In the first quarter of this century, Kaluza-Klein theory has been an attempt to unify Electromagnetism and General Relativity. It is still of interest today because it has survived many changes in theoretical models in Physics.

Our own interest in this theory originates in its strong geometric structure, which undoubtedly explains its many reappearances. It also leads the way to the construction of many interesting examples of Riemannian metrics having special curvature properties, the so-called Einstein metrics. These metrics are defined on total spaces of bundles, i.e., on spaces having a local product structure, a feature which makes them interesting both on mathematical and on physical grounds. The notion of bundle is indeed by now accepted as a major landmark in the quest for globality in Geometry, and was also recognized somewhat later as fundamental in Gauge Physics. Some of these examples have been obtained only recently, and, as a result, this subject is connected with present day research.

All along this article, we will be moving in the realm of Riemannian geometry. More precisely, our point of view will be the following. On a given \( n \)-dimensional manifold \( M \) (most of the time compact), we are looking for metrics which fit this manifold as well as possible. For surfaces, constant curvature metrics are a perfect match. In higher dimensions \( (n \geq 3) \), things cannot be so simple since space forms (i.e., manifolds admitting constant curvature metrics) do not by far exhaust all possible differentiable types. One fruitful idea (that mathematicians probably borrowed from physicists) comes
then to mind: pick the metric as a critical point of a functional having a geometrical content. The simplest functional involving the curvature is the total scalar curvature $S$ which at a metric $g$ takes the value $S(g) = \int_M s_g v_g$

where $v_g$ denotes the volume element defined by the metric, and $s_g$ the scalar curvature function, i.e., the function one obtains from the Riemann curvature 4-tensor by two successive contractions. Since this functional is sensitive to enlarging the metric by a constant factor (a mere homothety), it is convenient to restrict it to the space of metrics $g$ having a given total volume, say $\int_M v_g = 1$. An Einstein metric is then a critical point for this restricted $S$, and it is characterized by the equation

$$r_g = \frac{1}{n} s_g g$$

where $r_g$ denotes the Ricci curvature of the metric $g$, i.e., the symmetric 2-tensor field obtained by contracting the Riemann curvature tensor once. (In this definition, there is only a sign ambiguity because of the special symmetries of the Riemann curvature 4-tensor. We kill it by requiring that $r_g$ and $s_g$ be positive for the standard metric on the sphere.) As we will recall in Section I, this equation coincides with the field equation proposed by Albert Einstein for General Relativity in the context of Lorentzian metrics, hence the name given to a metric satisfying this equation.

*Kaluza-Klein theory gives a recipe to construct special solutions to this equation, and opens the way to an interesting family of geometric objects. This is especially useful since examples of Einstein metrics on compact manifolds remained for a long time rather scarce. Indeed, the only examples of Einstein metrics known until 1975 were obtained by purely algebraic methods on irreducible homogeneous spaces. Later, geometric constructions, and powerful analytic methods, provided us with more examples†.

Our little venture into Physics will give us the opportunity of comparing the attitudes of mathematicians and physicists in this type of problems, and hopefully to propose some guidelines for later developments.

This expository article is organised as follows. In section I we present the theory introduced by Kaluza, and later by Klein in its original context, which can be coined as a 5-dimensional version of General Relativity. In present day mathematical terminology, it corresponds to studying the Riemannian  

†For an up-to-date survey on Einstein metrics, the reader should consult [2]. A more complete description of mathematical aspects of Kaluza-Klein theory can also be found in [3].
geometry of circle bundles as we explain in Section II. We move on in Section III to considering non-abelian versions of Kaluza-Klein theory, and their links to gauge theories. Various interesting examples of Einstein metrics in the Kaluza-Klein spirit will be given in Section IV. We close this review by a side by side comparison of concepts used by mathematicians and physicists in our context.

It is a pleasure to take the opportunity of this preface to thank the organizers of the meeting "Partial Differential Equations and Geometry" for their efficiency in setting up this conference, and for inviting us to report on this topic for a mixed audience of analysts and geometers.

I. Kaluza-Klein theory: the early days.

As was said in the introduction, Kaluza-Klein theory was an attempt to unify Electromagnetism and General Relativity. Let us begin by recalling the most basic features of both of these theories.

a) The mathematical structure of Electromagnetism

In 1873, James Clerk Maxwell proposed the set of equations known today under his name and the basic concept of an electromagnetic field \( \omega \). In the language of exterior algebra (which Hermann Grassmann was just developing at the time of Maxwell), \( \omega \) is an exterior differential 2-form over the space \( \mathbb{R}^4 = \{ x | x = (x^0, x^1, x^2, x^3), x^i \in \mathbb{R} \} \) viewed as space-time. The 2-form \( \omega \) incorporates the electrical field \( \vec{E} \) and the magnetic field \( \vec{B} \) as follows

\[
\omega = \vec{E} \wedge dx^0 + \sum B_i dx^i dx^k
\]

where the last summation is taken over even permutations \( \{ i, j, k \} \) of \( \{ 1, 2, 3 \} \). This point of view incorporates the later unification between the notions of space and time made by Special Relativity, and more fundamentally the introduction of a metric of signature \( (- + + +) \) over \( \mathbb{R}^4 \) due to Hermann Minkowski after earlier work by Hendrik A. Lorentz. In this setting, the Maxwell equations say that

\[
\begin{align*}
\delta \omega &= 0 \\
d \omega &= \mathbf{i}
\end{align*}
\]

where \( d \) is the exterior differentiation operator, \( \delta \) its formal adjoint using the volume element defined by the metric, and \( \mathbf{i} \) denotes the current, a 4-vector
having the charge density as time component, and the current density as space content. In the vacuum, the second set of equations is of course $\delta \omega = 0$.

Notice that the first part of the system incorporates the Faraday law, and the law expressing the absence of magnetic sources.

