A TRANSFORMATION THEOREM FOR PERIODIC SOLUTIONS OF NONDISSIPATIVE SYSTEMS

Abstract. We prove a theorem of existence of periodic solutions for nonautonomous ordinary differential systems which are "dissipative" only with respect to some (but not all) components of the phase-space. The main result follows from a lemma for the computation of the fixed point index of Poincaré's operator of some autonomous equations in terms of the generating vector field.

1. Introduction

Let \( F = F(t, x) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \) be a continuous function which is \( T \)-periodic in the first variable. We assume that uniqueness and global existence in the future for the solutions of the Cauchy problems associated to

\[
(1.1) \quad x' = F(t, x),
\]

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is ensured and denote by $x(\cdot; t_0, z)$ the solution of (1.1) satisfying the initial condition $x(t_0) = z \in \mathbb{R}^m$.

The problem of existence of $T$-periodic solutions for equation (1.1) has been widely considered in the literature for its significance in the applications.

A particular class of systems which has been investigated in detail is that of the dissipative systems in the sense of Levinson, i.e. systems for which there exists a closed ball $B(0, R_0) = \{z \in \mathbb{R}^m : \|z\| \leq R_0\}$, such that for each $z \in \mathbb{R}^m$ there is a time $t_z$ with $x(t; 0, z) \in B(0, R_0)$, for all $t \geq t_z$.

Dissipative periodic equations were studied by Levinson [18] and successively investigated by Cartwright [6], Massera [20], with respect to the existence of harmonic (i.e. $T$-periodic) solutions for $m = 2$ and subharmonic solutions for $m > 2$ (see also Ezeilo [9], Pliss [25]).

Then, it was discovered by the application of the Browder fixed point theorem [3], that the existence of at least one $T$-periodic solution for arbitrary equations in $\mathbb{R}^m$ is always ensured under the above recalled dissipativity condition (see, Yoshizawa [32, 33], Krasnosel’skii [16]). Such an approach was then formalized in a more abstract setting by Hale, LaSalle and Slemrod [12] and Hale and Lopes [13] (for an extensive bibliography, see also [11, 33], as well as the references at the end of the present paper).

On the other hand, in spite of a large number of existence results dealing with the non-dissipative case (see, e.g., Mawhin [21] and the references in [22]), a general theory concerning such kind of equations seems far to be completed.

In this work, we study the situation in which the dissipativity condition is satisfied only with respect to some components of the phase - space $\mathbb{R}^m$, while, with respect to the remaining coordinates, we assume a condition which is in some sense "dual" to the dissipativity (cf. [2]), that is we suppose that the origin is a repeller (observe that dissipativity can be explained as a condition of repulsivity of $\infty$).

We recall that for autonomous dissipative systems of the form

\[ x' = g(x), \]

it is known that the Brouwer degree of $g$ with respect to any open ball $B(0, R)$ containing the attractor, can be computed as

\[ \deg(g, B(0, R), 0) = (-1)^m, \]
Applications of this formula to ODEs arising in population dynamics and mathematical ecology can be found in [15].

Our first main result (Lemma 1 of Section 2) provides a similar computation; more precisely, we prove:

\[
\text{deg}(g, \mathcal{R}, 0) = (-1)^{m-j},
\]

where \( \mathcal{R} \) is a rectangle in the \( \mathbb{R}^m = \mathbb{R}^{m-j} \times \mathbb{R}^j \)-space, that is, the product of an open ball in \( \mathbb{R}^{m-j} \) with an open ball in \( \mathbb{R}^j \) and the system is supposed to be dissipative along the first \( m-j \) coordinates. The proof is based on Leray product formula combined with a result about the computation of the degree of the displacement operator obtained in [5]. Namely, recalling such result [5, Corollary 2] for the reader's convenience, we have

\[
(-1)^m \text{deg}(g, G, 0) = \text{deg} \left( I_{\mathbb{R}^m} - Q_T, G, 0 \right),
\]

where \( Q_T \) denotes the Poincaré map at the time \( T \) corresponding to system (1.2) and \( G \) is any open bounded subset of \( \mathbb{R}^m \) such that

\[ Q_T(z) \neq z, \quad \text{for all } z \in \partial G. \]

