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GORENSTEIN ALGEBRAS AND THE CAYLEY-BACHARACH THEOREM

Summary: Some new theorems on zero-dimensional subschemes of projective space, results discovered in collaboration with A. Geramita, P. Maroscia and F. Orecchia.

Introduction

This lecture consists of an exposition of certain results obtained jointly with A. Geramita, P. Maroscia and F. Orecchia, and contained in the papers [DGM], [DM] and [DGO]. The conceptual center of these results - articulated only in [DGO] - is the connection between the classical Cayley-Bacharach theorem for complete intersections in $\mathbb{P}^2$ and properties of Gorenstein algebras. Technically they all depend on a very simple idea, namely, that of reducing the consideration of certain questions about arithmetically Cohen-Macaulay subschemes of projective space to the study of an appropriately defined artinian algebra.

Notation. Absent any statement to the contrary, $\mathbb{P}$ denotes $\mathbb{P}^n(k)$, $k$ an algebraically closed field, and $X$ denotes a 0-dimensional subscheme of $\mathbb{P}: X = \text{Proj} A$, where $A = K/P$, $K$ a homogeneous coordinate algebra for

Conferenza tenuta dall'Autore presso il Seminario Matematico il 9 febbraio 1984.
\( P, \ P \) "the" ideal of \( X \) in \( K \). So \( A \) is a **standard graded** \( k \)-algebra; \( A \) is \( \mathbb{Z} \)-graded with \( A_t = 0 \quad (t < 0) \); \( A_0 = k \); \( A = k [A_1] \); \( /A_1/ \) is finite. (We use \( /-\) to denote the length function on \( k \)-modules.) Let \( \sim \) denote (for \( K \)-modules) reduction mod a general linear form in \( K \). Since \( A \) is Cohen-Macaulay and 1-dimensional, a sufficiently general linear form in \( K \) is \( A \)-regular. Hence \( \tilde{A} \) is artinian and \( /\tilde{A}/ = \delta (A) \). We use \( \delta \) to denote "multiplicity of standard graded \( k \)-algebras" and "degree of closed subschemes of \( P \). So \( \delta (X) = \delta (A) \).

**Hilbert function.** Let \( H (X, t) = b^0 (O_\ell (t)) - b^0 (\mathcal{I}_X (t)) = /K_t /- /P_t / \). For any standard graded \( k \)-algebra \( S \), let \( H (S_- \) denote its **Hilbert function**: \( H (S, t) = /S_t / \). So \( H (X,--) = H (A,-- \) and since a sufficiently general linear form in \( K \) is \( A \)-regular, \( H (\tilde{A},--) = \Delta H (A,--) \). (Here \( \Delta \) denotes the difference operator on maps from \( \mathbb{Z} \) to \( \mathbb{Z} \) : \( \Delta f (t) = f (t) - f (t - 1) \).) It follows that there is a nonnegative integer \( \tau (X) \) such that: \( H (X, t) = = \delta (X) (\tau (X) \leq t) \); \( \Delta H (X, t) > 0 \quad (0 \leq t \leq \tau (X)) \). Proof: \( \tau (X) = \max \{ t : A_t \neq 0 \} \). So if \( Y \neq X \) is a subscheme of \( X \) and \( t \geq \tau (X) \), then \( H (Y, t) \neq \neq H (Y, t) \). Clearly: \( d < \tau (X) \) if \( X \) has \( CBP (d) \). We say that the Cayley-Bacharach theorem is **valid for** \( X \) if \( X \) has \( CBP (\tau (X) - 1) \). It is almost a trivial observation (see §2) that this is the case if \( A \) is a Gorenstein algebra. In particular then, the Cayley-Bacharach theorem is valid for \( X \) if \( X \) is a (scheme-theoretic) complete intersection, i.e., if \( P \) is generated by a regular sequence.

