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P. Diaconis(*) – **W. Fulton**(**)

A GROWTH MODEL, A GAME, AN ALGEBRA, LAGRANGE INVERSION, AND CHARACTERISTIC CLASSES

Dedicated to Paolo Salmon on his sixtieth birthday

Abstract. From a suitable probability-like function on pairs of points in a set, a commutative algebra is constructed. The product in this algebra can often be realized as a growth model, or a game whose outcome is independent of choices. A special case yields a simple proof of the Lagrange inversion formula, and intriguing formulas for characteristic classes on projective spaces.

1. Introduction

We make several related constructions, each starting from a set X together with a function $p : X \times X \longrightarrow R$ with the property that, for any x in X, there are only finitely many y with $p(x,y)$ nonzero. Here R can be any commutative ring, but for simplicity in this introduction we take R to be the real numbers, and, moreover, we assume that $p(x,y)$ are probabilities:

$$p(x,y) \geq 0 \qquad \text{for all } x \text{ and } y \text{ in } X \; ;$$
$$\sum_{y \in X} p(x,y) = 1 \qquad \text{for all } x \text{ in } X \; .$$

Regard the points of X as sites, and $p(x,y)$ as the probability that a particle starting at x will move to y. For example, X could be the integer lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, with equal probability $1/2^n$ to move to each of the nearest neighbors.

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For each finite subset S of X, and each x in S and y not in S, define $p(x, S, y)$ to be probability that, if a particle starts at x and moves from site to site according to the probability p, then y is the first site outside S that the particle lands on. That is,

$$p(x,S,y) = p(x,y) + \sum_{u \in S} p(x,u)p(u,y)$$
$$+ \sum_{u_1,u_2 \in S} p(x,u_1)p(u_1,u_2)p(u_2,y) + \dots ,$$

a series which can also be evaluated in finite terms by inverting a matrix. Our $p(x, S, y)$ are known as taboo probabilities in the classical theory of Markov chains. Chung [Ch, §I.9] gives a systematic development.

To simplify the discussion in this introduction, we assume that it is possible to escape from any finite set, i.e.,

(*) For any nonempty finite subset S of X there is some x in S and some y not in S with $p(x, S, y) > 0$.

In the text, the topics discussed in this introduction will be treated in more detail, and without this assumption.

A growth model. Fix a finite subset S of X, and think of the sites of S as *inhabited* and those outside S as *uninhabited*, and replace the particles by explorers. We send out an explorer, who starts from a point x in S, travelling always according to the probabilities, until reaching an uninhabited site, where the explorer settles. This is repeated, successively sending a new explorer from the point x, each travelling until arriving at a site outside S not occupied by a previous explorer. If T is a set with k sites which is disjoint from S, let $q(T)$ be the probability that T becomes inhabited by the first k explorers. If $k = 1$, so $T = \{y\}$, then $q(T) = p(x, S, y)$. For $k > 1$,

$$q(T) = \sum_{y \in T} q(T - \{y\}) \cdot p(x, S \cup T - \{y\}, y).$$

These probabilities can be quite complicated, even for simple models. For example, if $X = \mathbb{Z}^n$ is the standard lattice in $\mathbb{R}^n$, and $p(x, y)$ is $1/2^n$ if the distance between x and y is 1, and 0 otherwise, it is not easy to give closed formulas for $q(T)$ if $n > 1$. However, Lawler, Bramson, and Griffeath [LBG] have shown that the limiting shape, as the process is repeated over and over, starting with $S = \{x\}$, is a round ball centered at x.

The growth model described above corresponds to repeatedly multiplying by a fixed element in the algebra to be described below. There are several other possibilities. For example, the random set can be repeatedly squared. Starting from a fixed point, the number of sites doubles each time. Probabilistically, if S_n is the random set at time n, enumerate the points in S_n as $s_1, \ldots, s_{2^n}$. Start an explorer out at s_1, and add the first unexplored site it hits, say s'_1. Then start an explorer out at s_2 and add the first point hit outside $S_n \cup \{s'_1\}$. Continuing in this way gives S_{n+1} of size 2^{n+1}. One consequence of the results proved below is that the growth distribution is independent of the method of enumeration. A second consequence is that the distribution of S_n in the random squaring model is the same as the distribution of S_{2^n} in the model where single points are added each time.

There is a natural generalization, when each site x comes equipped with a positive integer $d(x)$, thought of as a *tolerance* (the preceding case being when all $d(x) = 1$). Now if we are given a finite set S with a population $n(x)$ of individuals at each site x in S, we can choose any site x with more individuals than the tolerance ($n(x) > d(x)$), and send out an explorer from x, moving according to the probability p as before, but settling now as soon as it reaches a site y with $n(y) < d(y)$. This process can be repeated until there are no overpopulated sites. In this setting, the key fact is that *the probability of a given outcome is independent of the order of choices.*

The growth model is carefully described in section 3, which also develops probability results for the simplest case of random walk on $\mathbb{Z}$.

A game. Suppose at each site in a set X we are given an infinite deck of cards, each labelled with some site in X. Assume for simplicity that for any infinite subset S of X there are infinitely many cards at sites in S labelling sites not in S. Suppose a finite number of chips are arranged on X, with a certain number at each site. A *legal move* is the choice of x in X with more than one chip; one of the chips at x is moved to the site labelled by the top card of the deck at x, and this card is thrown away. More generally, if each site has a tolerance, one can restrict legal moves to those x for which the number of chips is greater than the tolerance. A *game* is a sequence of legal moves, until no legal move is possible. The claim is that the game always terminates, and *the number of moves and the final position are independent of choices.*

This is proved in section 4.

Given a probability function p as before, the decks can be randomly arranged so that $p(x, y)$ is the probability that a card at site x labels site y. The fact that the outcome of the game is uniquely determined implies the fact that the probabilities $q(T)$ of the preceding discussion are independent of choices.

This game is a close relative of a variety of chip-firing games on graphs, as in [BLS] and [Moz]. In these games, chips are moved around a graph with some apparent freedom of choice until there is one chip per vertex. The number of moves taken and the final position turns out to be independent of the choices. Eriksson [E] has introduced a game which includes both a version of our game and that of [BLS]. As far as we know, there is not currently a connection with the "olympiad" game in [Moz].

An algebra. With X and p as at the beginning, let Λ be the vector space with basis the finite subsets S of X. We will define a product on Λ, which makes it into an associative and commutative R-algebra, graded by the cardinalities of the sets. The product is uniquely determined by the rules

(1) $S \cdot T = S \cup T$ if S and T are disjoint ;

(2) $\{x\} \cdot S = \sum_{y \notin S} p(x, S, y)(S \cup \{y\})$ if $x \in S$.