As a next step, using a continuation theorem recently developed in [5], we obtain a result of existence of \( T \)-periodic solutions for equation (1.1). Such existence theorem (Theorem 1 of Section 3) is based on a deformation (homotopy) of equation (1.1) to an autonomous equation of the form (1.2) and the estimate of the degree obtained in (1.3) is crucial for our proof.

This paper is intended as a short note containing the more abstract results. Accordingly, for the moment we confine ourselves just to the statement and proof of Lemma 1 and Theorem 1. It is reasonable to observe that various applications could be obtained at least in the following directions: a) using Liapunov - like functions to get attractivity of the origin for some components of the flow and repulsivity along the remaining coordinates; b) using arguments based on the computation of the fixed point index, via the formula (1.3), in order to get also uniqueness of the \( T \)-periodic solutions.
for equation (1.1) (see, e.g., [16, 23, 1, 10]). Further developments in these directions will be possibly discussed in a future work.

2. A lemma for the computation of the degree

Let $g : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function and suppose that uniqueness and global existence in the future for the solutions of the Cauchy problems associated to $x' = g(x)$, is ensured. We denote by $x(\cdot; z)$ the (unique) solution of the Cauchy problem

\begin{align}
\tag{2.1} x' &= g(x), \\
\tag{2.2} x(0) &= z,
\end{align}

for any $z \in \mathbb{R}^m$.

According to the assumptions, $x(t; z)$ is defined for all $t \geq 0$. Moreover, the map $(t, z) \mapsto x(t; z)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^m$. In particular, for every $t \geq 0$, the $t-$ Poincaré's operator

$$Q_t : \mathbb{R}^m \to \mathbb{R}^m,$$

is continuous, with

$$Q_0 = I_{\mathbb{R}^m} \quad (= \text{the identity in } \mathbb{R}^m).$$

We consider now the decomposition of $\mathbb{R}^m$ as

$$\mathbb{R}^m = \mathbb{R}^j \times \mathbb{R}^{m-j}$$

and, correspondingly, we set

$$z := (u, v), \quad \text{with } u \in \mathbb{R}^j, \ v \in \mathbb{R}^{m-j};$$
\( x(t; z) := (\varphi(t; z), \psi(t; z)) \), with \( \varphi \in \mathbb{R}^j \), \( \psi \in \mathbb{R}^{m-j} \).

For simplicity, we confine ourselves to the case:

\[ 1 \leq j \leq m - 1. \]

If \( j = 0 \) or \( j = m \), we easily can modify the arguments which are developed below; consequently, such situations are not examined in detail.

The following conditions are considered on the components \( \varphi \) and \( \psi \) of the solution \( x(\cdot) \).

There are constants

\[ T > 0, \quad 0 < A < a < L, \quad 0 < K < b, \]

such that, for each \( z = (u, v) \in \mathbb{R}^m \), we have:

\[(i_1) \quad \| u \| = a, \quad \| v \| \leq b \implies \| \varphi(t; z) \| \geq A, \quad \forall 0 \leq t \leq T; \]

\[(i_2) \quad \| u \| = a, \quad \| v \| \leq b \implies \| \varphi(T; z) \| \geq L; \]

\[(i_3) \quad \| u \| \leq a, \quad \| v \| = b \implies \| \psi(T; z) \| \leq K. \]

Let \( \mathcal{R} \subset \mathbb{R}^m = \mathbb{R}^j \times \mathbb{R}^{m-j} \), be the open rectangle:

\[ \mathcal{R} := B(0, a) \times B(0, b), \]

where, as usual, \( B(0, r) \) is the open ball of center the origin and radius \( r > 0 \).