**Cayley-Bacharach Property.** Given \( d \) in \( \mathbb{Z} \), we say that \( X \) has \( CBP (d) \) provided that: for any \( t \leq d \), and any subscheme \( Y \) of \( X \) with \( \delta (Y) = = \delta (X) - 1 \), \( H^0 (\mathcal{I}_Y (t)) = H^0 (\mathcal{I}_X (t)) \), i.e., \( H (Y, t) = H (X, t) \). Clearly: \( d < \tau (X) \) if \( X \) has \( CBP (d) \). We say that the Cayley-Bacharach theorem **is valid** for \( X \) if \( X \) has \( CBP (\tau (X) - 1) \). It is almost a trivial observation (see §2) that this is the case if \( A \) is a Gorenstein algebra. In particular then, the Cayley-Bacharach theorem is valid for \( X \) if \( X \) is a (scheme-theoretic) complete intersection, i.e., if \( P \) is generated by a regular sequence.

**Historical note.** The theorem published by Cayley (1843) concerns a group \( X \) of \( ab \) distinct points in the plane (i.e., \( n = 2 \), \( \delta (X) = ab \), \( X = X_{\text{red}} \)) which is the complete intersection of a curve of degree \( a \) and a curve of degree \( b \). It asserts that \( X \) has \( CBP (a + b - 3) \). Cayley's proof has a significant gap: in effect it assumes the validity of the theorem. Bacharach (1885) published a correct proof based on Noether's \( AF + B \Phi \) theorem.
It went unnoticed by the mathematical community for almost 80 years that Cayley’s theorem is a simple consequence of an algebraic identity published by Jacobi (1835). This, despite the facts: (i) the proof that Jacobi’s result implies Cayley can easily be understood by a student of the first course in linear equations; (ii) both results were quite well known; (iii) Jacobi (1836) published a second paper explicitly pointing out the significance of his identity for complete intersections in the plane. The validity of Cayley’s theorem for the famous case \( a = b = 3 \) was asserted by Euler (1744) in a letter to Cramer; but the question of whether Euler then knew a correct proof is moot: his letter responding to the letter in which Cramer asks for a proof avoids the issue.

There is at least some internal evidence in the Euler-Cramer correspondence to suggest that Euler’s “proof” was similar to Cayley’s.

§1. Maroscia’s Conjecture.

Suppose \( n = 2 \), and let \((a, b) \in \mathbb{N}^2\) with \( 0 < a \leq b \). We say that \( X \) is a CI \((a, b)\) in case \( P = (F, G) \), \( F \in K_a \), \( G \in K_b \). In this event: since \( F \) is \( K \)-regular, \( H(K/FK, t) = H(K, t) - H(K, t-b) \); and since \( G \) is \( K/FK \)-regular, \( H(K/(F, G), t) = H(K/FK, t) - H(K/FK, t-b) \); whence \( H(X, t) = H(a, b, t) \) \( = H(a, b, t) \) \( = \mathbb{K} / /K_{t-a} / - /K_{t-b} / + /K_{t-a-b} / \). In case \( H(X, -) = H(a, b, -) \), we say that \( X \) is a HCI \((a, b)\). From the explicit formula for \( H(a, b, t) \), for example, one sees that if \( X \) is a HCI \((a, b)\), then \( \delta(X) = ab \) and \( \tau(X) = a + b - 2 \).

Obviously CI \((a, b) \Rightarrow\) HCI \((a, b)\); but except for the trivial case \( a = 1 \), the converse implication is false. Natural Question: HCI \((a, b) + (?) = CI(a, b)\)? In [DGM] we really do better than answer that question: in effect we prove a structure theorem for HCI \((a, b)\)’s. Now CI \((a, b) \Rightarrow \delta = \) \( = ab \) and \( CBP(a + b - 3) \); but these two properties, which clearly determine \((a, b)\), do not even imply “complete intersection”: e.g., any group of 9 distinct points on an irreducible conic has \( CBP (3) \). Natural Question [GH]: \( \delta(X) = ab + X \) has \( CBP(a + b - 3) \) \( + (?) \) \( = X \) is a CI \((a, b)\)? Maroscia [M] combines the two questions into a formal conjecture for groups of distinct points in the plane, and observes that it is valid in case \( a \leq 3 \). We subsequently obtained the affirmation of that conjecture as a corollary of the HCI \((a, b)\) analysis of [DGM]:
(1.1) Theorem [DGM, (4.21)]. Suppose $n = 2$ and $X = X_{\text{red}}$. Then $X$ is a CI $(a, b)$ if, and only if, $X$ is a HCI $(a, b)$ and has $\text{CBP} (a + b - 3)$. In [DM], using more strongly the HCI $(a, b)$ analysis of [DGM], we show that (1.1) is valid without “reduced”; in fact we indicate there how to prove the sharper criterion:

