REMARKS ON \( k \)-SPANNEDNESS FOR ALGEBRAIC SURFACES

Introduction

Let \( L \) be a line bundle on a smooth connected projective surface \( X \). The concept of \( k \)-spannedness for a line bundle \( L \), first defined in [BFS], is a natural notion of higher order embedding for the map associated to \( L \). So, for instance, 0-spanned is equivalent to \( L \) being spanned by global sections, and 1-spanned is equivalent to being very ample.

In the first four sections of this paper, following methods used in [B1] and [B2] to investigate 0 and 1 spannedness, we study pairs \((X, L)\) where \( X \) is either a rational or a ruled surface, and \( L \) is a \( k \)-spanned line bundle on it. In the last section, we examine the same problem for the case of elliptic fibrations.

We work over the field of the complex numbers. Throughout the paper, \( X \) always denotes a smooth connected projective surface, and \( K_X \) its canonical bundle. As usual, we do not distinguish between a divisor and its associated line bundle. A divisor \( D \) is said to be \( Q \)-effective if, for some positive integer \( n \), \( nD \) is effective.

Classificazione per soggetto AMOS(MOS, 1980):14J29
§1.

Let $L$ be a line bundle on $X$ and $k$ be a non-negative integer, then $L$ is said to be $k$-spanned if for any effective 0-cycle $Z = \sum_{j=1}^{r} k_j z_j$ on $X$, with $\deg(Z) = k + 1$, the map on global sections

$$\Gamma(X, L) \longrightarrow \Gamma(Z, L \otimes O_z)$$

is onto (see [BFS] (0.4)).

The following result is a direct consequence of [BFS] Theorem (3.1):

**THEOREM 1.1** Let $L$ be a line bundle on a smooth surface $X$, assume that:

(a) $L$ is $\mathbb{Q}$-effective;

(b) $L^2 \geq 4k + 5$;

(c) $(L - E) \cdot E \geq k + 2$ for every effective divisor $E$ such that $L - 2E$ is $\mathbb{Q}$-effective then, $L + K_X$ is $k$-spanned.

1.2. Now, let $M = L - K_X$, and set $D_M = \{\text{effective divisors } E : M - 2E \text{ is } \mathbb{Q}\text{-effective}\}$. Let $E = E_1 + \ldots + E_p$ be an effective divisor, with $E_j$ $j = 1, \ldots, p$ irreducible and reduced. We say that $E \in E_k$ if either $p = 1$ or, if $p > 1$, then the following conditions are satisfied:

i) $\sum_{j=1}^{p} (E - E_j) \cdot E_j \geq (p - 1)(k + 2) + 1$;

ii) if $F_1 + F_2 = E$ is a non-trivial decomposition of $E$, then $F_1 \cdot F_2 \geq (k + 3)/2$.

Let $X, L, M, \text{ and } k$ be as above, one has the following lemmas which are straightforward generalizations of lemmas (1.2.3) and (1.3.1) of [B1].

**LEMMA 1.3** Suppose that for every $E \in E_k \cap D_M$ we have $(M - E) \cdot E \geq k + 2$, then the same is true for all $E \in D_M$.

**LEMMA 1.4** If $E = E_1 + \ldots + E_p \in E_k$, and $g(E) = \text{arithmetic genus of } E$, then $g(E) \geq 0$, and if $p \geq 2, g(E) \geq (1/2)(p - 1)(p + 1)$.

**REMARKS 1.5** If $E \in E_k$, then:

(a) $(M - E) \cdot E = L \cdot E - 2g(E) + 2$, therefore the inequality in (1.3) is equivalent to $L \cdot E \geq 2g(E) + k$;

(b) If $g(E) \geq k/2$ then $E$ is reduced and irreducible. So, if $g(E) = 0$, then $E \in E_k$ if and only if $E$ is smooth, and $L$ $k$-spanned implies $L \cdot E \geq k$.