For arbitrary S and T, if $S \cap T = \{x_1, \ldots, x_n\}$, then, using (2) inductively,

$$S \cdot T = \{x_1\} \cdot (\ldots \cdot (\{x_n\} \cdot (S \cup T)) \ldots).$$

In this language, the basic fact is that the result is independent of the choice of ordering of the points $x_1, \ldots, x_n$. One can also generalize this to the case when there are tolerances $d(x)$, a basis for the algebra becoming finite sets with multiplicities not exceeding the tolerances.

We take this algebra Λ as the basic object of study. Its properties are carefully described in section 2. In particular, we will give generators and relations for Λ as an R-algebra.

Sinks. It is important to be able to deal also with the situation when assumption (*) fails. Call a nonempty finite set Z of X a *sink* if $p(x, y) = 0$ for all $x \in X$, $y \notin Z$ and if Z is minimal with this property. An explorer landing in a sink can never leave it, and once all the sites in a sink have reached their tolerance levels, one can only keep track of the total number of individuals in the sink: movements within a full sink are ignored. Final states of the growth model, or basic elements in the algebra Λ, are given by a *marked set*, which is a finite number of sites x with a positive integer $m(x)$ no larger than the specified tolerance $d(x)$, together with a finite number of sinks Z with a positive integer $e(Z)$ such that $m(x) = d(x)$ for all x in Z; this indicates that each site in Z has an excess $e(Z)$ beyond its full tolerance of inhabitants, which are at some unspecified site of Z. As a less refined alternative which is also useful, one can simply ignore all sinks when they become full, regarding them as "black holes", and setting the corresponding classes equal to zero in the algebra.

Lagrange inversion. A very special interesting example is a "circular" case, when X is the set $\{1, 2, \ldots, n\}$ with probabilities

$$p(1,2) = p(2,3) = \ldots = p(n-1, n) = p(n, 1) = 1\,,$$

and all other $p(i,j) = 0$. If $q(1), q(2), \ldots$ are any real numbers, following Hirzebruch one defines $\Phi(x)$ to be the unique power series with constant term 1 such that the coefficient of x^n in $\Phi(x)^{n+1}$ is $q(n)$ for all $n = 1, 2, \ldots$. By calculating in the ring Λ constructed from this X and p, one gets a useful formula for the first n coefficients of $\Phi(x)^{n+1}$. This is worked out in section 5, where the Lagrange inversion formula is deduced as a corollary.

Characteristic classes. With X and p as in the preceding paragraph, one can associate to each subset S of X the projective subspace V_S of the projective space $\mathbb{P}^{n-1}$ which is the intersection of corresponding hyperplanes:

$$V_S = \{[z_1 : z_2 : \ldots : z_n] \in \mathbb{P}^{n-1} : z_i = 0 \text{ for all } i \in S\}\,.$$

These are the invariant subvarieties for the natural action of the torus $T = (\mathbb{C}^*)^n/\mathbb{C}^*$ on $\mathbb{P}^{n-1}$. In the language of toric varieties, each S corresponds to a face of the corresponding convex polytope lying in the hyperplane with equation $x_1 + \ldots + x_n = 0$ in $\mathbb{R}^n$. The ring Λ constructed above makes the invariant cycles into a ring, which maps onto the cohomology of $\mathbb{P}^{n-1}$. Numbers $q(1), q(2), \ldots$ correspond to a characteristic class, namely that class whose value on $\mathbb{P}^k$ is $q(k)$ for all k. The formula referred to in the preceding paragraph gives a lifting of this class from the cohomology ring to this ring of cycles, where it has a much simpler expression. For example, the Todd class corresponds to the choice $q(k) = 1$, and this gives the coefficient of V_S as the fraction of the space spanned by the corresponding face which is cut out by the cone over that face. This is discussed in section 6. Unfortunately, however, we know no way to generalize any of this to general toric varieties, so it remains an intriguing curiosity.

Acknowledgments. The formula for the Todd class of projective spaces was discovered several years ago by the second author in collaboration with Joe Harris and Bob MacPherson. At that time Ira Gessel pointed out how our combinatorial formula could be deduced from the Lagrange inversion formula. Alan Landman suggested generalizations to other characteristic classes. The attempt to make the invariant cycles on a toric variety into a ring led to the ring Λ for this "circular" case. The procedure for including sinks benefited from a discussion with S. Garoufalidis. The game above evolved in conversations with David Aldous, see [A]. We also thank Laurel Beckett, Bob Keener, Ken Ross, the authors of [LBG] and [E], who responded to an earlier version of this note.

2. Construction of the algebra

In this section we make an algebra out of linear combinations of marked sets. The construction depends on four elementary lemmas, whose proofs are postponed to §7. Let X be a set, R a commutative ring with unit, and $p : X \times X \longrightarrow R$ a function such that, for all $x \in X$, $p(x,y) = 0$ for all but finitely many y. A *sink* is a nonempty finite subset Z of X such that $p(x,y) = 0$ for all $x \in Z$ and $y \notin Z$, and such that Z is minimal with this property. Since the intersection of two sets with this property also has it, any two sinks must be disjoint. Call a finite subset *good* if it contains no sinks.

For a finite subset S of X, let P_S be the matrix $(p(u,v))_{u,v \in S}$, and I_S the identity matrix $(\delta(u,v))_{u,v \in S}$; these are n by n matrices, $n = |S|$, with rows and columns indexed by u and v in S (an ordering of S is irrelevant). Our basic assumptions are:

(2.1) For any good S the matrix $I_S - P_S$ is invertible;

(2.2) For any sink Z and $z \in Z$, we have $\sum_{y \in Z} p(x,y) = 1$.

Condition (2.1) means that for each good S there is a matrix $B_S = (b_S(u,v))_{u,v \in S}$ such that, for any u and w in S,

$$\sum_{v \in S} b_S(u,v) \cdot (\delta(v,w) - p(v,w)) = \begin{cases} 1 & \text{if } u = w \\ 0 & \text{otherwise} \end{cases} .$$

Define $p(x,S,y)$, for each good set S and $x \in S, y \notin S$, by the formula

$$p(x,S,y) = \sum_{u \in S} b_S(x,u) \cdot p(u,y) . \tag{2.3}$$

We will see in the next section that, when $R = \mathbb{R}$ and p is a probability function, conditions (2.1) and (2.2) are implied by condition (*) of the introduction, and formula (2.3) agrees with the intuitive definition described there.

LEMMA 2.1. *Fix a good set S and an element $y \notin S$. The elements $p(x,S,y)$, for $x \in S$, are the unique solutions of the equations*

$$p(x,S,y) = p(x,y) + \sum_{u \in S} p(x,u)p(u,S,y) . \tag{2.4}$$

The right side of (2.4) can be used to extend the definition of $p(x,S,y)$ to the case where x is not in S.