In modern mathematical terms, the electromagnetic field $\omega$ in the vacuum is a harmonic 2-form.

Moreover, if the space-time is contractible (as $\mathbb{R}^4$ is), or if we content ourselves with working locally, thanks to the Poincaré lemma one can write $\omega = d\alpha$ where $\alpha$ is a differential 1-form. This result will prove of importance later in our discussion.

**b) General Relativity**

In 1913, A. Einstein (in [5] written jointly with Marcel Grossmann) proposed a new theory of gravitation which was very geometric in nature. In it, he replaced the basic scalar field introduced by Newton to describe the effect on the geometry of the distribution of mass by a Lorentzian metric $\ell$, a 10-component field, on a manifold viewed as space-time. In this paper, he failed to find the proper field equations, but we feel that this jump was the crucial step. As soon as one works in this context, the field equations are in some sense inevitable. The field equations, called the *Einstein equations*, found by Einstein in 1915, state that

\[ r_\ell - \frac{1}{2} s_\ell = T \]

where again $r_\ell$ and $s_\ell$ are respectively the Ricci and scalar curvatures of the Lorentzian metric $\ell$, and $T$ the stress-energy 2-tensor field which describes the Physics of the situation (outside the effects of gravity). (One should notice the parallel with the Maxwell equations, where the left hand side has a strong geometric content whereas the right hand side describes the Physics). For the vacuum the right hand side vanishes, hence by taking the trace of equation (4) one obtains the *Einstein vacuum equations*

\[ r_\ell = 0 \]

of which equation (1) is a Riemannian generalization. For some time, Einstein himself considered the Lorentzian version of (1), calling $\lambda$ the *cosmological constant*, but later, out of physical grounds, he viewed this modification as a blunder.

The systems of Partial Differential Equations (1) and (4) are non-linear. Indeed, in a coordinate system ($x^i$) in which the metric $g$ can be expressed
as \( g = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j \), the expression of the Ricci curvature \( r_g \) is the given by

\[
\sum_{i,j=1}^{n} r_{ij} dx^i dx^j
\]

with

\[
\sum_{k,l=1}^{n} \frac{1}{2} g^{kl} \left( \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} \right) + \text{quadratic terms in } \frac{\partial g}{\partial x}
\]

(in this formula, \( g^{ij} \) denotes the inverse matrix of \( g_{ij} \)).

The key difference between the Riemannian and the Lorentzian settings is of course that in the first case the system is elliptic, and in the second it is hyperbolic. On the space of metrics, the left hand side of the Einstein field equation has a variational nature. It is indeed the differential of the total scalar curvature functional \( S \). Since this fact was discovered by David Hilbert, \( S \) is often referred to as the Einstein-Hilbert Lagrangian.

c) The original Kaluza-Klein theory

In 1919, Theodor Kaluza proposed a 5-dimensional version of General Relativity. Although at first sight the theory presented some interesting features, Kaluza’s paper (cf. [7]) was only accepted for publication by Einstein in 1921.

Kaluza considered \( \mathbb{R}^5 \) with coordinates \( (x^0, x^1, x^2, x^3, \xi) \) as model for an extended space-time, which can be projected by a map \( p \) onto the ordinary space-time \( \mathbb{R}^4 \) with coordinates \( (x^0, x^1, x^2, x^3) \). The main trick, the so-called Kaluza Ansatz, is to consider a Lorentzian metric \( \bar{\ell} \) on \( \mathbb{R}^5 \) of the form

\[
\bar{\ell} = \phi^a (p^* \ell + \phi (\alpha + d\xi) \otimes (\alpha + d\xi))
\]

where \( \phi : \mathbb{R}^4 \rightarrow \mathbb{R} \) is a function, \( a \in \mathbb{R} \), \( \ell \) is a Lorentzian metric on \( \mathbb{R}^4 \) and \( \alpha \) is a 1-form on \( \mathbb{R}^4 \).

Provided one takes \( \phi \equiv 1 \), the key fact is then

Kaluza’s discovery. – The Einstein vacuum equations for \( \bar{\ell} \) contain the Maxwell equations for the electromagnetic field \( \omega = d\alpha \), and the Einstein equations for \( \ell \) with right hand side \( T \) taken to be the interaction term coming from the electromagnetic field \( \omega \).
This approach immediately raised the question of the physical meaning of the fifth dimension corresponding to the coordinate $\xi$ that nobody had so far observed. What also appeared awkward was the necessity of putting $\phi \equiv 1$.

Some years later, in 1926, at a time when Quantum Mechanics was formulated, Oscar Klein (cf. [9]) took up the Kaluza model having in mind to exploit it in connection with the newly established quantum theory. He in particular suggested that the extra-dimension be taken compact, i.e., a circle. The extended space-time $\tilde{M}$ was then $\tilde{M} = M \times S^1$. Thanks to this reduction one can see the Kaluza Ansatz as the first term in a Fourier series expansion of the field with respect to the fifth variable.

It is of interest to know that the 5-dimensional Lagrangian $\tilde{S}(\tilde{\ell}) = \int_{\tilde{M}} \tilde{s}_\ell v_\ell$ (provided one assumes that the integral makes sense) can be expressed as an effective Lagrangian where, after performing an integration along the fibres $S^1$, the integral is taken along the ordinary space-time $M$, namely

$$S(\ell) = \int_M (s_\ell - \phi |d\alpha|^2 - \frac{1}{6} \phi^{-2} |d\phi|^2) v_\ell,$$

(Here, we have set $\alpha = -\frac{1}{3}$ as physicists usually do for dimensions reasons). If we had brought in the fundamental constants of Physics measuring the relative strengths of the gravitational and electromagnetic interactions, one would have seen that the size of the circle is to be taken of the order of $10^{-35}$m, a good reason for not having yet observed the fifth dimension.