(c) Since $M^2 = 4E \cdot (M - E) + (M - 2E)^2$, we have $E \cdot (M - E) \geq 2 + k$ if
and only if $M^2 \geq 5 + 4k + (M - 2E)^2$. Assuming that

$$
(1.5.1) \quad \begin{cases} 
M^2 \geq 5 + 4k \\
(M - E) \cdot E \leq 1 + k
\end{cases}
$$

then

$$
(1.5.2) \quad (M - 2E)^2 \geq 1.
$$

**Lemma 1.6** With the above notation, suppose $E \in E_k \cap D_M$, $E^2 \geq 0$, $(M - 2E) \cdot E \geq 0$, and (1.5.1) verified, then

$$
(1.6.1) \quad M \cdot E - (1 + k) \leq E^2 < M \cdot E/2 < (1 + k) + (1/4)[1 - (4k + 5)^{1/2}].
$$

**Proof.** By the Hodge index theorem and the assumptions, we get that

$$
E^2 \leq E^2(M - 2E)^2 \leq (E \cdot (M - 2E))^2 \leq (1 + k - E^2)^2 < (1 + k + 1) + (1/2)[1 - (4k + 5)^{1/2}].
$$

Moreover, we have $M \cdot E > 2E^2$. For, if $M \cdot E = 2E^2$ then, by the Hodge index theorem, we would get $M - 2E \equiv \lambda E$ ("\(\equiv\" = numerical equivalence), $\lambda \in Q$, and by (1.5.2), we would get $E^2 = 0$. Again, by the Hodge index theorem, $M \equiv \mu E$, $\mu \in Q$, and so $M^2 = 0$ contradicting (1.5.1).

**Remarks 1.7** (a) Suppose that a smooth surface $X$ satisfies the following property:

$$
(1.7.1) \quad \text{If } A, B \text{ are effective divisors such that } A^2 \geq 0, \ B^2 \geq 0 \text{ then } A \cdot B \geq 0; \text{ then the same property is true for a surface obtained from } X \text{ by blowing-up one point.}
$$

(b) If $X = P^2$ or $X = \text{geometrically ruled surface}$, then (1.7.1) is true. It follows from (a) that the same is true for every rational or ruled surface.

(c) The inequality $(M - 2E) \cdot E \geq 0$ stated in lemma (1.6) is a consequence of (1.5.1) and the assumption $E^2 \geq 0$.

§2.

We want to discuss $k$-spannedness for line bundles on rational surfaces. Let $x_1, \ldots, x_s$ be distinct points in $P^2$, and let $\pi : X \rightarrow P^2$ be the blowing up of $P^2$ at $x_1, \ldots, x_s$. On $X$ every divisor is linearly equivalent to a divisor of the form $mr - \sum_{j=1}^{s} n_j P_j$, where $r =$ class of $\pi^*(O_{P^2}(1))$ and $P_i =$ class of $\pi^{-1}(x_j) \ j = 1, \ldots, s; m, n_1, \ldots, n_s \in Z$. In particular, if $L \equiv dr - \sum_{j=1}^{s} t_j P_j$ and so $M = L - K_X \equiv (d + 3)r - \sum (t_j + 1)P_j$, by (1.5(b)) we can assume $t_1 \geq t_2 \geq \ldots \geq t_s \geq k, d \geq k.$
REMARKS 2.1 (cfr. [B1] §2) (a) It follows from the theorem of Riemann-Roch and the fact that \( d \geq k \), that if \( M^2 > 0 \) then \( M \) is \( Q \)-effective.

(b) If \( E = y - \sum_{j=1}^{s} a_j P_j \in D_M \) then one has, \( 0 \leq y \leq (d + 3)/2 \) and \( y - 1 - (d - t_j)/2 \leq a_j \leq y \). In particular, one gets:

(i) If \( M^2 \geq 5 + 4k \) and \( y = (d + 3)/2 \), then \( (M - E) \cdot E \geq k + 2 \);

(ii) If \( E \in E_k \) and \( y = 0 \), then \( E - P_j \) for some \( j = 1, \ldots, s \); and \( (M - P_j) \cdot P_j = t_j + 2 \geq k + 2 \);

(iii) If \( E \in E_k \) and \( y \geq 1 \), then \( a_j \geq 0, j = 1, \ldots, s \), and if \( y \geq 2 \), then for all \( j \), \( a_j \leq y - 1 \);

(iv) If \( E \in E_k \), set \( E' = y - \sum b_j P_j \), where \( b_j = \min\{((t_j + 1)/2, a_j\}. \) Then, \( E' \in E_k \), and \( (M - E') \cdot E' \leq (M - E) \cdot E \). So it is enough to consider divisors \( E \) with \( a_j \leq (t_j + 1)/2 \).