Assumptions \((i_2)\) and \((i_3)\) can be described as a kind of repulsivity of the origin along the \( u \)-component and attractivity with respect to the \( v \)-coordinate. Hypothesis \((i_1)\) means that the components of the flow which are repelled, leave the ball \( B(0, a) \) without passing through the origin before (see Example 1).

Then we have:
LEMMA 1. Assume \((i_1), (i_2), (i_3)\). Then

\[ \text{deg}(g, R, 0) = (-1)^{m-i}. \]

Proof.

First of all, we observe that by \((i_2)\) and \((i_3)\), the \(T\)–Poincaré map associated to \((2.1)\) is fixed point free on the boundary \(\partial R\) of \(R\), that is,

\[ z \neq Q_T(z), \quad \text{for all } z \in \partial R. \]

Hence, we also have \(g(z) \neq 0\), for all \(z \in \partial R\).

Now, we can use the above recalled result in [5, Corollary 2] according to which (see formula (1.4)), we can conclude that

\[ \text{deg}(g, R, 0) = (-1)^m \text{deg} \left( I_{R^m} - Q_T, R, 0 \right). \]

Observe that, by our definitions,

\[ Q_T(z) = Q_T(u, v) = (\varphi(T;(u,v)), \psi(T;(u,v))). \]

Now we introduce the (continuous) homotopy

\[ H : R^m \times [0,1] \rightarrow R^m, \]

\[ H : ((u,v), \lambda) \mapsto (\lambda u - \varphi(T;(u,\lambda v)), v - \lambda \psi(T;(\lambda u,v))) \]

and observe that

\[ H(z, 1) = z - Q_T(z), \quad H(z, 0) = (-\tilde{\varphi}(T;u), v), \]

where in order to simplify the notations, we have set for all \(u \in R^j\) and \(t \geq 0\),

\[ \tilde{\varphi}(t;u) := \varphi(t;(u,0)). \]
We claim that

$$H(z, \lambda) \neq 0, \quad \text{for all } z \in \partial\mathcal{R}, \ \lambda \in [0, 1].$$

Indeed, assume by contradiction that there are $\tilde{z} = (\tilde{u}, \tilde{v}) \in \partial\mathcal{R}$ and $\tilde{\lambda} \in [0, 1]$, such that $H(\tilde{z}, \tilde{\lambda}) = 0$. From the definition of $H(\cdot, \cdot)$, we immediately obtain that

$$\lambda\tilde{u} = \varphi(T; (\tilde{u}, \lambda\tilde{v})), \quad \tilde{v} = \lambda\psi(T; (\lambda\tilde{u}, \tilde{v})),$$

with

$$\|\tilde{u}\| = a, \quad \|\lambda\tilde{v}\| \leq b \quad \text{or} \quad \|\lambda\tilde{u}\| \leq a, \quad \|\tilde{v}\| = b.$$ 

In any case, passing to the norm in (2.4) and using, respectively $(i_2)$ or $(i_3)$, we obtain a contradiction. Thus (2.3) is proved as well.

By the homotopic invariance of the degree we get

$$\deg \left( f_{\mathbb{R}^m} - \mathcal{Q}_T, \mathcal{R}, 0 \right) = \deg (H(\cdot, 0), \mathcal{R}, 0)$$

$$= \deg (-\varphi(T; \cdot), B(0, a), 0) \times \deg \left( f_{\mathbb{R}^{m-j}}, B(0, b), 0 \right)$$

$$= (-1)^j \deg (\varphi(T; \cdot), B(0, a), 0).$$

A comparison of the former identity for the degree with the latter one, yields

$$\deg (g, \mathcal{R}, 0) = (-1)^{m-j} \deg (\varphi(T; \cdot), B(0, a), 0).$$

Our final step is the evaluation of $\deg (\varphi(T; \cdot), B(0, a), 0)$. To this end, we consider the (continuous) retraction

$$r : \mathbb{R}^j \rightarrow B[0, a] := \overline{B(0, a)} \subset \mathbb{R}^j,$$

$$r(u) := \begin{cases} u, & \text{for } \|u\| \leq a, \\ \frac{u}{\|u\|}, & \text{for } \|u\| > a. \end{cases}$$
Observe that
\[ r(u) = 0 \quad \text{if and only if} \quad u = 0. \]