II. The Riemannian geometry of circle bundles.

In this section we take up the Kaluza-Klein original model from a mathematical point of view. It is then convenient to consider only Riemannian metrics. We begin with a proposition.

PROPOSITION 1. – If $(\tilde{M}, \tilde{g})$ has a circle action by isometries with orbits of the same length, then all orbits are geodesics, and the action can be made free by going, if necessary, to a covering of the circle acting.

The proof, which is straightforward, relies on appropriately studying the covariant derivative of the vector field $V$ defined by the action.

REMARK. – If we had inserted a non-constant function $\phi$ in the expression of the metric as Kaluza did in his first attempt, we would have forced the fibres
not to be totally geodesic. Then, the geometric description of the situation would have been more involved. We will say more on that in the more general situation of general bundles that we address in section III.

The whole geometry of the situation is then captured by the following data. It is first convenient to normalize \( \bar{g} \) so that \( V \) has length 1. One then introduces the differential 1-form \( \bar{a} \) dual to the vector field \( V \) via the metric \( \bar{g} \) (its kernel is perpendicular to the orbits), and the space of orbits \( M = \bar{M}/S^1 \) which inherits a metric \( g \) since the bilinear form \( g_H \) defined on \( \bar{M} \) by \( g_h = \bar{g} - \bar{a} \otimes \bar{a} \) satisfies \( \mathcal{L}_V g_h = 0 \). Of course, if \( p \) denotes the projection from \( \bar{M} \) to \( M \), \( g_h = p^* g \).

As a result, one can view \( \bar{M} \) as an \( S^1 \)-principal bundle over \( M \) on which \( \bar{a} \) is a connection form.

One can also run the construction backwards, i.e., start from a Riemannian manifold \((M, g)\), and consider an \( S^1 \)-principal bundle \( p : \bar{M} \rightarrow M \) over it. To any choice of an \( S^1 \)-connection with 1-form \( \bar{a} \) on \( \bar{M} \), one can associate the Riemannian metric

\[
\bar{g} = p^* g + \bar{a} \otimes \bar{a}.
\]

The fibres are automatically totally geodesic for the metric \( \bar{g} \) by Proposition 1. Therefore, we proved the following

**Theorem 2.** The space of \( S^1 \)-invariant Riemannian metrics on the total space of an \( S^1 \)-principal bundle for which the fibres are totally geodesic is in 1-1 correspondence with the product of the space of Riemannian metrics on the base by the space of \( S^1 \)-connections over the bundle.

It is this fact that S. Kobayashi rediscovered in 1963 (cf. [10]). Of course, what was of interest to him (and remains to us) is the description of the curvature of \( \bar{g} \).

If we denote as above by \( V \) a unit vertical vector field, and by \( X \) and \( \bar{X} \) respectively a tangent vector to \( M \) and its horizontal (i.e., \( \bar{g} \)-perpendicular to \( V \)) lift, we have the following basic formulas for \( \bar{r} \), the Ricci curvature of \( \bar{g} \),

\[
\begin{align*}
\bar{r}(V, V) &= \frac{1}{4} |\omega|^2 \\
\bar{r}(\bar{X}, V) &= -\delta \omega(X) \\
\bar{r}(\bar{X}, \bar{Y}) &= r(X, Y) + \frac{1}{2} \omega \circ \omega(X, Y)
\end{align*}
\]

where \( \omega \) is the curvature 2-form of the connection form \( \bar{a} \) (i.e., \( p^* \omega = d\bar{a} \)), \( \delta \) denotes the codifferential operator on \((M, g)\), and the notation \( \omega \circ \omega \) means
the composition of $\omega$, viewed as a linear map, with itself (i.e., in a coordinate system, $(\omega \circ \omega)_{ij} = \sum_{kl} g^{kl} \omega_{ik} \omega_{lj}$).

From (8), one easily sees that the formula for the scalar curvature of $\bar{g}$ is $\bar{s} = s - \frac{1}{4} |\omega|^2$, hence that the formula for the Lagrangian $\tilde{S}$ appears indeed as the sum of the Lagrangian $S$ plus the energy term coming from the connection.

Hence, one can derive the precise version of Kaluza’s discovery (also the content of Kobayashi’s remark) from (8). Physicists like to see this as a superposition of energies for the extended system (often called minimal coupling).

**THEOREM 3.** On the total space $\tilde{M}$ of an $S^1$-bundle $p: \tilde{M} \to M$, the Riemannian metric $\bar{g} = p^*g + \alpha \otimes \bar{\alpha}$ associated with the $S^1$-connection form $\bar{\alpha}$ with curvature $\omega$ is Einstein for the constant $\tilde{\lambda}$ if and only if:

i) $\omega$ is a harmonic 2-form with constant length $2\sqrt{\tilde{\lambda}}$;

ii) $r = \tilde{\lambda}g - \frac{1}{2} \omega \circ \omega$.

**REMARK.** Because of i), the Einstein constant $\tilde{\lambda}$ is necessarily positive unless the bundle is flat in which case $\bar{g}$ is Ricci-flat. The equations we have been writing hold equally well in the Lorentzian case. We come up with the same constraint on the sign of $\tilde{\lambda}$ provided the fibres are space-like (as Kaluza assumed). Hence, we see that the vacuum Einstein equations force the bundle to be trivial, and also, what makes matters worse, the electromagnetic field to vanish identically. This phenomenon is often referred to as the Kaluza-Klein inconsistency.

