G. Anichini - G. Conti*

EXISTENCE OF SOLUTIONS OF A BOUNDARY VALUE PROBLEM THROUGH THE SOLUTION MAP OF A LINEARIZED TYPE PROBLEM

Abstract. A boundary value problem of the form:

\[
(BV) \quad \begin{cases} 
x' = f(t, x), & t \in I \subseteq \mathbb{R}, \ x \in \mathbb{R}^n, \\
x \in S, 
\end{cases}
\]

where \( f \in C(I \times \mathbb{R}^n, \mathbb{R}^n), S \subseteq C(I, \mathbb{R}^n) \), is considered. Its possible solutions are investigated by studying the properties of the solution mapping of some linearized type problem associated to the given one. Two cases, a Neumann and a Dirichlet boundary value problem, are considered in the absence of the hypotheses which usually ensure the convexity of the values of the multifunction defined by the solution mapping.

Introduction

Usually, when a boundary value problem for ordinary differential equations of the form:

*Work done under the auspices of MPI-Research Project "Equazioni Differenziali".
\[ (BV) \quad \begin{cases} \dot{x} = f(t, x), & t \in I \subseteq \mathbb{R}, \ x \in \mathbb{R}^n \\ x \in S, \end{cases} \]

where \( f \in C(I \times \mathbb{R}^n, \mathbb{R}^n), S \subseteq C(I, \mathbb{R}^n) \), is considered, the most probable approach may consist of finding fixed points of the so called solution map \( \Sigma \) i.e. the map which associates the solutions of some easier to deal boundary value problem (the “linearized type problem”) originated from the (BV) problem (see e.g. [1], [2], [3], [7] or [8] for a wider reference).

Then the essential features of that approach consist of checking if the mapping \( \Sigma \) is in some sense a regular one: it is worthwhile to recall that this mapping is not always a single-valued mapping, but it may be also a multivalued function; then, in the first case, we have to check just the continuity property of the mapping (and some standard conditions as compactness, a priori boundedness and so on), but in the latter case we have also to consider, beyond the regularity assumption (usually upper semicontinuity), the “geometry” of the images of the set-valued mapping. As matter of fact it is well known that if these images are convex (non empty and compact), then it is not hard to get fixed points for the set-valued mapping; if the convexity assumption is not given, then a different approach has to be taken into account.

The convexity condition is obtained if the hypotheses we put on the linearization of the function \( f \) are in some sense strong: if we want to relax them it happens that the convexity property is sometime missed.

In that paper we want to consider two very classical problems, the Neumann problem and the Dirichlet problem, and show that it is possible to get the existence of (at least) one solution without requiring the strong assumptions needed in order to apply some fixed point theorem for which the convexity is an essential argument.

Notations and preliminary results

**DEFINITION 1.** ([15]): Let \( F^o = (F^n)_{n \in N} \) denote the \( \check{\text{C}} \)ech cohomological functor with coefficients in \( Z \), defined on the category of topological pairs
(Y, A). We denote by $F^n_*$ the reduced cohomology. Then, given a topological space $Y$, we shall say that it is acyclic if for all $n \in \mathbb{N}$ we have $F_n^*(Y) = 0$.

One of the most important examples of acyclicity is given by convex sets. As a topological property of acyclic sets we know that they are connected and even simply connected.

DEFINITION 2. Let $X$ and $X'$ be metric spaces. A set valued mapping $G : X \rightarrow X'$ is said to be upper semicontinuous (u.s.c.) at $x \in X$ if for any neighborhood $V \supset G(x)$ there exists a neighborhood $U$ of the point $x$ such that $G(y) \subset V$ for any $y \in U$.

If, for every $x \in X$, $G$ is u.s.c. at $x$ and $G(x)$ is a compact set, then $G$ is said to be upper semicontinuous on $X$.

REMARK. If an u.s.c. mapping $G$ sends bounded sets into relatively compact sets then it is said to be compact.

We want to recall here that $G$ is called a closed graph operator if from $x_n \rightarrow x_o, \ y_n \rightarrow y_o, \ y_n \in G(x_n)$ it follows $y_o \in G(x_o)$. If $G(x)$ is a closed set for all $x \in X$ and $G(X)$ is a relatively compact set, then $G$ is u.s.c. if and only if $G$ is a closed graph operator.

