generated by quadrics, and, for } n = 3 \text{ it is generated by 3 quadrics and 1 cubic. So, for } n \geq 3, \text{ the general set of } \ell = 2n + 1 \text{ points in } \mathbb{P}^n \text{ has minimal free resolution which differs, in the first term, from that of } \Gamma \text{ above.}

\text{Now apply Lemma 14 with } d = \ell = 2n + 1. \text{ We obtain a set } \Gamma \cup \Gamma', \text{ where } \Gamma \text{ is the set above and } \Gamma' \text{ is a general set of } 2n + 1 \text{ points. The set } \Gamma \cup \Gamma' \text{ is minimal free resolution of the ideal of } \Gamma \text{ above.}
has UPP because all of its subsets have generic Hilbert function. But $\Gamma$ and $\Gamma'$ are two subsets of $\Gamma \cup \Gamma'$ with the same cardinality and different minimal free resolutions. Hence this is a set without URP.

Contrary to an assertion in [B], there also exist examples in $\mathbb{P}^2$. From [Ge-Mig, Thm. 4.3] we see that among all sets $\Gamma$ of 12 distinct points in $\mathbb{P}^2$ with generic Hilbert function and UPP there are two possibilities for the degrees of a minimal set of generators for the ideal of $\Gamma$. Let $\Gamma$ be such a set with non-generic degrees of generators. Add 12 generic points to $\Gamma$ using Lemma 14 and you obtain a set of 24 points in $\mathbb{P}^2$ with UPP but not URP.

**Question 17.** It is well-known (see e.g. [Ge-Mig]) that certain Hilbert functions are impossible for points in $\mathbb{P}^n$ with UPP. It is also known precisely which numerical equivalence classes in $\text{Hilb}_{\mathbb{P}^2}$ contain a point corresponding to a set of $\ell$ distinct points in $\mathbb{P}^2$ with URP. However, already in $\mathbb{P}^3$ there are several unanswered questions. We shall give just two of them here.

a) The most fundamental unanswered question here is a characterization of the Hilbert function of a set of points with UPP. The most recent discussion of this problem can be found in [GKR].

b) Suppose that $H$ is known to be the Hilbert function of some set of points in $\mathbb{P}^n$ with UPP. If a numerical equivalence class "has" $H$ as the Hilbert function of all its points and that class contains a point corresponding to a set of points in $\mathbb{P}^n$ with UPP, must that class also contain a point corresponding to a set of points with URP?

We know of no instance where this is not the case, but we suspect there should be such examples in $\mathbb{P}^n$ (and even perhaps when $n = 3$).

In fact, if $H$ is the generic Hilbert function then for $\ell = \binom{d+3}{3}$ or $\binom{d+3}{3} \pm 1$, there is only one numerical equivalence class which contains a point corresponding to $\ell$ distinct points in $\mathbb{P}^3$ with UPP [L2, §3].

We think the search for examples here will prove quite interesting.

REFERENCES


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