Condition ii) does not imply that the base metric is Einstein. It is nevertheless under this special assumption that explicit examples were found by Kobayashi in [10]. If $r = \lambda g$, then we must have

$$\omega \circ \omega = 2(\tilde{\lambda} - \lambda)g,$$

so that, if $\tilde{\lambda} \neq 0$, the curvature form must be non degenerate. This is of course the case if $(M, g)$ is Kählerian with a Kähler form being a multiple of the curvature form $\omega$ of the bundle. A typical example of this situation is $M = \mathbb{C}P^m$ with its Fubini-Study metric, the form $\omega$ being then a generator of $H^2(M, \mathbb{Z})$. In that case, the manifold $\tilde{M}$ is a standard sphere $S^{2m+1}$. The construction can be carried over with other Hermitian symmetric irreducible spaces. The main drawback of these examples is that the metrics obtained turn out to be homogeneous isotropy irreducible, hence belong to a well-known family.
In the hands of McKenzie Wang and W. Ziller (cf. section IV and [14]), this construction provides an infinite family of new examples of Einstein metrics provided one does not take the base metric to be itself Einstein.

III. Non-abelian bundle theory.

a) The Kaluza-Klein approach to gauge theories

In 1954, C.N. Yang and R.L. Mills proposed in [15] a variational approach as classical model for strong interactions (the ones responsible for the cohesion of the nucleus). Later, this model was also used to describe classically weak interactions (connected to the β-decay). This proposal was known to be only a part of the story since, due to their ranges, these interactions have to be dealt with quantum mechanically.

This variational theory turns out to be identical to the theory of connections on G-bundles, as was (much) later recognized. The group G to be considered here is the group on invariance of the interactions, as of today, $U_1$ for electromagnetism, $SU_2$ for weak interactions, $SU_3$ for strong interactions. An elementary particle subject to a given interaction being attached to an irreducible representation of the invariance group, these groups can be determined experimentally by letting interact different particles subject to a given interaction, and comparing the outcome of the interaction with the tables of representations of the presupposed symmetry group.

A compact Lie group $G$ being given, we consider $p : \tilde{M} \rightarrow M$ a $G$-bundle. The field to be extremized in this setting is a $G$-connection $\alpha$ for the Yang-Mills functional $\mathcal{YM}$ defined as

\begin{equation}
\mathcal{YM}(\alpha) = \frac{1}{2} \int_M |\Omega^\alpha|^2 v_g
\end{equation}

where $\Omega^\alpha$ denotes the curvature of the connection $\alpha$. The volume element $v_g$ refers to a Riemannian metric $g$ supposedly given on the space-time manifold $M$. Due to the use of this model as a first perturbation term towards a Quantum theory, it is indeed a Riemannian metric that physicists pick.

How does the Kaluza-Klein Ansatz interfer with this problematic? This was first contemplated by B. De Witt in 1964, and later developed by A. Trautman and R. Kerner (cf. [8]). One more time, one can play the game of using the data to construct a metric on the total space of the bundle. This can be done by using the same recipe as before, namely by setting

\begin{equation}
\tilde{g} = p^* g + \bar{g}
\end{equation}
where $\tilde{g}$ is a chosen $G$-invariant metric on the model space of the fibers (necessarily a space where $G$ acts) that can be extended as a bilinear form on the whole of $T\tilde{M}$ thanks to the connection. Indeed, for a tangent vector $Z$ to $\tilde{M}$, we decompose it first into its horizontal and vertical components thanks to the connection, say $Z = Z_v + Z_h$, and set

$$\tilde{g}(Z, Z) = g(Tp(Z),Tp(Z)) + \tilde{g}(Z_v,Z_v).$$

The first remarkable fact is then that the $G$-bundles can be characterized in terms of metrics like $\tilde{g}$ that we just defined. We come to this a little later. Another fact of concern to us is of course the expression of the Lagrangian $\tilde{S}$ for the metric $\tilde{g}$. Anticipating a bit curvature calculations that we develop later, we have

$$\tilde{S}(\tilde{g}) = \int_{\tilde{M}} \tilde{s} \, v_{\tilde{g}} = \int_{\tilde{M}} (s \circ p + \tilde{s} - |\Omega^0|^2) \, v_{\tilde{g}}$$

where $\tilde{s}, s$ and $\tilde{s}$ denote respectively the scalar curvatures of the metrics $\tilde{g}, g$ and $\tilde{g}$. In other words

**FACT 4.** - The Hilbert-Einstein Lagrangian of General Relativity evaluated on the extended space-time according to the Kaluza-Klein prescription is the sum of the Hilbert-Einstein Lagrangian of the base and fibers and of the Yang-Mills functional of the connection.

In the Physics literature, this fact is often referred to as minimal coupling.

**b) The Riemannian geometry of fibre bundles**

A systematic study of the Riemannian geometry of fibre bundles was considered only recently (see [13]) if one compares to the time of existence of Riemannian geometry itself.

In this setting, the basic notion is that of a Riemannian submersion, i.e., a map $p : (\tilde{M}, \tilde{g}) \to (M, g)$ so that, at each point $\tilde{m}$ of $\tilde{M}$, the restriction of $T_{\tilde{m}}p$ to the orthogonal space to its kernel at $\tilde{m}$ is an isometry from the metric induced by $\tilde{g}$ to $(T_{p(\tilde{m})}M, g_{p(\tilde{m})})$. To stick to the discussion we started from the Physics side, we will assume that $p : (\tilde{M}, \tilde{g}) \to (M, g)$ is a fibre bundle whose fibre is denoted by $F$. Note however that completeness (a fortiori compactness) and connectedness of $(\tilde{M}, \tilde{g})$ ensure that property. (This is a theorem of R. Hermann). At this point, it is important to stress that it would be inconsistent to assume that $F$ is endowed with a Riemannian metric. We come back to this point later.
The key observation due to B. O'Neill is that two 3-tensor fields are enough to describe the geometry of the situation. They measure the difference of \( \bar{D} \), the Levi-Civita connection of \( \bar{g} \), and its projection onto the vertical and horizontal subbundles respectively. If \( h \) and \( v \) denote the orthogonal projections onto the horizontal and vertical spaces respectively, for tangent vector \( Z_1, Z_2 \) to \( \tilde{M} \), we have

\[
\begin{align*}
A_{Z_1}Z_2 &= h(\bar{D}_{h(Z_1)}v(Z_2)) + v(\bar{D}_{h(Z_1)}h(Z_2)) \\
T_{Z_1}Z_2 &= h(\bar{D}_{v(Z_1)}v(Z_2)) + v(\bar{D}_{v(Z_1)}h(Z_2)).
\end{align*}
\]