Now, denote by \( T_k \) the set of all divisors \( E = y - \sum_{j=1}^{s} a_j P_j, E \in E_k \cap D_M \) such that \( 1 \leq y \leq (d + 2)/2 \) and \( \max\{0, y - 1 - (d - t_j)/2 \leq a_j \leq \) either 1, if \( y = 1 \), or \( \min\{y - 1, (t_j + 1)/2\} \) if \( y \geq 2 \), then by (1.1), (1.6), (1.7(c)), and (2.1) one has the following:

THEOREM 2.2 Let \( X, L, \) and \( M \) be as in (2.0), \( k \geq 0 \). Assume that:

1) \( M^2 \geq 5 + 4k \);

2) \( (M - E) \cdot E \geq k + 2 \) for any \( E \in T_k \) such that \( E^2 < 0 \).

Then, \( L \) is \( k \)-spanned unless there exists \( E \in T_k \) such that \( E^2 \geq 0 \) and (1.6.1) holds for \( E \).

Also, using the same argument as in the proof [B1] Theorem (2.4), one can show:

THEOREM 2.3 If \( X, L, \) and \( M \) are as in (2.0), and if \( d \geq k + \sum_{j=1}^{s} t_j, k \geq 0, \) then \( L \) is \( k \)-spanned.

REMARK 2.4 In the case when the points \( x_1, \ldots, x_s \) lie on a line, the inequality stated in the above theorem is in fact the best possible.

§3.

3.0. We will use the definition of general position for 0-cycles in \( P^2 \) given in [B1]. Denoting by \( m_j(F) \) the multiplicity of \( F \in H^0(P^2, \mathcal{O}_{P^2}(y)) \) at the point
$x_j \in P^2$, if $X$ and $L$ are as in the previous sections, we say that the points $x_1, \ldots, x_s$ are in general position w.r.t. $L$ if for any $E \in |O_{P^2}(y)|$ such that:

(i) $E$ is irreducible and reduced;
(ii) $1 \leq y \leq (d + 2)/2$;
(iii) $m_j(E) \leq (t_j + 1)/2, j = 1, \ldots, s$; then:

$$(\frac{1}{2}) \sum_{j=1}^{s} m_j(E)(m_j(E) + 1) \leq h^0(E) - 1 = y(y + 3)/2.$$ 

If $E \equiv y - \sum_{j=1}^{s} a_j P_j \in E_k$ with $y \leq (d+2)/2$ and, for all $j$, $a_j \leq (t_j+1)/2$, as in [B1] Lemma (3.1), one gets:

$$(\frac{1}{2}) \sum_{j=1}^{s} a_j(a_j + 1) \leq y(y + 3)/2 \text{ or equivalently } E \cdot (E - K_x) \geq 0.$$ 

**Proposition 3.1** Let $X, L$, and $M$ be as in the previous sections. Assume that $M^2 \geq 5 + 4k$ and $x_1, \ldots, x_s$ are point in general position w.r.t. $L$. If $E \equiv y - \sum_{j=1}^{s} a_j P_j \in E_k \cap D_M$ is such that $g(E) \geq 1$, and if $E \cdot (M - E) \leq 1 + k$ then $g(E) \leq 1 + k$ and (1.6.1) holds.

**Theorem 3.2** Let $X, L$, and $M$ be as above. Assume that:

(i) $M^2 \geq 5 + 4k$;
(ii) $x_1, \ldots, x_s$ are in general position w.r.t. $L$;
(iii) For any $E \in E_k \cap D_M$ such that $g(E) \leq k + 1$, one has $L \cdot E \geq 2g(E) + k$; then, $L$ is $k$-spanned.

**Theorem 3.3** Let $X, L$, and $M$ be as above. Suppose (i) and (ii) of (3.2) are satisfied. If

$$(3.3.1) \quad d > 1/4[9 + (9 + 8k)^{1/2}]t_1 + (2 + 6k)/[3 + (9 + 8k)^{1/2}]$$

then, $L$ is $k$-spanned.

*Proof.* This is a direct consequence of Theorem (3.2). In fact, if $E \equiv y - \sum_{j=1}^{s} a_j P_j$ and $g(E) \leq k + 1$, then $L \cdot E < 2g(E) + k$ would contradict (3.3.1).
§4.

(4.0.) Let us examine now the case of ruled surfaces ([II] V §2). Let $p_1, \ldots, p_s$ be distinct points on a geometrically ruled surface $S$. Let $\pi: X \to S$ be the blowing-up of $S$ at $p_1, \ldots, p_s$, and let $P_1, \ldots, P_s$ be the exceptional curves.