Suppose we also have a "tolerance" function $d : X \longrightarrow \mathbb{N}^+$. Let $\mathcal{Z}$ be the set of all sinks in X. Define a *marked set* to be pair (m,e) of functions $m : X \longrightarrow \mathbb{N}$ and $e : \mathcal{Z} \longrightarrow \mathbb{N}$ each of which is zero on all but finitely many elements, and satisfying:

$$m(x) \leq d(x) \text{ for all } x \in X; \tag{2.5}$$

$$e(Z) = 0 \text{ if there is any } z \in Z \text{ with } m(z) < d(z). \tag{2.6}$$

Call a sink Z *full* for the marked set (m,e) if $m(z) = d(z)$ for all $z \in Z$. We think of $e(Z)$ as measuring the number of extra markers in Z. Define the *degree* of (m,e) to be

$$\sum_{x \in X} m(x) + \sum_{Z \in \mathcal{Z}} e(Z) .$$

Note that, when there are no sinks, and the tolerance function is identically 1, a marked set is simply a finite subset of X, and the degree is its cardinality.

Each x in X may be identified with the marked set $(\delta_x, 0)$, where $\delta_x : X \longrightarrow \mathbb{Z}$ is the Kronecker function whose value on x is 1 and on other y is 0. The empty set $\emptyset$ is represented by the pair (0,0). For x in X, let $\chi_x : \mathcal{Z} \longrightarrow \mathbb{N}$ take a sink Z to 1 if Z contains x and to 0 otherwise.

Let $\Lambda = \Lambda(X, p, d)$ be the free-module with basis the marked sets. We will make Λ into a commutative associative R-algebra, graded by degree, with unit 1_Λ corresponding to the empty set. To do this we must in particular define the product $x \cdot (m, e)$ in Λ for each marked set (m, e) and each x in X. Each marked set (m, e) determines a decomposition of X into four disjoint subsets A, B, C, and D, with all but the first finite:

- A is the set of all x such that $m(x) < d(x)$;
- B is the union of all sinks that are full for (m, e);
- C is the set of x such that $m(x) = d(x)$, and x belongs to a sink $Z(x)$ which is not full for (m, e);
- D is the set of x such that $m(x) = d(x)$, and x is not in any sink.

The definition of $x \cdot (m, e)$ depends on which of these sets contains x; in each case, it corresponds to what happens if a marker is dropped at x, and it moves according to the probability until it gets to a place which has room for it, or it gets to a sink which is already full, in which case it adds one to the number of extra markers on the sink:

(a) $x \cdot (m, e) = (m + \delta_x, e)$ if $x \in A$;

(b) $x \cdot (m, e) = (m, e + \chi_x)$ if $x \in B$;

(c) $x \cdot (m, e) = \sum_{z \in Z(x) \cap A} p(x, Z(x) - Z(x) \cap A, z)(m + \delta_z, e)$ if $x \in C$;

(d) $x \cdot (m, e) = \sum_{y \notin D} p(x, D, y)(y \cdot (m, e))$ if $x \in D$, where, for $y \notin D$, $y \cdot (m, e)$ is defined by (a), (b), or (c).

Multiplication by x determines an R-linear map $x\cdot : \Lambda \longrightarrow \Lambda$. The key fact is the commutativity of these operators:

LEMMA 2.2. *For any marked set (m, e) and any x, y in X,*

$$y \cdot (x \cdot (m, e)) = x \cdot (y \cdot (m, e)) \text{ in } \Lambda.$$

It is easy to see that any marked set can be realized by successively operating on the empty set $\emptyset$ by appropriate elements of X. It follows that the product on Λ is uniquely determined by the above definition. In order to see that this product is well defined, we use a roundabout but simpler procedure. Let $R[X]$ be the polynomial ring on the variables $x \in X$. By Lemma 2.2, there is a homomorphism from $R[X]$ to the ring of endomorphisms on the R-module Λ, which takes x to the above operator $x\cdot$. Our program is to find an ideal I in the kernel of this mapping, and then to show that the mapping from $R[X]/I$ to Λ that takes a polynomial to its effect on the empty set $\emptyset$ determines an isomorphism of $R[X]/I$ onto Λ. The algebra structure on Λ is then evident. For this we need another lemma:

LEMMA 2.3. *If (m,e) is a marked set, and x is an element of X with $m(x) = d(x)$, then*

$$x \cdot (m,e) = \sum_{y \in X} p(x,y)(y \cdot (m,e)) \text{ in } \Lambda\,.$$

Note that the product of any (m,e) by x is a linear combination of terms (m',e') with $m'(x) = \min(d(x), m(x)+1)$. It follows from Lemma 2.3 that, for any x in X, and any (m,e),

$$\left(x^{d(x)+1} - \sum_{y \in X} p(x,y) x^{d(x)} y\right) \cdot (m,e) = 0\,. \tag{2.7}$$

Similarly, if Z is a sink, the product of any (m,e) by the product $\prod_{z \in Z} z^{d(z)}$ is a linear combination of marked sets for each of which Z is full. It follows that for all sinks Z and all x and y in Z,

$$\left((x-y) \prod_{z \in Z} z^{d(z)}\right) \cdot (m,e) = 0\,. \tag{2.8}$$

Now we define I to be the ideal in $R[X]$ generated by all elements of the form

(i) $\left(x - \sum_{y \in X} p(x,y) y\right) x^{d(x)}$ for all x in X;

(ii) $(x-y)\prod_{z\in Z} z^{d(z)}$ for all sinks Z and all $x, y \in Z$.

By (2.7) and (2.8), the ideal I acts trivially on Λ, so we have a homomorphism of R-modules

$$R[X]/I \longrightarrow \Lambda\,, \quad P \longrightarrow P\cdot\emptyset\,.$$

If Z is a sink and e a nonnegative integer, and M is any monomial in the variables x in X which contains each $z \in Z$ to the power at least $d(z)$, we write $M \cdot Z^e$ in $R[X]/I$ in place of the element $M \cdot z^e$, where z is any element in Z; by (ii), this is independent of choice of z. The assertions about Λ are part of the following theorem.

THEOREM. (a) *The ring $R[X]/I$ is a free R-module with basis the images of monomials of the form*

$$\prod_{x\in X} x^{m(x)} \prod_{Z\in \mathcal{Z}} Z^{e(Z)}$$

where (m, e) varies over all marked sets.