Next, we define the map
\[ r \circ \bar{\varphi} : \mathbb{R}^j \times [0, T] \longrightarrow \overline{B(0, a)}. \]

We claim that
\[ (2.6) \quad r \circ \bar{\varphi}(t; u) \neq 0, \quad \text{for all } u \in \partial B(0, a), \quad t \in [0, T]. \]

Indeed, it is sufficient to observe that
\[ r \circ \bar{\varphi}(t; u) = 0 \iff \bar{\varphi}(t; u) = \varphi(t; (u, 0)) = 0 \]

and then use property (i1) which ensures that
\[ \| \bar{\varphi}(t; u) \| \geq A > 0, \quad \text{for } \| u \| = a \quad \text{and} \quad t \in [0, T]. \]

Thus (2.6) easily follows.

Therefore, by the homotopic invariance of the degree and also recalling that
\[ r \circ \bar{\varphi}(0; u) = r(\varphi(0; (u, 0))) = r(u) = u, \quad \text{for all } u \in B[0, a], \]

we get
\[ \deg (r \circ \bar{\varphi}(T; \cdot), B(0, a), 0) = \deg \left( I_{B(0,a)}, B(0,a), 0 \right) \]

(2.7)
\[ = \deg \left( I_{\mathbb{R}^j}, B(0,a), 0 \right) = 1. \]
By the Leray product formula (see [19]), we can write*:

\[(2.8) \quad \text{deg}(r \circ \varphi, B(0, a), 0) = \sum_k \text{deg}(r, \Delta_k, 0) \times \text{deg}(\varphi, B(0, a), p_k),\]

where, the \(\Delta_k\)'s are the connected components of \(M \setminus \varphi(\partial B(0, a))\), \(p_k \in \Delta_k\) for each \(k\) and \(M\) is an arbitrary open bounded set containing \(\varphi(B[0, a])\).

Let \(\Delta_0\) be the connected component of \(M \setminus \varphi(\partial B(0, a))\) containing the origin of \(R^J\).

By assumption (\(i_2\)) it follows that

\[\| \varphi(u) \| \geq L > a, \quad \text{for all } u \in \partial B(0, a),\]

so that

\[\varphi(\partial B(0, a)) \cap B(0, a) = \emptyset.\]

Now, taking \(M = B(0, R)\), with \(R > a\) and sufficiently large, we obtain that

\[B(0, a) \subset M \setminus \varphi(\partial B(0, a))\]

and therefore we have

\[B(0, a) \subset \Delta_0.\]

On the other hand, as \(\Delta_k \cap \Delta_0 = \emptyset\), for each \(k \neq 0\) (if any of such \(k\) exists), we have

\[B(0, a) \cap \Delta_k = \emptyset, \quad \forall k \neq 0.\]

Since

\[\text{deg}(r, \Delta_0, 0) = \text{deg}(r, B(0, a), 0) = \text{deg} \left( I_{R^J}, B(0, a), 0 \right) = 1,\]

while

\[\text{deg}(r, \Delta_k, 0) = 0, \quad \text{for } k \neq 0\]

*In order to simplify further the notation we set \(\varphi(\cdot) := \varphi(T \cdot)\).
(in fact, \( r(u) \neq 0 \), for all \( u \in \Delta_k \), with \( k \neq 0 \)), from (2.8) we finally obtain

\[
(2.9) \quad \text{deg} (r \circ \varphi, B(0, a), 0) = \text{deg} (\varphi, B(0, a), p_0) = \text{deg} (\varphi, B(0, a), 0),
\]

(note that the choice \( p_0 = 0 \) is allowed as \( 0 \in \Delta_0 \)).