The 3-tensor \( A \) measures the obstruction to integrability of this horizontal distribution. It can be identified with the curvature of the connection. This can be done as follows. The 3-tensor \( A \) is defined over \( \tilde{M} \), but one easily sees that \( A_X Y = v([X,Y]) \), so that \( A \) gives rise to a 2-form on \( M \) taking its values in vector fields along the fibre. It can therefore be identified with \( \Omega^\alpha \). The horizontal part of \( A \), namely \( A_X V \) for a vertical vector \( V \), can be recovered from the previous one since \( \bar{g}(A_X \bar{Y}, V) = -\bar{g}(A_X V, \bar{Y}) \).

The 3-tensor \( T \) measures the deviation of the fibres from being totally geodesic. The importance of \( T \) is best captured by the following theorem.

**THEOREM 5 (R. HERMANN, cf. [6]).** - If the fibres of a Riemannian submersion \( p : (\tilde{M}, \tilde{g}) \to (M, g) \) are totally geodesic, then all fibres are isometric, and the bundle is a \( G \)-bundle where \( G \) is a group of isometries of the fibre.

The key observation which makes the proof work is that, given a curve \( \gamma \) joining two points \( m \) and \( m' \) in the base \( M \), the map from the fibre \( F_m \) to the fibre \( F_{m'} \), obtained by lifting horizontally the curve \( \gamma \) is an isometry. This in turn follows from the fact that \( L_X(\bar{g}|_F) = 0 \) for the horizontal lift \( \bar{X} \) of a vector field \( X \) on \( M \). This forces all fibres to be isometric, and in this case one can speak of the fibre \( F \) as a Riemannian manifold as we did in paragraph a). We denote this metric by \( \bar{g} \), and quantities related to it with a bar above it.

Then, the 3-tensor \( A \), when identified with \( \Omega^\alpha \), takes its values in the infinitesimal isometries of \( (F, \bar{g}) \).

This shows that the situation of interest to physicists can be characterized mathematically by the geometric property that the Riemannian submersion has totally geodesic fibres, generalizing what we had seen in Proposition 1 for \( S^1 \)-bundles.
REMARK. – Before going on working under this assumption, let us remark
that without it the Hilbert-Einstein Lagrangian \( \tilde{S} \) takes the value

\[
\tilde{S}(\tilde{g}) = \int_M (s \circ p + \tilde{s} - |A|^2 - |T|^2 - |N|^2) v_\tilde{g}
\]

where \( N \) is the mean curvature vector of the fibre. We indeed have the reduc-
tion to (14) when the fibres are totally geodesic. Notice also that, when one vari-
est \( \tilde{S} \) among the metrics on \( \tilde{M} \) of the type we considered (namely,
fibered metrics with totally geodesic fibres), one does not get the Einstein
equation as Euler-Lagrange equation. This accounts for the so-called Kaluza-
Klein inconsistency. The Einstein equations on \( \tilde{M} \) are indeed stronger than
these equations, hence the fact that sometimes physicists drop some of the
conditions that the Einstein equations imply. One then speaks of a truncated
equation.

We can now come to the expression of the Ricci curvature for \((\tilde{M}, \tilde{g})\) in
terms of the geometric data. For horizontal vectors \( U, V \), and horizontal lifts
\( \tilde{X}, \tilde{Y} \) of tangent vectors \( X, Y \) to \( M \), at a point \( m \) we have

\[
\begin{align*}
\bar{\pi}(U, V) &= \tilde{\pi}(U, V) + \sum_{i,j=1}^{n} \tilde{g}(\Omega^o_{e_i,e_j}, U) \tilde{g}(\Omega^o_{e_i,e_j}, V) \\
\bar{\pi}(\tilde{X}, U) &= \tilde{\pi}((\delta \Omega^o)_{X}, U) \\
\bar{\pi}(\tilde{X}, \tilde{Y}) &= \pi(X, Y) - 2 \sum_{i=1}^{n} \tilde{g}(\Omega^o_{X,e_i}, \Omega^o_{Y,e_i}),
\end{align*}
\]

where \( (e_i) \) is an orthonormal basis of the \( n \)-dimensional vector space \( T_mM \).

From these formulas, we draw the following conclusions of interest to us.

THEOREM 6. – The Riemannian metric induced on the total space \( \tilde{M} \) of a
connected \( G \)-bundle \( p : \tilde{M} \rightarrow M \) defined by a \( G \)-invariant Riemannian metric
\( \tilde{g} \) on the fibre \( F \), a Riemannian metric \( g \) on the base and a \( G \)-connection \( \alpha \) is
Einstein with constant \( \lambda \) if and only if:

i) the connection \( \alpha \) is Yang-Mills and its curvature form \( \Omega^\alpha \) has constant
norm;

ii) the curvature form \( \Omega^\alpha \) is connected to the Ricci curvature of the fibre
by the relation

\[
\sum_{i,j=1}^{n} \tilde{g}(\Omega^o_{e_i,e_j}) \otimes \tilde{g}(\Omega^o_{e_i,e_j}) + \bar{\pi} = \lambda \tilde{g},
\]
and the fibre \((F, g)\) has constant scalar curvature;

iii) the curvature form \(\Omega^\alpha\) is connected to the Ricci curvature of the base by the relation

\[
 r - 2 \sum_{i=1}^{n} \bar{g}(\Omega^\alpha_{e_i}, \Omega^\alpha_{e_i}) = \bar{\lambda} g
\]

and the base \((M, g)\) has constant scalar curvature.