REMARK. The spaces we shall consider in the sequel will be the space $X = C([a, b], \mathbb{R}^n)$ of all continuous functions defined on the real interval $[a, b]$ and the space $Y = C^1([a, b], \mathbb{R}^n)$ of all continuously differentiable functions defined on $[a, b]$; these (Banach) spaces will be equipped with the usual norms i.e.

$$||x|| = \sup\{|x(t)|, t \in [a, b]\}, \text{ whenever } x \in X,$$

$$||y||_1 = \sup\{|y(t)| + |y'(t)|, t \in [a, b]\}, \text{ for all } y \in Y.$$ 

DEFINITION 3. A map $T : X \rightarrow X'$ is said to be a proper mapping if the inverse image of any compact subset of $X'$ is a compact subset of $X$. It is worthwhile to remark that if $T$ is a compact mapping then $I - T$, if restricted to bounded sets, is a proper mapping.

In the sequel the open ball with radius $r$ and centered in $x_0$ will be denoted by $B_r(x_0)$.
A notation very useful in the sequel is the following:

Let $Q$ be any set related to some linearized type boundary value problem: we shall denote by $\Sigma : Q \to S$ the multivalued operator which defines a correspondence between any $q \in Q$ and the set of solutions of the linearized type problem.

Let us recall the following result of [5].

**PROPOSITION 1.** Let $X$ be a metric space and let $Y$ be a Banach space. Let $\{F_k\}$ be a sequence of continuous and proper mappings from $X$ into $Y$. Then, if the equation $F_k(x) = y$ admits an unique solution for each $y_0 \in Y$ and for each $y \in B_r(y_0)$ for a suitable $r$, the set $F^{-1}(y_0)$ is an acyclic set.

A result we will use very often in order to prove our results is the following:

**PROPOSITION 2.** Consider the second order differential problem

\[
(BV1) \begin{cases}
  x'' = f(t,x,x'), & t \in I = [a,b] \subset R, \\
  x \in S,
\end{cases}
\]

where $f \in C(I \times R \times R, R)$ and $S \subset C(I,R)$. Let $g \in C(I \times R^4, R)$ such that $g(t,c,c,d,d) = f(t,c,d)$ for all $(t,c,d) \in I \times R^2$ and assume that there exist sets $Q \subset C^1(I,R)$ and $S_1$ satisfying the following hypotheses:

- $Q$ is a closed and convex subset;
- $S_1 \subset Q \cap S$ is a bounded subset.

Let us consider the (linearized type) differential problem

\[
(BVq1) \begin{cases}
  x'' = g(t,x(t),q(t),x'(t),q'(t)) & t \in I, \\
  x \in S_1.
\end{cases}
\]

Then, if we assume that the solution set $\Sigma(q)$ of the latter system is an acyclic set for any $q \in Q$, the differential problem $(BV1)$ has at least one solution.

(The proof of this result can be found in [2] and it relies essentially on a straightforward application of the Eilenberg-Montgomery theorem ([9])).
Results

THEOREM 1. Let us consider the following Neumann problem:

\[ \begin{cases} -u'' + g(u) = h(t, u, u'), & t \in [0, \pi], \quad u \in \mathbb{R}, \\ u'(0) = u'(\pi) = 0. \end{cases} \quad (N) \]

Let assume the following conditions hold:

i) \((t, u, u') \rightarrow h(t, u, u')\) is a continuous and bounded function such that for all \(q \in C^1([0, \pi])\) we have:

\[
\frac{1}{\pi} \int_0^\pi h(s, q(s), q'(s))ds \in Int(Im(g)),
\]

where \(Int(Im(g))\) stands for the (relative) interior of the image of the function \(g\).

ii) \(u \rightarrow g(u)\) is a continuous, bounded, increasing function such that \(ug(u) > 0, \ u \neq 0\) and \(\sup\{|g(u)|, \ u \in \mathbb{R}\} > \sup\{|h(t, u, u')|, \ (t, u, u') \in [0, 1] \times \mathbb{R}^2\}\).

Then the problem \((N)\) has at least one solution.

Proof: Let any \(q \in C^1([0, \pi])\) be given and let us consider the following problem:

\[ \begin{cases} -u'' + g(u) = h(t, q(t), q'(t)), & t \in [0, \pi], \\ u'(0) = u'(\pi) = 0. \end{cases} \quad (NL) \]

We show firstly that there exists a ball \(Q \subset Y = C^1([0, \pi])\) such that every possible solution of \((NL)\) belongs to \(Q\) for each \(q \in Q\).

If we denote by \(S\) the set \(\{y \in Y : y'(0) = y'(\pi) = 0\}\) then, by following the notations of Proposition 2, the set \(S \cap Q\) will be the set \(S_1\).