If $L$ is a line bundle on $X$, then $L \equiv \mathcal{O}_X(D)$ where $D \equiv aC + bf - \sum_{j=1}^{s} t_j P_j$ with $C$ and $f$ equal to respectively the proper transform of the minimal section and of a fiber. We want to discuss $k$-spannedness for $L$, so w.l.o.g., we can assume $t_1 \geq t_2 \geq \ldots \geq t_s \geq k \geq 0$, $a \geq k$. Set $M = L - K_X \equiv (a + 2) + C(b - 2q + 2 + e)f - \sum_{j=1}^{s} (t_j + 1)P_j$, where $-e = C^2$ and $q = \text{irregularity}$ of $X$.

As in remark (2.1), one checks that if $M^2 > 0$, then $M$ is $Q$-effective.

(4.1.) Let $E \equiv xC + yf - \sum_{j=1}^{s} a_j P_j \in E_k \cap D_M$, and let $M - 2E = zC + wf - \sum_{j=1}^{s} b_j P_j$. Set $\Gamma_0 = \{(x, e) \in \mathbb{Z} \times \mathbb{Z}: \text{either } x = 0, 1 \text{ or } 2 \leq x \leq (a + 2)/2 \text{ and } e \geq 0\}$ and $\Gamma_1 = \{(x, e) \in \mathbb{Z} \times \mathbb{Z}: 2 \leq x \leq (a + 2)/2 \text{ and } e < 0\}$.

Then, the following inequalities must be satisfied ([B2] (1.1.2)):

1) $0 \leq x \leq (a + 2)/2$;

2) $y \geq \begin{cases} 0 & \text{if } (x, e) \in \Gamma_0 \\ ex/2 & \text{if } (x, e) \in \Gamma_1 \end{cases}$ and

$$y \leq \begin{cases} (b + 2 + e - 2q)/2 & \text{if } (z, e) \in \Gamma_0 \\ (b + 2 + e - 2q - ze/2)/2 & \text{if } (z, e) \in \Gamma_1 \end{cases}$$

3) $a_j \leq \begin{cases} x + y & \text{if } (x, e) \in \Gamma_0 \\ x + y - ex/2 & \text{if } (x, e) \in \Gamma_1 \end{cases}$ and

$$a_j \geq \begin{cases} (t_j + 1 - z - w)/2 & \text{if } (z, e) \in \Gamma_0 \\ (t_j + 1 - z - w + ze/2)/2 & \text{if } (z, e) \in \Gamma_1 \end{cases}.$$

REMARK 4.2 With the same notation as in (4.0), if $M^2 \geq 5 + 4k$, one has the following results (cfr. [B2] lemmas (1.1.4) and (1.2)):

(4.2.1.) If either $z = 0$ or $z > 0$ and $w \leq ze/2$, then $(M - E) \cdot E \geq 2 + k$.

(4.2.2.) If $E = xC + yf - \sum_{j=1}^{s} a_j P_j \in E_k$, set $E' = xC + yf - \sum_{j=1}^{s} b_j P_j$ where $b_j = \min\{a_j, (t_j + 1)/2\}$, then, if $z \geq 1$ and $w > ze/2$, we get that:
(i) \( x = 0 \) implies \( 0 \leq y \leq 1; \)
(ii) \( x = 0 = y \) implies \( \sum_{j=1,\ldots,s} a_j = -1 \) and, for all \( j, a_j \leq 0; \)
(iii) \( x \geq 1 \) implies that, for all \( j = 1,\ldots,s, 0 \leq a_j \leq x; \)
(iv) \( E' \in E_k \) and \( (M \cdot E') \cdot E' \leq (M - E) \cdot E; \) so it enough to consider divisors \( E \) with \( a_j \leq (t_j + 1)/2. \)

**DEFINITION 4.3** A divisor \( L = aC + bJ \) is said to satisfy property \((P_k)\) if for every effective divisor \( D = f - \sum_{j=1,\ldots,s} a_j P_j \), with \( 0 \leq a_j \leq 1, \) one has

\[
L \cdot D = a - \sum_{j=1,\ldots,s} a_j t_j \geq k.
\]