Thus, from (2.5), (2.7) and (2.9) we get

\[
\text{deg}(g, R, 0) = (-1)^{m-j}.
\]

The proof is complete. 

REMARK 1. The assumption \((i_1)\) is crucial; it is independent with respect to \((i_2),(i_3)\) and cannot be completely omitted.

An elementary example of this fact is the following:

EXAMPLE 1. Let \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \) and consider the equation

\[
(2.10) \quad x_1' = 1, \quad x_2' = -x_2,
\]

inducing the flow

\[
x(t; z) = (u + t, ve^{-t}); \quad z = (u, v).
\]

In this case, we have:

\[
\varphi(t; z) = u + t, \quad \psi(t; z) = ve^{-t}.
\]

In particular,

\[
\lim_{t \to +\infty} \| \varphi(t; z) \| = +\infty,
\]

uniformly in \( v \) and for \( u \) bounded, and

\[
\lim_{t \to +\infty} \| \psi(t; z) \| = 0,
\]
uniformly in $u$ and for $v$ bounded. It is easy to check that, given arbitrary

$$0 < a < L, \quad 0 < K < b,$$

it is possible to find $T > 0$ such that $(i_3)$ and $(i_3)$ are fulfilled. However, for any such $T$, $(i_1)$ never holds.

Note that in this case, for any open rectangle $R$ containing the origin, we have

$$\text{deg}(g, R, 0) = 0,$$

for $g(x_1, x_2) = (1, -x_2)$, as $g(x) \neq 0$, for all $x \in \mathbb{R}^2$ (indeed, the degree is zero with respect to any open bounded subset of the plane).

The same example also shows that no condition of the form

$$\lim_{t \to +\infty} \| \varphi(t; z) \| = +\infty$$

(even if assumed to hold uniformly with respect to the $v$-coordinate) is itself sufficient to guarantee the existence of periodic solutions for systems like $x' = F(t, x)$ or $x' = g(x)$, without adding further conditions implying the validity of some property of the form like $(i_1)$.

**REMARK 2.** From the proof of Lemma 1 it is possible to check that $(i_1)$ can be replaced by some weaker condition, like, for instance:

$$(i_1^*) \quad z = (u, 0), \quad \| u \| = a \implies \| \varphi(t; z) \| \geq A, \quad \forall 0 \leq t \leq T.$$

### 3. Existence of periodic solutions

Let $f = f(t, x; \lambda) : \mathbb{R} \times \mathbb{R}^m \times [0, 1] \to \mathbb{R}^m$ be a continuous function which is periodic of a fixed period $T > 0$ in the $t$-variable and suppose that uniqueness and global existence in the future for the solutions of the Cauchy problems associated to $x' = f(t, x; \lambda)$, is ensured.
We denote by $x(\cdot; z, \lambda)$ the (unique) solution of

\[
(3.1_\lambda) \quad x' = f(t, x; \lambda),
\]

\[
(3.2) \quad x(0) = z \in \mathbb{R}^m,
\]

which is defined on $[0, T]$.

We also set

\[
F(t, x) := f(t, x; 1).
\]

As in the preceding section, we consider now the decomposition of the space $\mathbb{R}^m$ by

\[
\mathbb{R}^m = \mathbb{R}^j \times \mathbb{R}^{m-j}
\]

and, correspondingly, we set

\[
z := (u, v), \quad \text{with } u \in \mathbb{R}^j, \; v \in \mathbb{R}^{m-j};
\]

\[
x(t; z, \lambda) := (\varphi(t; z, \lambda), \psi(t; z, \lambda)), \quad \text{with } \varphi \in \mathbb{R}^j, \; \psi \in \mathbb{R}^{m-j}.
\]

The following conditions are now considered on the components $\varphi$ and $\psi$ of the solution $x(\cdot)$.