(b) *The homomorphism $R[X]/I \longrightarrow \Lambda, P \longrightarrow P\cdot\emptyset$, is an isomorphism. It takes the displayed monomial to the marked set (m, e).*

For the proof, the last statement in (b) is easy from the definition, and it follows from this that the monomials are R-independent in $R[X]/I$. To complete the proof it must be verified that these monomials generate $R[X]/I$. For this we need a recipe to decrease exponents in monomials, to get them below the tolerance levels.

LEMMA 2.4. *Suppose $n : X \longrightarrow \mathbb{N}$ vanishes outside a finite set. If S is a good set, and $n(v) \geq d(v)$ for all $v \in S$, and x is any element of S, then*

$$\prod_{v\in X} v^{n(v)} - \sum_{w\notin S} p(x, S, w) \prod_{v\neq x} v^{n(v)} x^{n(x)-1} w$$

is in I.

Given a monomial $\prod_{v\in X} v^{n(v)}$, let S be the set of elements v in X such that v is not in a sink and $n(v) \geq d(v)$. Lemma 2.4 shows how to decrease exponents of any x which is not in a sink to tolerance levels, without increasing exponents of any element not in a sink. To complete the proof of the theorem, it suffices to see how to decrease exponents in a monomial $M = \prod_{z \in Z} z^{n(z)}$ when Z is a sink (noting that there is no mixing of terms in a sink with those outside). If $n(z) \geq d(z)$ for all $z \in Z$, then M is equivalent modulo I to $\prod_{z \in Z} z^{d(z)} \cdot Z^{\Sigma n(z) - d(z)}$. If not, to decrease the exponent of x if $n(x) \geq d(x)$, apply Lemma 2.4, with S the set of z in Z such that $n(z) \geq d(z)$. Using the fact that $p(x, S, w) = 0$ if w is not in Z (cf. Lemma 7.2), the sum in Lemma 2.4 needs only be taken over those w in Z - $Z \cap S$, and the terms in this sum have lower exponents of x. This completes the proof of the Theorem from Lemmas 2.1-2.4.

Let $\overline{\Lambda}$ be the free R-module on the functions $m : X \longrightarrow \mathbb{N}$ such that $m(x) \leq d(x)$ for all x and which vanish outside some finite subset, and such that, for any sink Z there is some $z \in Z$ with $m(z) < d(z)$. The product we have constructed determines a product on $\overline{\Lambda}$, by identifying m with the marked set $(m, 0)$, and throwing away any marked sets in the results for which some sink is full. To see this, let $J \subset \Lambda$ be the set of R-linear combinations of marked sets (m, e) for which some sink is full. It follows from the definition that J forms a homogeneous ideal in Λ, so Λ/J is a commutative, associative, graded R-algebra. It follows from this construction and the Theorem that Λ/J is free on the marked sets of the form $(m, 0)$, with m as above, so it is canonically isomorphic to $\overline{\Lambda}$, and gives $\overline{\Lambda}$ the required algebra structure.

COROLLARY. *With this product, $\overline{\Lambda}$ is a commutative, associative, graded R-algebra. It is canonically isomorphic to $R[X]/\overline{I}$, where $\overline{I}$ is the ideal generated by the elements*

(i) $(x - \sum_{y \in X} p(x, y) y) x^{d(x)}$ for all x in X ;

(ii) $\prod_{z \in Z} z^{d(z)}$ for all sinks Z .

Note that when the tolerance function d is identically 1, $\overline{\Lambda}$ is free on the finite good sets, and the relation (i) becomes simply

$$x^2 - \sum_{y \in X} p(x, y) xy = 0 .$$

3. The Markov model

Consider the case where $R = \mathbb{R}$ and p is a probability:

$$0 \le p(x,y) \le 1 \text{ and } \sum_{y \in X} p(x,y) = 1 \text{ for all } x \text{ in } X .$$

For a sequence of elements $u_1, u_2, \ldots, u_r$ of X, set

$$p(u_1, \ldots, u_r) = p(u_1, u_2)p(u_2, u_3) \cdot \ldots \cdot p(u_{r-1}, u_r) .$$

In this setting there are several ways to say that a set is good:

PROPOSITION 3.1. *For a finite subset S of X, the following are equivalent:*

(i) For every finite nonempty subset T of S, the matrix $I_T - P_T$ is invertible;
(ii) S contains no sink;
(iii) for all x in S there is a y not in S such that either $p(x,y) > 0$ or there is a sequence $u_1, \ldots, u_r$ in S such that $p(x, u_1, \ldots, u_r, y) > 0$;
(iv) the sum $\sum_{r=1}^{\infty} \sum_{u_1,\ldots,u_r \in S} p(u_1, \ldots, u_r)$ converges.

Proof. $(i) \Rightarrow (ii)$: If there is no escape from a subset T of S, then the vector $(1, \ldots, 1)$ is in the kernel of $I_T - P_T$.

$(ii) \Rightarrow (iii)$: If T is the set of all x in S for which there is no y as in (iii), then there is no escape from T.

$(iii) \Rightarrow (iv)$: From (iii) there is a positive integer r and a $\rho < 1$ such that for all $x \in S$,

$$\varphi(x,r) = \sum_{u_1,\ldots,u_r \in S} p(x, u_1, \ldots, u_r) \le \rho .$$

Then $\varphi(x, mr) \le \rho^m$, and (iv) follows.

$(iv) \Rightarrow (i)$: The matrix $(b_T(u,v))_{u,v \in T}$ is inverse to $I_T - P_T$, where

$$b_T(u,v) = \sum_{r=0}^{\infty} \sum_{u_1,\ldots,u_r \in T} p(u, u_1, \ldots, u_r, v) . \tag{3.1}$$

∎

It follows from (3.1) that for any good set S, with x in S and y not in S,

$$p(x,S,y)=\sum_{r=0}^{\infty}\sum_{u_1,\ldots,u_r\in S}p(x,u_1,\ldots,u_r,y). \tag{3.2}$$

Suppose we are also given a tolerance function $d: X \longrightarrow \mathbb{N}^+$. If $n: X \longrightarrow \mathbb{N}$ is any function which is zero outside a finite set, we can write, in the ring Λ constructed in §2,

$$\prod_{v\in X} v^{n(v)}=\sum q(m,e)\cdot(m,e), \tag{3.3}$$

the sum over all marked sets (m,e). The coefficient $q(m,e)$ is the probability that the marked set (m,e) results when explorers are repeatedly sent from those sites x with $n(x) > d(x)$, travelling according to the probability p, stopping at a site with population less than its tolerance, or at a site in a full sink. The fact that the product is well defined – which comes from the commutativity of Lemma 2.2 – implies that the coefficients are independent of the order of choice of explorers. The case when the monomial is x^k, and there are no sinks, is that discussed in the introduction.