**REMARKS 4.4** (a) Let \( L = xC + yf - \sum_{j=1,\ldots,s} a_j P_j \in E_k \cap D_M. \) If \( x = 0 = y, \) then by

\[(4.2.2) \quad (M - E) \cdot E \geq 2 + k.\]

(b) If \( L \) does not satisfy property \((P_k), \) then \( L \) is not \( k \)-spanned; and if \( L \) satisfies \((P_k), \) then \( (M - D) \cdot D \geq 2 + k \) for every \( D \) as in definition \((4.3).\)

**4.5.** Let \( T_k \) be the set of all \( L = xC + yf - \sum_{j=1,\ldots,s} a_j P_j \in E_k \cap D_M \) such that:

(i) \( 1 \leq x < (a + 1)/2; \)

(ii) \( (b - 2q + 2 + e - (xe + 1)/2)/2 \geq y \geq \begin{cases} 0 & \text{if } (x,e) \in \Gamma_0 \\ xe/2 & \text{if } (x,e) \in \Gamma_1 \end{cases} \]

(iii) \( \min\{x, (t_j + 1)/2\} \geq a_j \geq \begin{cases} \max\{0, t_j + 1 - z - w)/2\} & \text{if } (z,e) \in \Gamma_0 \\ \max\{0, (t_j + 1 - z - w + (ze/2))/2\} & \text{if } (z,e) \in \Gamma_1 \end{cases} \)

**THEOREM 4.6** Let \( X, L, \) and \( M \) be as in (4.0). Assume that:

1) \( M^2 \geq 5 + 4k; \)
2) \( L \) satisfies \((P_k); \)
3) \( (M - E) \cdot E \geq 2 + k \) for every \( E \in T_k \) such that \( E^2 < 0. \) Then, \( L \) is \( k \)-spanned unless there exists \( E \in T_k \) such that \( E^2 \geq 0 \) and \((1.6.1)\) holds for \( E. \)

Also, arguing as in the proof of \([B2]\) Theorem (1.3), one can show:
**THEOREM 4.7** Let $X, L, M$ be as in (4.0), and $e \geq 0$. If $b \geq ae + 2q + k + \sum_{j=1,...,s} t_j$ and $L$ satisfies $(P_k)$, then $L$ is $k$-spanned.

**REMARK 4.7.1** In particular, if $s = 0$, i.e., if $X$ is a geometrically ruled surface; $e \geq 0$, $a \geq k$ and $b \geq ae + 2q + k$ implies that $L$ is $k$-spanned (compare with [BS] Prop. (3.3)).

(4.8.) Set

$$\delta_k = \begin{cases} 
\max\{0,1/4(s_0 - 3)\} & \text{if } a = 0 \\
\max\{ae/2,-7/6 - 1/3s_1 + 1/6s_0 - k/3\} & \text{if } a = 1 \\
\max\{ae/2,-1/8((11 + 4k + 7s_2 + 4s_1 - s_0))\} & \text{if } a = 2 \\
\max\{ae/2 - 1 - k/2 - s + s_1/2 + 11/10s_0\} & \text{if } a \geq 3
\end{cases}$$

where $s_j$ is the number of indexes $h \in \{1,\ldots,s\}$ such that $t_h = j$. Then, as in [B2] Theorem (1.4), one can show:

**THEOREM 4.9.** Let $X, L, M$ be as in (4.0), and $e < 0$. If $L$ satisfies $(P_k)$, and if $b \geq ae/2 + 2q + k + \sum_{j=1,...,s} t_j + \delta_k$; then $L$ is $k$-spanned.

**REMARK 4.10.** With the above notation, if $s = 0, e < 0, a \geq k$, and $b \geq ae/2 + 2q + k + \max\{ae/2, -1 - k/2\}$, then $L$ is $k$-spanned (cfr. [BS] Prop. (3.4)).

§5.

Let $p : X \to B$ be an elliptic fibration [BPV]. As in [BO] we assume $(X, B, p)$ irreducible; i.e., if $F_t = p^{-1}(t)$ is a fibre of $p$, then either $F_t$ is irreducible, or $F_t = nf_t$, where $n \in \mathbb{N}$, and $f_t$ is irreducible. The following is a generalization of [BO] theorem (1.1).

**THEOREM 5.1.** Let $(X, B, p)$ be an irreducible elliptic fibration, and let $f$ be the reduced component of a fibre of maximum multiplicity. If $G$ is a $Q$-effective divisor, $m$ a sufficiently large integer, and $G \cdot f \geq k + 2$, then $L = G + mf$ is a $k$-spanned line bundle.