We begin to say that, from i) and ii), there exists a positive constant \(L_3\) such that

\[ |u''(t)| \leq L_3, \quad \text{for} \quad t \in [0, \pi]. \]
From the equality \( u'(t) = \int_0^t u''(s) \, ds \) it follows at once the existence of \( L_2 > 0 \) such that \( |u'(t)| \leq L_2 \) for all \( t \in [0, \pi] \).

Finally we want to show that there exists a positive constant \( L_1 \) such that every solution of \((NL)\) verifies:

\[
(*) \quad |u(t)| \leq L_1.
\]

From ii) we have that there exists a positive constant \( L_1 \) such that

\[ g(u) - h(t, q(t), q'(t)) > 0, \quad \text{for all} \quad |u| > L_1. \]

So Lemma 1.2 of [10] allows us to say that the positive constant \( L_1 \) satisfies \((*)\).

So, from the above setting, we can say that the set \( Q \) is given by:

\[ Q = \{ u \in Y : ||u||_1 \leq L_1 + L_2 \}. \]

In [13] is shown that the application \( \Sigma : Y \rightarrow Y \) such that \( \Sigma(q) \) is a solution of the problem \((NL)\) is a set valued function whose images are acyclic sets: so, by using Proposition 2, this is enough to conclude that the problem \((NL)\) has at least one solution.

THEOREM 2. Let us now consider the following Dirichlet problem:

\[
(D) \quad \begin{cases} 
-u'' - u + g(u) = h(t, u, u') & t \in I = [0, \pi], \quad u \in \mathbb{R}^n, \\
u(0) = u(\pi) = 0.
\end{cases}
\]

Let assume the following conditions hold:

i) \( u \rightarrow g(u) : \mathbb{R} \rightarrow \mathbb{R} \) is a continuous and increasing function;

ii) \( (t, u, u') \rightarrow h(t, u, u') \) is a continuous function and such that \(|h(t, u, u'))| \leq H \) for all \((t, u, u') \in J \times \mathbb{R}^2\). Moreover there is a positive constant \( M \) such that:

\[ -u + g(u) - H > 0 \quad \text{for} \quad u > M \]

and

\[ -u + g(u) - H < 0 \quad \text{for} \quad u < -M. \]

Then the problem \((D)\) has at least one solution.
REMARK. In order to establish some existence results, the Dirichlet problem with increasing nonlinear perturbation has been extensively considered in the literature. We refer to [11] and [12] and the references therein for a wider source.

In order to prove the above Theorem 2 we need firstly to prove the following:

**THEOREM 2.** Consider the second order differential problem:

\[(BV) \]

\[
\begin{align*}
-u'' - u + g(u) &= f(t), & t &\in I = [0, \pi], \\
u(0) &= u(\pi) = 0.
\end{align*}
\]

where \( f : [0, \pi] \to \mathbb{R} \) is a continuous and bounded function and \( g : \mathbb{R} \to \mathbb{R} \) is a continuous and increasing function such that there is a positive \( M \) such that:

\[-u + g(u) - F > 0 \quad \text{for} \quad u > M\]

and

\[-u + g(u) - F < 0 \quad \text{for} \quad u < -M,\]

where

\[F = \sup \{|f(t)|, \quad t \in [0, \pi]\}\]

Then the set of solutions of \((BV)_f\) is acyclic.

**Proof:** Since the function \( g \) ranges over the whole real space and it is increasing, a well known result of [4] allows us to say that the problem \((BV)_f\) has solutions; so, as a standard result (see, for instance, Lemma 1.2 of [10]), we can claim that the above set of solutions is bounded (i.e. \(|u(t)| \leq M\) for all \( t \in [0, \pi] \)). Now let \( \Omega = B_r(0) \subset X \) be such that the solutions of \((BV)_f\) belong to \( \Omega \) and define the operator:

\[L : D(L) \subset X \to X \quad \text{as} \quad Lu = u''\]

where \( D(L) = \{u \in C^2([0, \pi]) : u(0) = u(\pi) = 0\} \).