The simplest examples of the product lead to interesting probabilistic problems. Consider $X=\mathbb{Z}$, with $p(x,x+1)=1/2=p(x,x-1)$. The Markov chain is simply random walk on $\mathbb{Z}$. Construct a random set by repeatedly multiplying the point zero. Using probability language: Begin by lighting up the one point set $\{0\}$. Then choose one of the neighbors of 0 and light it up. After n steps, there will be an interval of n steps, including 0, lit up. At step $n+1$, a random walker starting at 0 wanders until exiting the sites lit up at step n; this new site is lit up.

A natural question is: how does this random set grow? If the random set at time n is the interval $[-a,b]$, the classical gambler's ruin problem, see e.g. [Fe], says that the chance that the next site lit up is negative is $\frac{b+1}{a+b+2}$. Thus the random walker tends to exit from the shorter side, and so tends to equalize the number of positive and negative sites. The next result shows that the number of negative sites is $\frac{n}{2}$ plus approximately normal fluctuation at scale $\sqrt{n}$.

Proposition 3.2. *Let N_n be the number of negative sites at time n for the random set on $\mathbb{Z}$ generated by simple symmetric random walk. Then, as n tends to infinity,*

$$P\left\{\frac{N_n - \frac{n}{2}}{\sqrt{n/12}} \le t\right\} \longrightarrow \frac{1}{\sqrt{2\pi}}\int_0^t e^{-x^2/2}dx .$$

Proof. The growth of negative sites has an alternative description as an urn model: Begin with an urn containing one black and one white ball. At stage $k, k \ge 1$, draw a ball out at random and replace it, together with a ball of the opposite color. Let N_k be the number of black balls in the urn at stage k. This N_k process has the same distribution as the number of lit-up negative sites, for $k = 0, 1, 2, \ldots$. The N_k process has been studied as Bernard Freedman's urn by David Freedman [Fr], who proves the proposition in this setting. ■

In recent work, Bramson, Griffeath, and Lawler [LBG] have shown that the growth model based on nearest neighbor random walk in $\mathbb{Z}^m$ has a limiting shape and that this shape is round. They show in fact that, given any $\varepsilon > 0$, then, with probability 1, there is an N_0 such that for $n > N_0$ the set S_n with n points is contained in the ball of radius $(1+\varepsilon) \cdot \sqrt[m]{n/B_m}$ and contains all lattice points of the ball of radius $(1-\varepsilon) \cdot \sqrt[m]{n/B_m}$, where B_m is the volume of the ball of radius 1 in $\mathbb{R}^m$. This is the first example where a spherical shape has been determined. For comparison, in Richardson's growth model a random set is grown by considering all points at distance one from the current set and adding one of them with probability proportional to the number of neighbors in the current set. It was shown that Richardson's model has a limiting shape, and computer simulations suggest that it is round, but it has been shown that in very high dimensions the shape is not a ball. See Durrett [D] for this.

4. The game

Suppose at each site in a set X we are given an infinite deck of cards, each labelled with some site in X. Let Z be a subset of X with the property that, for any finite subset S of X not contained in Z there are infinitely many cards at sites in S labelling sites not in S. Suppose $d : X \longrightarrow \mathbb{N}^+$ is a function. Suppose a finite number of chips are arranged on X, with $n(x)$ chips at site x. A *legal move* is the choice of x in X - Z with $n(x) > d(x)$; one of the chips at x is moved to the site labelled by the top card of the deck at x, and this card is thrown away. A *game* is a sequence of legal moves, until no legal move is possible.

PROPOSITION 4.1. *Any game terminates in a finite number of moves. The final position, i.e., the number of chips on each site, the number of moves, and the number of times each site is chosen in a game are independent of choices.*

Proof. A strategy for a terminating game is to choose a chip on any x not in Z with $n(x) > d(x)$, and follow it until it lands on a site y with $n(y) < d(y)$ or it lands in Z. Since this decreases the number of sites outside Z with excess chips, the procedure must terminate. Let $(x) = (x_1, \ldots, x_n)$ be a terminating sequence of legal moves. To prove the rest, we follow Eriksson [E], cf. [A]. It suffices to show that any other sequence of legal moves is a permutation of a subsequence of this one. If not, there is a sequence of legal moves $(y_1, \ldots, y_k, v)$ such that $(y) = (y_1, \ldots, y_k)$ is a permutation of a subsequence of (x), but the sequence with v on the end is not. The number of times v occurs in the sequence (x) is therefore the same as the number of times v occurs in (y), and all other sites occur at least as many times in (x) as in (y). It follows that the number of chips on v after (x) is at least as large as the number of chips after (y). Since this number after x is no larger than $d(v)$, the same is true after (y), so v cannot be a legal move after (y). ∎

5. The circle and Lagrange inversion

One of the simplest examples is $X = \{1, 2, \ldots, n\}$, with

$$(5.1) \qquad p(1,2) = p(2,3) = \ldots = p(n-1, n) = p(n, 1) = 1\,,$$

and all other $p(i,j) = 0$. We regard the set X as arranged circularly, i.e., identify X with $\mathbb{Z}/n\mathbb{Z}$, so that $n + 1 = 1$. The coefficient ring R can be any commutative ring containing the rational numbers $\mathbb{Q}$. The only set which is not good is X itself. The tolerance function is taken to be identically 1. Note that in this case the corollary in §2 is particularly obvious: it says simply that the algebra

$$\overline{\Lambda} = R[x_1,\ldots,x_n]/(x_i^2 - x_i x_{i+1}, 1 \le i \le n, \text{ and } x_1 \cdot \ldots \cdot x_n) \tag{5.2}$$

has basis of all monomials $x_S = \prod_{i \in S} x_i$ for all proper subsets S of X, including the empty set: $x_\emptyset = 1$. This simple fact, which is easy to prove directly, is all that will be used in this section.