(1.2) Theorem [DM, (4.4)]. Suppose $n = 2, X$ has $\text{CBP} (a + b - 3)$, and $\Delta H(X, t) = \Delta H(a, b, t)$ for $b - 2 \leq t \leq a + b - 2$. Then $X$ is a: CI $(a, b)$ if $a \neq 1$; CI $(1, \delta (X))$ if $a = 1 \neq b$. (Correction: In [DM] we neglected to point out the obvious exception $a = b = 1$.)

And in the reduced case there is yet another criterion sharper than (1.1):

(1.3) Theorem [DM, (4.4)]. Suppose $n = 2, X = X_{\text{red}}, \delta (X) = ab, X$ has $\text{CBP} (a + b - 3)$. Then $X$ is a CI $(a, b)$ if, and only if, for some $a - 1 \leq t \leq b$, $H(X, t) = H(a, b, t)$.

The analogue of Maroscia’s conjecture is false for $n > 2$ - but not if one regards (1.1) as a criterion for “arithmetically Gorenstein” rather than “complete intersection”. It just happens that the two notions coincide if $n = 2$. (In an appendix we show how to deduce that well known fact from the HCI $(a, b)$ analysis of [DGM].) Recall that $X$ is arithmetically Gorenstein provided that $A$ is a Gorenstein ring, equivalently, $\tilde{A}$ is a Gorenstein ring, equivalently, since $\tilde{A}$ is artinian, $\text{ann}_1\tilde{A} = 1$. We need the following bit of “Gorenstein duality” - which in only elementary linear algebra.

(1.4) Proposition. Let $S$ be an artinian standard graded Gorenstein $k$-algebra, and let $N = \max \{t : S_t \neq 0\}$. Then:

(a) $S_i \times S_j \rightarrow S_{i+j}$ (bilinear map induced by multiplication) is nonsingular if $i + j \leq N$ and $0 \leq i, j$. Hence:

(b) $/S_t/ = /S_{N-t}/ (0 \leq t \leq N)$.

(c) Let $I$ be a homogeneous ideal of $S$, and let $J = \text{ann} I$. Then $I = \text{ann} J$, and $/I_t/ = /J_{N-t}/ = /S_t/ (0 \leq t \leq N)$.

We say that $X$ has symmetric Hilbert function provided that $\Delta H(X, t) = \Delta H(X, \tau (X) - t)$ for $0 \leq t \leq \tau (X)$. In view of (1.4), $X$ has symmetric
Hilbert function if $X$ is arithmetically Gorenstein. In case $n=2$, $X$ has symmetric Hilbert function if, and only if, $X$ is a $HCl(a,b)$ for appropriate $(a,b)$—see appendix. So the following theorem valid for arbitrary $n$ includes (1.1).

(1.5) Theorem [DGO, (5)]. Suppose $X=X_{\text{red}}$. Then $X$ is arithmetically Gorenstein if, and only if, it has symmetric Hilbert function and the Cayley-Bacharach theorem is valid for $X$.

Remark. We don’t know whether or not (1.5) is valid, as in case $n=2$, without “reduced”. The point where that hypothesis enters our argument is in the use of the “length of the conductor” criterion for Gorenstein. This is quite different from our $n=2$ argument, which is essentially 2-codimensional rather than 0-dimensional.

§ 2. Gorenstein $\Rightarrow$ Cayley-Bacharach, Two Views.