**Proof.** The statements contained in i), ii) and iii) are obvious consequences of the system (17) except for the constancy of \(|\Omega^\alpha|^2, \bar{s}\) and \(s\).

To establish these facts one needs only the following clever remark. Taking the trace of (19) gives the constancy of \(|\Omega^\alpha|^2\) along a fibre, and that since \(\bar{s} = \bar{s} + s \circ \pi - |\Omega^\alpha|^2\), by moving along a fibre again, \(\bar{s}\) must be a constant. By taking a trace of the first formula in the system (17), one obtains that \((n + p) \bar{\lambda} = \bar{s} + |\Omega^\alpha|^2\) (if \(p\) denotes the dimension of \(F\)), hence that \(|\Omega^\alpha|^2\) is also a constant on \(M\). The constancy of \(s\) then follows immediately.

**Remark.** Of course, as in the case of \(S^1\)-bundles, this does not imply that either the fibre, or the base are Einstein. On some occasions this implication has been unduly claimed in the Physics literature.

Conditions (18) and (19) seem a priori rather difficult to satisfy. In section IV we give a few examples of situations where this Kaluza-Klein approach does give new examples of Einstein metrics. One more time, the most convenient case to study is when both the base and the fibres are Einstein.

**Corollary 7.** Any Einstein metric arising on the total space of a bundle via a Kaluza-Klein type construction has a positive constant \(\lambda\) unless the bundle is flat, and its total space is covered by the product of a covering of the base and of the fibre both endowed with Einstein metrics having the same negative constant.

**Proof.** As for the case of \(S^1\)-bundles the proof relies again on the formula giving the Ricci curvature in purely vertical directions, i.e., here on formula (18). If \(\bar{\lambda}\) is negative, then necessarily \(\bar{r}\) is negative since the curvature contribution is positive. But, by a well-known theorem of S. Bochner, this implies that \((F, g)\) has no infinitesimal isometries. Since we know that the curvature \(\Omega^\alpha\) takes its values in the vector space of infinitesimal isometries, this forces \(\Omega^\alpha\) to vanish, i.e., the connection \(\alpha\) is flat, and \((F, g)\) is Einstein. As a consequence, the manifold \(\tilde{M}\) is covered by a product of the fibre and a covering of
the base. By using (19), one sees that \((M, g)\) is also Einstein. In the absence of the curvature term, both \(g\) and \(\bar{g}\) must have the same negative constant.

IV. Some new examples of Einstein metrics obtained by bundle constructions.

a) The canonical variation of a Riemannian submersion.

As explained in the last Section, solving system (17) is not an easy task. We single out one case where the construction does provide new examples of Einstein metrics.

Given a Riemannian submersion \(p : (\tilde{M}, \tilde{g}) \to (M, g)\), it is possible to make a simple-minded change in the metric \(\tilde{g}\) which turns out to be geometrically meaningful, namely to multiply uniformly the metric \(\tilde{g}\) in the fibres by a factor \(t^2 (t > 0)\). (If we refer to Kaluza’s original Ansatz, this means taking \(\phi\) to be a constant different from 1). We denote the resulting metric on \(\tilde{M}\) by \(\tilde{g}_t\). For any \(t > 0\), the map \(p : (\tilde{M}, \tilde{g}_t) \to (M, g)\) remains a Riemannian submersion. (Indeed, nothing has been changed about orthogonality between horizontal and vertical directions, and the induced metric on the horizontal spaces has been kept the same.) Moreover, if the fibers of \(p\) are totally geodesic for \(\tilde{g}\), they remain so for \(\tilde{g}_t\). This construction is often referred to as the canonical variation of the Riemannian submersion.

We can then ask the following (a priori peculiar) question. Is it possible that more than one metric in the family \(\tilde{g}_t\) be Einstein? The answer of course is contained in comparing the expression of the system (17) for the various Einstein metrics \(\tilde{g}_t\). It is reasonable to assume that \(\tilde{g} = \tilde{g}_t\) is Einstein with constant \(\lambda\). Since \(r_{\tilde{g}_t} = r = \tilde{r}\) the metric \(\tilde{g}_t\) will then be Einstein with constant \(\lambda_t\) provided

\[
\tilde{r} = \frac{t^2}{1 - t^4}(\lambda_t - t^2 \lambda)g
\]

(which follows from (18)) and

\[
r = \frac{1}{1 - t^2}(\lambda_t - t^2 \lambda)g
\]

(which follows from (19)).

We therefore have
THEOREM 8 (L. BÉRARD BERGERY) – Let \( p : (\tilde{M}, \tilde{g}) \to (M, g) \) be a Riemannian submersion with totally geodesic fibres and base and fibre metrics \( g \) and \( \tilde{g} \) Einstein for positive constants \( \lambda \) and \( \tilde{\lambda} \). The metric \( \tilde{g} \) is Einstein with constant \( \tilde{\lambda} \) if and only if

i) the connection \( \alpha \) is a Yang-Mills and its curvature \( \Omega^\alpha \) has constant length \( \epsilon \) and satisfies

\[
\sum_{i,j=1}^{n} \tilde{g}(\Omega_{e_i,e_j}^\alpha) \otimes \tilde{g}(\Omega_{e_i,e_j}^\alpha) = \frac{\epsilon}{k} \tilde{g} \\
\sum_{i=1}^{n} \tilde{g}(\Omega_{e_i}^\alpha, \Omega_{e_i}^\alpha) = \frac{\epsilon}{n} g
\]

where \( k \) denotes the dimension of \( F \);

ii) the Einstein constants are related by \( \lambda = \tilde{\lambda} + \epsilon(\frac{2}{n} + \frac{1}{k}) \).