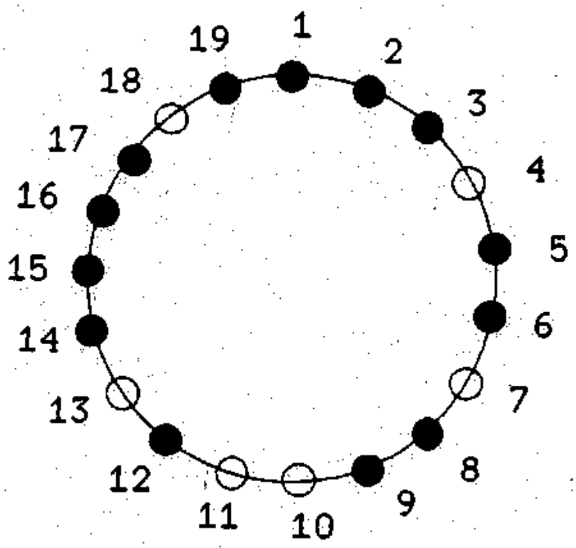
We consider proper subsets S of X also as arranged in a circle, so that 1 follows n. The *components* of S are defined to be the maximal subsets of consecutive integers. Given any sequence of elements $q(1), q(2), \ldots$ in R, set

$$q(S) = \prod \frac{q(|T|)}{|T|+1}, \tag{5.3}$$

the product over all components T of S. The *type* of S is the sequence $(a_1, a_2, \ldots)$ where a_t is the number of components of S of cardinality t. So

$$q(S) = \prod_t \left(\frac{q(t)}{t+1}\right)^{a_t} . \tag{5.4}$$

For example, with, with $n = 19$ and S as shown by black dots:



the type is (1,2,0,2), and $q(S) = \frac{q(1)}{2}\left(\frac{q(2)}{3}\right)^2\left(\frac{q(4)}{5}\right)^2$.

First we have an elementary count:

Lemma 5.1. *The number of proper subsets of X of type $(a_1, a_2, \ldots, a_r)$ is*

$$\frac{n}{n-k}\binom{n-k}{a_1 a_2 \ldots a_r} = \frac{n}{n-k}\,\frac{(n-k)!}{(n-k-a)!a_1! \cdot \ldots \cdot a_r!},$$

where $k = a_1 + 2a_2 + \ldots + ra_r$ is the cardinality of the subsets, and $a = a_1 + a_2 + \ldots + a_r$ is the number of components.

Proof. We first count the number N of subsets S of type $(a_1, \ldots, a_r)$ that do not contain n. For such S write down a sequence of $n-k$ integers, by listing in order, for each of the integers from $\{1, \ldots, n\}$ which are not in S, the number of integers immediately preceding it which are in S. This list consists of a_1 1's, a_2 2's,..., a_r r's, and $n-k-a$ 0's. Conversely, any such sequence of integers comes from a unique such subset. The number of such sequences is

$$N = \frac{(n-k)!}{a_1! \cdot \ldots \cdot a_r! \cdot (n-k-a)!}.$$

To count all subsets, consider for each S which does not contain $\{n\}$ the n sets obtained by rotating S around the circle clockwise. Each set of type $(a_1, a_2, \ldots, a_r)$ occurs $n-k$ times in this way, so the total number of sets is $\dfrac{n}{n-k}N$, as asserted. ∎

From this we can express the sum of all $q(S)$ over subsets of given cardinality, as a coefficient on a power series: let

$$Y(x) = \sum_{n=1}^{\infty} q(n-1)\frac{x^n}{n}, \tag{5.5}$$

where we set $q(0) = 1$. For a power series $P(x)$ in x, let $\langle x^k \rangle (P(x))$ denote the coefficient of x^k in $P(x)$.

Lemma 5.2. *For $k < n$, $\sum_{|S|=k} q(S) = \dfrac{n}{n-k}\langle x^n \rangle (Y(x)^{n-k})$.*

Proof. The left side is, by Lemma 5.1,

$$\frac{n}{n-k}\sum_{a=0}^{n-k}\binom{n-k}{a}\sum\binom{a}{a_1\,a_2\,\ldots\,a_r}\left(\frac{q(1)}{2}\right)^{a_1}\left(\frac{q(2)}{3}\right)^{a_2}\cdot\;\cdots\;\cdot\left(\frac{q(r)}{r+1}\right)^{a_r}$$

where the inner sum is over all $(a_1,\ldots,a_r)$ with $a_1+a_2+\ldots+a_r=a$ and $a_1+2a_2+\ldots+ra_r=k$. The right side is

$$\frac{n}{n-k}\langle x^k\rangle(1+\frac{q(1)}{2}x+\frac{q(2)}{3}x^2+\ldots)^{n-k}$$

$$=\frac{n}{n-k}\sum_{a=0}^{n-k}\binom{n-k}{a}\langle x^k\rangle\left(\frac{q(1)}{2}x+\frac{q(2)}{3}x^2+\ldots\right)^a,$$

and the equality is evident by the binomial theorem. ∎

Following Hirzebruch [H], let $\Phi(x)$ be the unique power series in $R[[x]]$ with constant term 1 such that

$$\langle x^n\rangle(\Phi(x)^{n+1})=q(n)\quad\text{for } n=1,2,\ldots. \tag{5.6}$$

The ring $\overline{\Lambda}$ can be used to give a simple proof of the following combinatorial identity:

PROPOSITION 5.1. *For* $k<n, \langle x^k\rangle(\Phi(x)^n)=\sum_{|S|=k}q(S)$.

Proof. We claim that, in the ring $\overline{\Lambda}$, we have the identity

$$\prod_{i=1}^{n}\Phi(x_i)=\sum_S q(S)x_S\,. \tag{5.7}$$

The proposition follows from (5.7) by applying the canonical homomorphism of graded R-algebras from $\overline{\Lambda}$ to $R[x]/(x^n)$ which maps each x_i to x (noting that the relations in (5.2) map to zero).

To verify (5.7), let $r(S)$ be the coefficient of x_S on the left side. Note that in $\overline{\Lambda}$ any product of x_i and other monomials is either zero or of the form x_U with U a set containing i. It follows that $r(S)$ is unchanged if one omits any terms $\Phi(x_i)$ with $i \notin S$ from the product. Likewise, $r(S) = \Pi r(T)$, where T varies over the components of S. So it suffices to look at connected S, and by symmetry we may take S to be $\{1, \ldots, k\}$. By the same reasoning, for this S, $r(S)$ will be independent of n for $n > k$, since we need only look at the product of $\Phi(x_i)$ for $1 \leq i \leq k$, and all monomials x_U with U containing any integer larger than k are ignored. In particular, we may take $n = k + 1$. But now specializing all x_i to x as above, we see that the sum of $r(S)$, over the n subsets with $n - 1$ elements, is the coefficient of x^{n-1} in $\Phi(x)^n$, which is $q(n - 1)$ by definition. Since these $r(S)$ are all equal, $r(S) = q(n-1)/n$, and since $q(S)$ is defined to be $q(|S|)/(|S| + 1)$, we see that $r(S) = q(S)$. ■

There are some consequences of the proposition which are not obvious from the definition of Φ. For example, if $R = \mathbb{R}$, and all $q(k)$ are nonnegative, then all coefficients $\langle x^k\rangle(\Phi(x)^n)$ are nonnegative for $k < n$. Another consequence is that, if $R = \mathbb{Q}$ and all $q(k)$ are integers, then when n is prime, each coefficient $\langle x^k\rangle(\Phi(x)^n)$ is integral at the prime n and is divisible by n for $0 < k < n - 1$. This follows from the fact that, when n is prime, no rotation of a proper subset of the circle is equal to itself.