**Proof.** Set $M = L - K_X$. Then, $M \equiv G + \psi(m)f$; for, as a $Q$-divisor, $K_X \equiv \alpha f$. Since $\psi(m)$ is an increasing function of $m$, arguing as in the proof of [BO] Theorem (1.1), one can easily show that, for $m \gg 0, M$ is an ample divisor. Also, $M^2 \geq 4k + 5$ if $m \gg 0$. So, in order to show that for $m \gg 0, L$ is a
$k$-spanned line bundle on $X$, by [BFS] Theorem (1.1), it suffices to show that
one cannot find an effective divisor $E$ on $X$ such that $M - 2E$ is $Q$-effective, and

$$(5.1.1) \quad M \cdot E - k - 1 \leq E^2 < M \cdot E/2 < k + 1.$$ 

Now, let $E \equiv D + \lambda f$, where $D \geq 0$ is a divisor supported away from the fibres, and $\lambda \geq 1$ is a rational number. Then:

a) If $E \cdot f = 0$, one has $D \cdot f = 0$, and so $D \geq 0$ implies $D = 0$. Thus, from
the first inequality in (5.1.1) one gets $\lambda(k + 2) \leq M \cdot E \leq k + 1$, which is false.

b) If $E \cdot f > 0$, let $p$ be the smallest positive integer such that $pG$ is an
effective divisor, and suppose $pG = \sum \lambda_j C_j, E = \sum \mu_j C_j + R,$
where, $\lambda_j > 0, \mu_j \geq 0$ are integers, and $R$ is an effective divisor supported
away from the $C_j, j = 1, \ldots, t$. Then,

$$G \cdot E \geq (1/p) \sum \lambda_j \mu_j C_j^2$$
$$\geq -(n/p) \sum \lambda_j \mu_j \text{ if } n \in \mathbb{Z} \text{ is such that for all } j \ C_j^2 \geq -n;$$
$$\geq -a(2k + 1), \text{ where } a \in \mathbb{Z} \text{ depends only on } G. \text{ This because } M$$
ample divisor amid (5.1.1) imply that $2k + 1 \geq M \cdot E \geq \sum \mu_j \geq \mu_j.$

Since (5.1.1) gives:

$$(5.1.2) \quad 2(k + 1) > M \cdot E = G \cdot E + \psi(m)f \cdot E$$
and $\psi(m)f \cdot E$ is an increasing function of $m$, one cannot find any effective
divisor $E$ satisfying the inequalities (5.1.1), hence the theorem is proved.

REMARK 5.2 In particular, if the surface $X$ in the above theorem, is an
Enriques surface (see [BPV], [C] for definitions), then, $p : X \rightarrow P^1$; so
$L = G + mf$, and $M = G + mf$. Thus, in this case we can give an estimate for
$\psi(m) = m$. We get that:

(a) If $pG$ does not contain nodal curves: i.e., smooth rational curves of self-
intersection equal to $-2$, then $m \geq 2k + 2$ (or, $m/2 \geq$ the last expression
in the inequalities (1.6.1));

(b) If $pG = \sum \lambda_j C_j + Q$ where, for all $j = 1, \ldots, t, \lambda_j$ is a positive
integer, and $C_j^2 = -2$ are all the nodal curves contained in $pG$; setting
$r = G \cdot f \geq k + 2$, then:

(i) $M^2 \geq 4k + 5$ implies that $m \geq m_1 = \sum (\lambda_j^2/p^2r) + 4k + (5/2r);$
(ii) $M \cdot C_j \geq 1$ implies $m \geq m_2 = 1 + 2 \max_{j=1,\ldots,t} \{\lambda_j/p\}$;

(iii) If $E = \sum_{j=1,\ldots,t} \mu_j C_j + R, \mu_j \geq 0$, and $\sum \mu_j \geq 1$; (5.1.2)) implies

$$m \geq m_3 = 2k + 2 + 2(2k + 1)(\sum \lambda_j)/p.$$ 

So, $m \geq \max\{m_1, m_2, m_3\}$.

REFERENCES


A. BIANCOFIORE, G.CERESA,

Dipartimento di Matematica - Università dell’Aquila
Via Roma - 67100 L’Aquila (ITALY)

Lavoro pervenuto in redazione il 27/1/1989