Moreover, another metric in the canonical variation of \( p : (\tilde{M}, \tilde{g}) \to (M, g) \) is Einstein if \( \lambda \neq \frac{1}{2} \tilde{\lambda} \) and it is unique.

Conversely, any Riemannian submersion with totally geodesic fibres having two Einstein metrics in its canonical variation belongs to the family described above.

There are many examples of the above situation the full discussion of which can be found in [2]. We first mention a large class, and then very specific examples.

If \( G \) is compact Lie group and \( H, K \) two compact subgroups of \( G \) with \( K \subset H \), then, as is well known, the natural projection \( \pi : G/K \to G/H \) is a fibration. If we choose an \( \text{Ad}_G(H) \)-invariant complement to the Lie algebra of \( H \) in the Lie algebra of \( G \), and on it an \( \text{Ad}_G(H) \)-invariant scalar product, we obtain a \( G \)-invariant Riemannian metric \( g \) on \( G/H \). By proceeding similarly for the pair \( (H, K) \), we obtain also an \( H \)-invariant Riemannian metric \( \tilde{g} \) on \( H/K \).

By taking the direct sum metric on the sum of the complements of the Lie subalgebras, we obtain a Riemannian metric \( \hat{g} \) on \( G/K \) which, by a theorem of L. Bérard-Bergery, is a Riemannian submersion with totally geodesic fibres.

Many important geometric examples fall under this category, such as the usual Hopf fibrations of spheres over projective spaces, and also the generalized Hopf fibrations of projective spaces over other projective spaces.

Then, Theorem 8 applied to this family gives “exotic” examples of Einstein spaces on simple manifolds. In particular, from the classical Hopf fibrations \( S^{4q+3} \to \mathbb{H}P^q \), one gets two Einstein metrics on \( (4q + 3) \)-spheres; from
the Hopf fibration \( S^{15} \to S^8 \), a third one on \( S^{15} \); from the generalized Hopf fibration \( \mathbb{C}P^{2q+1} \to \mathbb{H}P^q \), two Einstein metrics on odd-dimensional complex projective spaces.

This construction turns out to exhaust all homogeneous Einstein metrics on spheres and projective spaces, as was shown by W. Ziller (cf. [16]).

\[ b) \text{Other Riemannian submersions admitting Einstein metrics.} \]

Although the topic we discuss in this paragraph takes us away from ordinary Kaluza-Klein theory, it is worth being mentioned because the examples it gives are geometrically interesting.

Most constructions in this paragraph are due to L. Bérard Bergery (cf. [1]) after initial impetus given by a physicist, D. Page, who gave the simplest member of the family we are going to describe as a Riemannian counterpart to the Taub-NUT metric, one of the standard models in modern General Relativity.

We start from an \( S^1 \)-bundle \( P \) classified by an indivisible 2-cohomology class \([\alpha]\) dividing the first Chern class of a compact Kähler-Einstein manifold \( M \) (e.g., a Hermitian symmetric space). We set \( c_1(M) = q\alpha \).

We denote the \( k \)-fold cover of \( P \) by \( P_k \). As one easily sees, \( P_k \) is classified by \( k\alpha \). We then introduce \( \tilde{M}_k \) the \( S^2 \)-bundle over \( M \) associated with \( P_k \) for the usual \( S^2 = \mathbb{C}\cup\{\infty\} \). (An alternative geometric description of \( \tilde{M}_k \), more in the spirit of complex geometry, goes as follows. Take the complex line bundle over \( M \) whose first Chern class is \( k\alpha \); add to it a trivial bundle, and \( \tilde{M}_k \) is then the projective bundle of this rank 2 vector bundle).

**THEOREM 9 (L. BÉRARD BERGGERY).** - For \( k \) satisfying \( 1 \leq k < q \), the manifolds \( \tilde{M}_k \) admit a non homogeneous Einstein metric with positive scalar curvature.

For the proof we just refer to [2] page 273. Let us just say it involves putting a fibered metric on \( \tilde{M}_k \) as a bundle over a closed interval with fibres \( P_k \) degenerating at the end points of the interval with fibres \( M \). The metric is obtained as solution of a non linear system of differential equations involving two unknown functions scaling the base and the fibre metrics of the \( S^1 \)-bundle \( P_k \). Notice that this system is heavily overdetermined since, in order to obtain a smooth metric on \( M_k \), one has to fix the values of the two functions and at least of their first two derivatives at the boundary.

The metric obtained still retains an isometric group action in codimension 1. (One often speaks of metrics of cohomogeneity 1.) This method has
been sophisticated by N. Koiso and Y. Sakane (cf. [11]) to provide Einstein metrics of higher cohomogeneity, in particular on Kähler manifolds.

To close this paragraph, we mention that the basic example $\tilde{M}_1$ obtained for $M = \mathbb{C}P^1$ (one then necessarily has $k = 1$) can be identified with the non-trivial $S^2$-bundle over $S^2$ (this is easy via the second definition we suggested of the manifolds $\tilde{M}_k$), or as the manifold obtained from $\mathbb{C}P^2$ by blowing up one point. It therefore admits Kähler metrics although the metric we constructed is not Kähler. This is an example of a Kähler manifold admitting no Einstein-Kähler metric.

c) An infinite family of interesting $S^1$-bundles with Einstein metrics.

In this paragraph we take up the subject of Section II by describing an interesting family of Einstein metrics on $S^1$-bundles obtained by W. Ziller and M. Wang in 1987. Some of these examples were discovered at about the same time by the physicists L. Castellani, R. D'Auria and P. Fre, as parts of models of supergravity.