Let us denote by \( H : R(L) \to D(L) \) the inverse of the operator \( L|_{R(L)} \) and by \( N \) the Nemyckii operator associated with the function \((-f + g(u))\);
finally let us put \( P : X \to X \) the projection operator onto \( \text{ker}(L) = \text{span}\{\xi\} \) with \( |\xi| = 1 \) i.e. \( Pu = \langle u, \xi \rangle \xi \) where \( \langle \cdot, \cdot \rangle \) stands as the usual scalar product on \( L^2([0, \pi]) \). It is known ([6]) that the solutions of \((BV)_f\) are the fixed points of the operator \( T : X \to X \) defined as:

\[
Tu = Pu + H(I-P)N(u) + PN(u).
\]

This operator is compact and so the operator \( I - T \) is a proper operator. Now consider the sequence \( T_k : X \to X \) defined as \( T_ku = Pu + H(I-P)N_k(u) + PN_k(u) \) where \( N_k \) is the Nemyckii operator associated with the function \(-f + g(u) + (1/k)u\): so the fixed points of \( T_k \) are solutions of the problem:

\[
(BV)_k \begin{cases}
-u'' - u + g(u) + (1/k)u = f(t), & t \in [0, \pi], \\
u(0) = u(\pi) = 0.
\end{cases}
\]

We have \( \sup\{||T_ku - T_k(u)||, \ u \in \Omega\} < r_k \), where \( r_k \) is a sequence of real numbers which goes to zero as \( 1/k \). So we can say that \( T_k \) converges uniformly to \( T \) on \( \Omega \): moreover \( I - T \) is, for each positive integer \( k \), a proper mapping on \( \Omega \). Let us consider \( y \in B_1(0) \subset X \): the solutions of the equation \( u - T_ku = y \) are the solutions of the problem:

\[
(BV)_y \begin{cases}
-u'' - u + g(u) + (1/k)u = f(t) + y(t), & t \in [0, \pi] \\
u(0) = u(\pi) = 0.
\end{cases}
\]

Since the range of the function \( g_k(u) = g(u) + (1/k)u \) is \( R \) for each integer \( k \), we have that \((BV)_y\) has solutions ([4]). Moreover we have:

\[-u^2 + ug(u) + (1/k)u^2 - u(f(t) + y(t)) > 0\]

for \( |u| > M \) and so the solutions of \((BV)_y\) belong to \( \Omega \). It is also worthwhile to remark that each differential problem such the \((BV)_y\) has an unique solution for every \( y \in B_1(0) \); as matter of fact \( g_k(u) \) is a strictly increasing function for each \( k \) and so for every \( u \neq v \) we have:
\[
< u'' + u - g_k(u) - v'' - v + g_k(v), \ u - v > \\
= < L(u - v), \ u - v > + < g_k(v) - g_k(u), \ u - v < 0.
\]

Then, from Proposition 1, we can conclude the proof by saying that the set of solutions of \((BV)_f\) is acyclic.

**Proof of Theorem 2:** As in the proof of the Theorem 1 we can show that there is a constant \(D_1 > 0\) such that the solutions \(u\) of the following problem:

\[
(DL) \quad \begin{cases}
-u'' - u + g(u) = h(t, q(t), q'(t)), \ t \in [0, \pi], \\
u(0) = u(\pi) = 0.
\end{cases}
\]

satisfy \(|u(t)| \leq D_1\).

The boundedness of \(|u(t)|\) and the conditions i) and ii) imply that there exists a positive constant \(D_3\) such that \(||u''|| \leq D_3\). The Rolle theorem allows us to say that there is some point \(c \in (0, \pi)\) such that \(u'(c) = 0\); so \(u'(t) = \int_c^t u''(s)ds\) and we get the existence of a positive constant \(D_2\) such that \(|u'(t)| \leq D_2\).

Then by defining the set \(Q\) as \(\{u \in Y : ||u||_1 \leq D_2 + D_1\}\) we can say that there exists a ball \(B_\rho(0)\), with \(\rho = D_2 + D_1\), such that the solutions of the linear boundary value problem \((DL)\) belong to it. The set \(S_1\) of Proposition 2 is given by \(Q \cap S\) where \(S = \{u \in X = C([0, \pi], R) : u(0) = u(\pi) = 0\}\). By Proposition 3 the set of solutions of \((DL)\) is acyclic for every \(q \in C^1([0, \pi], R)\). Finally the thesis of the theorem is achieved.

**REFERENCES**


Giuseppe ANICIINI
Dipartimento di Matematica - Università di Modena
Via Campi 213/b, Modena, Italy.
Giuseppe CONTI
Istituto di Matematica, Facoltà di Architettura
Piazza Brunelleschi 2, Firenze, Italy.

Lavoro pervenuto in redazione il 24.11.90
e, in forma definitiva, il 9.4.91