For each $\lambda : 0 \leq \lambda < 1$, there are constants

\[
0 < A < a < L_\lambda, \quad 0 < K_\lambda < b,
\]

such that, for each $z = (u, v) \in \mathbb{R}^m$, we have:

\[
(j_1) \quad \| u \| = a, \quad \| v \| \leq b \implies \| \varphi(t; z, 0) \| \geq A, \quad \forall \, 0 \leq t \leq T;
\]

(actually, a weaker condition, in line with $(i_1^*)$ could be assumed here)

\[
(j_2) \quad \| u \| = a, \quad \| v \| \leq b \implies \| \varphi(T; z, \lambda) \| \geq L_\lambda, \quad \forall \, 0 \leq \lambda < 1;
\]
(j_3) \quad \| u \| \leq a, \quad \| v \| = b \implies \| \psi(T; z, \lambda) \| \leq K_\lambda, \quad \forall 0 \leq \lambda < 1.

Of course, nothing prevents the possibility that all the constants are independent of the parameter \( \lambda \). However, assumptions \((j_2), (j_3)\) allows us to deal with a slightly more general situation.

Let \( \mathcal{R} \subset \mathbb{R}^m = \mathbb{R}^j \times \mathbb{R}^{m-j} \), be the open rectangle:

\[ \mathcal{R} := B(0,a) \times B(0,b), \]

(as in Section 2). Then we have

**THEOREM 1.** Let \( f_0 : \mathbb{R}^m \to \mathbb{R}^m \) be a continuous function such that

\[ f(t,x;0) = f_0(x), \quad \forall 0 \leq t \leq T, \quad \forall x \in \mathbb{R}^m. \]

Assume \((j_1), (j_2), (j_3)\). Then the equation

\[ x' = F(t,x) \quad (3.3) \]

has at least one T-periodic solution \( \tilde{x}(\cdot) \) such that \( \tilde{x}(0) \in \overline{\mathcal{R}} \).

Clearly, all the assumptions of our theorem are meaningful in the case:

\[ 1 \leq j \leq m - 1. \]

If \( j = 0 \) (or, respectively, \( j = m \)), we consider \((j_1)\) and \((j_2)\) (or, respectively \((j_3))\) to be vacuously satisfied.

**Proof.** We use a recent continuation theorem developed in [5, Theorem 4], according to which for the validity of our statement it is sufficient to check that

\[ \text{deg}(f_0, \mathcal{R}, 0) \neq 0 \]

and

\[ z \neq x(T; z, \lambda), \quad \forall 0 \leq \lambda < 1, \quad \forall z \in \partial \mathcal{R}. \]
Now, the former property is satisfied by Lemma 1 applied to the function \( g = f_0 \), in virtue of the assumptions \((j_1), (j_2), (j_3) \) (for \( \lambda = 0 \)).

With respect to the latter property, we argue as follows:

Take any \( z = (u, v) \in \partial \mathcal{R} \) and consider, \( x(T; z, \lambda) = (\varphi(T; z, \lambda), \psi(T; z, \lambda)) \), for any \( \lambda \in [0, 1) \). Then, if \( u \in \partial B(0, a) \), by \((j_2)\) we have that \( || \varphi(T; z, \lambda) || > a \) and therefore \( \varphi(T; z, \lambda) \neq u \). On the other hand, if \( v \in \partial B(0, b) \), by \((j_3)\) we have that \( || \psi(T; z, \lambda) || < b \) and therefore \( \psi(T; z, \lambda) \neq v \).

In any case, we can conclude that \( z \neq x(T; z, \lambda) \). Then, Theorem 4 in [5] applies and the result is proved.

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