Combining the proposition with Lemma 5.2, we have:

COROLLARY. *With Y and Φ the power series defined by (5.5) and (5.6),*

$$n \cdot \langle x^n\rangle(Y(x)^{n-k}) = (n-k) \cdot \langle x^k\rangle(\Phi(x)^n)\,.$$

The preceding corollary is a version of the Lagrange inversion formula for power series in one variable. There are several equivalent forms of Lagrange inversion, one of which states that for any power series F,

$$n \cdot \langle x^n\rangle(F(Y(x)) = \langle x^{n-1}\rangle(F'(x) \cdot \Phi(x)^n)\,. \tag{5.8}$$

For $F(x) = x^p$ this follows from the corollary by setting $k = n - p$. The general case follows since both sides of (5.8) are linear in F.

If we take $F = \Phi$ we have on the right,

$$\langle x^{n-1}\rangle(\Phi'(x) \cdot \Phi(x)^n) = \frac{1}{n+1}\langle x^{n-1}\rangle\left(\frac{d}{dx}\Phi(x)^{n+1}\right)$$
$$\frac{n}{n+1}\langle x^n\rangle(\Phi(x)^{n+1}) = \frac{n}{n+1}q(n)\,.$$

By the definition of $Y(x)$, $\frac{1}{n+1}q(n) = \langle x^{n+1}\rangle(Y(x))$; so (5.8) specializes to the equation $\langle x^n\rangle(\Phi(Y(x)) = \langle x^{n+1}\rangle(Y(x))$, i.e., to

$$x \cdot \Phi(Y(x)) = Y(x)\,. \tag{5.9}$$

Equation (5.9) is usually used in place of (5.5) and (5.6) to give the relation between Φ and Y, cf. [Co].

6. Characteristic classes on projective spaces

Let $R \supset \mathbb{Q}$ be as in the preceding section, and let $q(1), q(2), \ldots$ be a sequence in R corresponding to a power series $\Phi(x)$ as in (5.6). The truncation of $\Phi(x)^n$ in $R[x]/(x^n)$ is the corresponding characteristic class of projective space $\mathbb{P}^{n-1}$, so the coefficient of x^k in $\Phi(x)^n$ is the term of codimension k, in $H^{2k}(\mathbb{P}^{n-1}, R) = R$, in this characteristic class, which we denote by $c\ell_k^{\Phi}(\mathbb{P}^{n-1})$. By definition, $q(m) = c\ell_m^{\Phi}(\mathbb{P}^m)$ is the corresponding genus. Proposition 5.1 gives pleasantly simple formulas for these characteristic classes on projective space:

$$c\ell_k^{\Phi}(\mathbb{P}^{n-1}) = \sum q(S)\,, \tag{6.1}$$

the sum of all $q(S)$, over all subsets S of $X = \{1, \ldots, n\}$ of cardinality k, with $q(S)$ as defined in (5.3) or (5.4).

For example, for any y in R, with

$$q(m) = 1 - y + y^2 - \ldots + (-1)^m y^m\,,$$

(Hirzebruch's "T_y-genus"), then $\Phi(x) = x + (y+1)x/(e^{(y+1)x} - 1)$. The Chern class, Todd class, and L-class correspond to the choices $y = -1$, $y = 0$, and $y = 1$ respectively, with $\Phi(x)$ being $1 + x$, $x/(1 - e^{-x})$, and $x/\tanh(x)$ respectively. For the Chern class, this just gives $q(S) = 1$ for all S, with total $\binom{n}{k}$, which is of course the coefficient of x^k in $(1+x)^n$. For the Todd class, $q(S) = (\Pi(|T|+1))^{-1}$, the product over components T of S as in §5. For the L-class, the formula is the same, but taken only over those components T with $|T|$ even. For example, with $R = \mathbb{Q}$, it follows for each of these classes that, if $n = p$ is prime, and $0 < k < p - 1$, then $\mathrm{ord}_p(c\ell_k^{\Phi}(\mathbb{P}^{p-1})) > 0$. It also follows for the general T_y-genus, with $R = \mathbb{R}$, that these classes $c\ell_k^\Phi(\mathbb{P}^{n-1})$ are positive for all n exactly when $y < 1$.

Let H be the hyperplane $x_1 + \ldots + x_n = 0$ in $\mathbb{R}^n$. Consider the simplex in H spanned by the to n vectors

$$(1,-1,0,\ldots,0),(0,1,-1,0\ldots,0),\ldots,(0,\ldots,1,-1),(-1,0,\ldots,0,1).$$

Identify X with the above vectors, so subsets S correspond to faces of the simplex. With the metric on H induced by the usual metric on $\mathbb{R}^n$, it is not hard to verify that the above number $q(S) = (\Pi(|T|+1))^{-1}$ is the *fraction of the space in the linear span of S which is cut out by the cone over S.*

By the general theory of toric varieties (cf. [O]), the convex polytope corresponds to the toric variety $\mathbb{P}^{n-1}$, with faces corresponding to subvarieties invariant by the torus, which can be identified with the intersection of the corresponding hyperplanes. This is the formula for the Todd class mentioned in the introduction. Recently Morelli [Mor] has given another proof of this formula, but he has also shown that one cannot find a metric on an arbitrary toric variety so that the Todd class is calculated by such fractions. For $\mathbb{P}^{n-1}$ the ring $\overline{\Lambda}$ constructed from the "circle" X makes a ring out of the invariant cycles, and it is the fact that calculation of $\Phi(x)^n$ can be lifted to this ring, so that every invariant variety has a definite contribution to the total class, that gives the simple formulas. Is this possible for other toric varieties?

7. Proofs of the lemmas

In this section we return to the lemmas from §2. The proofs are all elementary manipulations – they could hardly be otherwise with so few axioms – so we will omit some details.