One of the latest avatars of Kaluza-Klein theory in Physics has indeed been supergravity. This is an attempt to incorporate the ideas of supersymmetries into gravitation theory. Let us recall that in recent years physicists have been wandering about the possible existence of transformations associating bosons, represented by tensor fields, to fermions, represented by spinor fields that they call supersymmetric. These transformations would allow to organize fundamental particles into a small number of families transformed into one another by these supersymmetric transformations. The proper geometrical framework appropriate to deal with these objects is not yet firmly established.

Supergravity took ground in 1978 when a viable Lagrangian for it was proposed by E. Cremmer, B. Julia and J. Scherk (cf. [4]). It mixes metric data, with spinor data. For that purpose, there was good physical reasons to look for bundles whose total spaces are 11-dimensional, i.e., with 7-dimensional fibres. This dimension is indeed the smallest possible one on which acts transitively the group $U_1 \times SU_2 \times SU_3$ incorporating the symmetries of all known to day interactions. On the other hand, it is a theorem of W. Nahm (cf. [12]) that higher dimensional models would lead to ill-defined theories from a quantic point of view.

Physicists therefore put a great emphasis on constructing metrics (in fact mixed with more complicated fields such as a differential 3-forms) on 11-dimensional manifolds fibered over a 4-dimensional manifold with a 7-
dimensional fibre. Following the recipe we have been using ourselves, they looked for Einstein metrics on the fibres. For that purpose they studied extensively 7-manifolds acted upon by $U_1 \times SU_2 \times SU_3$, hence considered $S^1$-bundles over $\mathbb{C}P^1 \times \mathbb{C}P^2$, a non irreducible Hermitian symmetric space.

**THEOREM (M. WANG, W. ZILLER, [14]).** - Let $(M_i, g_i)$ be Kähler Einstein manifolds with positive Chern classes $c_i(M_i) = q_i \alpha_i$ where the $\alpha_i$ are indivisible cohomology classes. Any non trivial bundle $\tilde{M}$ over a product of Kähler-Einstein manifolds $M = \prod_{i=1}^{k} M_i$, whose characteristic class in $H^2(M, \mathbb{Z})$ is an integral linear combination of the $\alpha_i$'s carries an Einstein metric with positive constant as soon as $\pi_1(M)$ is finite.

The most interesting feature of the family we just described is the richness of examples of different Einstein metrics it provides on manifolds having a given topology. For that purpose, one needs to study carefully how the topology is affected by the integral linear combination used to construct $\tilde{M}$. To be specific, let us consider a few cases over $M = \mathbb{C}P^1 \times \mathbb{C}P^q$. The $S^1$-bundle $\tilde{M}$ is then specified by two integers $\ell_1$ and $\ell_2$ expressing the characteristic class of the bundle as $\ell_1 \alpha_1 + \ell_2 \alpha_2$ where $\alpha_1$ and $\alpha_2$ are respectively the generating classes in $\mathbb{C}P^1$ and $\mathbb{C}P^q$. One has:

i) for $q = 1, \tilde{M}$ is always diffeomorphic to $S^2 \times S^3$;

ii) for odd $q \geq 3, \tilde{M}$ is diffeomorphic to $S^2 \times S^{2q+1}$ when $\ell_2 = \pm 1$ and $\ell_1$ arbitrary, diffeomorphic to $S^2 \times S^{2q+1}$ if $\ell_1$ is even and $\ell_2 = \pm 1$, and to the non-trivial $S^{2q+1}$-bundle over $S^2$ if $\ell_1$ is odd and $\ell_2 = \pm 1$.

**COROLLARY.** - On $S^2 \times S^{2q+1}$ and on the non-trivial $S^{4q+1}$ bundle over $S^2$, there exist infinitely non isometric Einstein metrics with a positive constant.

In fact, this construction gives infinitely many homotopy types of manifolds admitting an Einstein metric in all but a finite number of dimensions $\geq 7$. The richness of this family goes as far as allowing some of them in dimension 7 to be homeomorphic without being diffeomorphic. More precisely, as shown by M. Kreck and S. Stolz, two $S^1$-bundles $\tilde{M}$ and $\tilde{M}'$ over $\mathbb{C}P^1 \times \mathbb{C}P^2$ corresponding to the same class over $\mathbb{C}P^2$ (for example $4\alpha_2$) are homeomorphic if and only if $\ell_1 \equiv \ell'_1$ mod (32), and diffeomorphic if and only if $\ell_1 \equiv \ell'_1$ mod (32.28). In fact these manifolds have 28 different differentiable structures (all obtained by taking connected sums with the 28 distinct 7-dimensional exotic spheres), and they turn out to be homogeneous spaces under the same group, solving in the negative a long standing question in group action theory.
To come back to our Riemannian geometric motivations, let us mention that if we normalize the Einstein metrics that we described, say on $S^2 \times S^{2r+1}$, to have volume 1, then the Einstein constants of metrics of the previous family go to zero, together with the lengths of the geodesic fibres. This shows that for the moduli space of Einstein metrics on a given differentiable manifold one should not expect compactness theorems to hold without making some geometric assumptions preventing the metric from collapsing. This also shows that the functional $S$ we have been considering does not satisfy condition (C), emphasizing the difficulty of finding Einstein metrics by purely analytical means.

V. Comparing some mathematical and physical natural assumptions.

To close this survey article, in which we stressed the numerous occasions on which mathematicians and physicists met by considering similar objects, we would like to show side by side some of the points on which they have as yet put different emphasis.

Some of these discrepancies may well be the basis of some future work aiming at bringing them closer.

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We hope to have by now convinced the reader that there is a lesson to be learned by mathematicians in the development of Modern Physics, and that, at any given moment, we should pay more attention to the questions certain physicists raise. More important certainly, we should make sure that we provide our students with enough exposure to the basic concepts and ideas coming from Physics that, when time comes, they will prove able to establish the necessary contacts and understand what the physicist, most of the time working next door, comes and asks.

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