We first prove Lemma 2.1. Since $I_S - P_S$ is invertible, the equations (2.3) are equivalent to the equations

$$p(x,y) = \sum_{u\in S}(\delta(x,u) - p(x,u))p(u,S,y),$$

which is the same as that in Lemma 2.1. ∎

For other lemmas we need two further identities:

LEMMA 7.1. *If S and T are disjoint, with $S \cup T$ good, $x \in S$, and $y \notin S \cup T$, then*

$$p(x, S \cup T, y) = p(x, S, y) + \sum_{t \in T} p(x, S, t) p(t, S \cup T, y) . \tag{7.1}$$

Proof. To see that (7.1) holds for all x in S, multiply by the invertible matrix $I_S - P_S = (\delta(u,v) - p(u,v))$, regarding each side of (7.1) as a vector whose x^{th} component is shown. On the left side we get a vector whose u^{th} component is

$$\begin{aligned} &\sum_{v \in S} (\delta(u,v) - p(u,v)) p(v, S \cup T, y) \\ &\qquad = p(u, S \cup T, y) - \sum_{v \in S} p(u,v) p(v, S \cup T, y) . \end{aligned} \tag{7.2}$$

On the right side we get similarly

$$\begin{aligned} p(u, S, y) - \sum_{v \in S} p(u,v) p(v, S, y) + \sum_{t \in T} p(u, S, t) p(t, S \cup T, y) \\ - \sum_{v \in S} \sum_{t \in T} p(u,v) p(v, S, t) p(t, S \cup T, y) . \end{aligned} \tag{7.3}$$

Applying Lemma 2.1 twice to (7.3), using the good set S, (7.3) becomes

$$p(u,y) + \sum_{t \in T} p(u,t) p(t, S \cup T, y) . \tag{7.4}$$

Lemma 2.1, applied this time to the set $S \cup T$, yields

$$\begin{aligned} p(u, S \cup T, y) = p(u,y) + \sum_{v \in S} p(u,v) p(v, S \cup T, y) \\ + \sum_{t \in T} p(u,t) p(t, S \cup T, y) . \end{aligned}$$

Substituting this in the right side of (7.2), we get (7.4), as required. ∎

The following identity is similarly a consequence of Lemma 2.1, using the assumption (2.2); we omit the proof.

LEMMA 7.2. *If S is good, Z is a sink, and $x \in S \cap Z$, then*
(a) $p(x,S,y) = p(x, S\cap Z, y)$ for all $y \notin S$;
(b) $p(x,S,y) = 0$ if $y \notin Z$;
(c) $p(x,S,y) = 0$ if $y \in Z$ and $Z \cap S = Z - \{y\}$.

We turn now to the commutativity Lemma 2.2. The definitions of $x\cdot(m,e)$ and $y\cdot(m,e)$ depend on which of the four sets A,B,C,D each of x and y belong to, and then, in considering the products $y\cdot(x\cdot(m,e))$ and $x\cdot(y\cdot(m,e))$, there are several subcases. Each is proved, however, by a short calculation using only Lemmas 2.1, 7.1, and 7.2. Since there is no difficulty beyond keeping track of all possibilities that can arise, we omit the tedious details. Instead, we consider the simpler case (which is all that was used in the rest of the paper) when the tolerance function d is identically 1, and we consider the equality in the module $\overline{\Lambda}$, which is free on the good sets S, as at the end of §2. The ideas are the same as in the general case, without so many cases to consider.

In $\overline{\Lambda}$, $x\cdot S$ is $S\cup\{x\}$ if $x \notin S$, and $x\cdot S = \sum_{y\notin S} p(x,S,y) S\cup\{y\}$ if $x\in S$, in both cases discarding any set which contains a sink. The equality $x\cdot(y\cdot S) = y\cdot(x\cdot S)$ is obvious if $x=y$ or if x and y are not in S. Suppose next that $x\in S$ and $y\notin S$. We have

$$x\cdot S = p(x,S,y)S\cup\{y\} + \sum_{w\notin S\cup\{y\}} p(x,S,w)S\cup\{w\}\,,$$

so $y\cdot(x\cdot S)$ is equal to

$$\Big[p(x,S,y)\Big(\sum_{w\notin S\cup\{y\}} p(y,S\cup\{y\},w)\Big) + \sum_{w\notin S\cup\{y\}} p(x,S,w)\Big] S\cup\{y,w\}\,.$$

Since $y\cdot S = S\cup\{y\}, x\cdot(y\cdot S)$ is equal to $\sum\limits_{w\notin S\cup\{y\}} p(x,S\cup\{y\},w)S\cup\{y,w\}$. Then Lemma 7.1, with $T=\{y\}$, implies that these are equal.

The remaining case is where x and y are in S, but $x\neq y$. We compute the coefficient of a good set $S\cup T$, with $T=\{w,z\}$, w and z a pair of distinct elements not in S. Since

$$x\cdot S = p(x,S,w)S\cup\{w\} + p(x,S,z)S\cup\{z\} + \sum_{u\notin S\cup T} p(v,S,u)S\cup\{u\}\,,$$

the coefficient of $S \cup T$ in $y \cdot (x \cdot S)$ is

$$p(x, S, w)p(y, S \cup \{w\}, z) + p(x, S, z)p(y, S \cup \{z\}, w) .$$

By Lemma 7.1, this coefficient is equal to

$$\begin{aligned} &p(x, S, w)(p(y, S, z) + p(y, S, w)p(w, S \cup \{w\}, z)) \\ &\qquad + p(x, S, z)(p(y, S, w) + p(y, S, z)p(z, S \cup \{z\}, w)) , \end{aligned}$$

and this is symmetric in x and y. ∎

The proof of Lemma 2.3 also uses Lemma 2.1, but it is simpler since there are only four cases to consider; again we omit details, noting that it is obvious in the special case of $\overline{\Lambda}$ considered in the preceding paragraph.

Finally, we prove Lemma 2.4 in the general setting. For this it suffices to show that

$$\prod_{v \in X} v^{d(v)}(x - \sum_{w \notin S} p(x, S, w)w) \in I$$

for every $x \in S$. Since S is good, $I_S - P_S = (\delta(u, x) - p(u, x))_{u, x \in S}$ is invertible, and multiplying by this matrix, we get a term for each $u \in S$:

$$\prod_{v \in X} v^{d(v)} \left[u - \sum_{x \in S} p(u, x)x - \sum_{w \notin S} p(u, S, w)w + \sum_{\substack{w \notin S \\ x \in S}} p(u, x)p(x, S, w)w \right] .$$

By Lemma 2.1, $p(u, S, w) = p(u, w) + \sum_{x \in S} p(u, x)p(x, S, w)$, so the term in square brackets simplifies to

$$\left[u - \sum_{x \in S} p(u, x)x - \sum_{w \notin S} p(u, w)w \right] = u - \sum_{y \in X} p(u, y)y .$$

Now for $u \in S$, $\prod_{v \in X} v^{d(v)}(u - \sum_{y \in X} p(u, y)y)$ is in I by definition. ∎

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Persi DIACONIS, Dept. of Mathematics, Harvard University,
Cambridge, MA 02138, U.S.A.

William FULTON, Dept. of Mathematics, University of Chicago,
Chicago, IL 60637, U.S.A.

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