For convenience, in this section we put aside the trivial case $\delta(X)=1$. Let $\phi \neq Y \neq X$ denote a subscheme of $X$, and let $I$ denote the ideal of $Y$ in $A$. Define: $\alpha(Y,X) = \min \{ t : H^0(\mathscr{J}_Y(t)) \neq H^0(\mathscr{J}_X(t)) \}$. So $\alpha(Y,X) = \alpha(I)$, where for any nonzero homogeneous ideal $Q$ of a standard graded algebra, $\alpha(Q) = \min \{ t : Q \neq 0 \}$. Since a sufficiently general linear form in $K$ is $A/I$-regular, $\alpha(I) = \alpha(I)$. Notice that the validity of the Cayley-Bacharach theorem for $X$ may be stated so: for every $Y$ such that $\delta(Y) = \delta(X) - 1$, $\alpha(Y,X) = \tau(X)$.

Recall that the $CM$-type of $A$ is the number $\text{ann}A_1^\perp$; by abuse of notation, we also call this number the $CM$-type of $X$. Since $\bar{A}_{\tau(X)} \subseteq \text{ann}A_1^\perp$, it is clear that: $CM$-type of $X \geq /A_{\tau(X)}^\perp = \Delta H(X, \tau(X))$; equality holds $\leftrightarrow \text{ann}A_1^\perp = \bar{A}_{\tau(X)}$ (In particular, equality must hold if $CM$-type of $X=1$, i.e., if $X$ is arithmetically Gorenstein.) These simple observations show that the Cayley-Bacharach theorem is valid for $X$ in case $X$ is, so to say, “half Gorenstein”:
(2.1) **Theorem** \([DGO,(4)]\). Suppose \(C\mathcal{M}\)-type of \(X = \Delta H (X, \tau (X))\). Then:

\[
\delta (Y) \leq \delta (X) - 1 + \alpha (Y, X) - \tau (X) \leq \delta (X) - 1.
\]

**Remark. (2.1)** remains valid for \(X\) of arbitrary dimension over arbitrary base field provided that: \(X\) and \(Y\) are arithmetically Cohen-Macaulay; \(\dim Y = \dim X\); \(\Delta\) is replaced by \(\Delta^{\dim X + 1}\). The proof remains the same provided that "-" is interpreted to mean reduction mod \(L\), where \(L\) is a general \(k\)-linear subspace of \(K_{1}\) with \(\dim L = \dim X + 1\). One needs infinite base field to be sure of the existence of a sufficiently general such \(L\), but the usual device of adjoining an indeterminate to the base field eliminates that problem.

Returning to the case \(\dim X = 0\), let \(J = \text{ann} I\). Since \(0\) is an unmixed ideal of \(A\), it follows that \(J\) is unmixed of height \(0\); i.e., \(A/J\) is 1-dimensional and Cohen-Macaulay - as are \(A\) and \(A/I\). Let \(Z = \text{Proj} (A/J)\). So \(Z\) is the subscheme of \(X\) "residual to \(Y\)."

The validity of the Cayley-Bacharach theorem for arithmetically Gorenstein \(X\) is also a consequence of the following theorem which tells what happens to Hilbert function under "liaison".

(2.2) **Theorem** \([DGO,(3)]\). Suppose \(X\) is arithmetically Gorenstein. Then:

\((a)\) \(\delta (X) = \delta (Y) + \delta (Z)\).

\((b)\) \(\Delta H (X, t) = \Delta H (Y, t) + \Delta H (Z, \tau (X) - t) = \Delta H (X, \tau (X) - t)\) \((0 \leq t \leq \tau (X))\).

\((c)\) \(\alpha (Y, X) + \tau (Z) = \alpha (Z, X) + \tau (Y) = \tau (X)\).

**Remark on (2.2)**. Observe that \((a)\) and \((c)\) are formal consequences of \((b)\). However, given \((a)\), one easily proves \((b)\) by passing to \(\overline{A}, \overline{I}, \overline{J}\), and applying (1.4). That is, the crucial observation is \((a)\), which requires "Gorenstein" only to know that for every member \(p\) of \(\text{Ass} A\), \(A_p\) is Gorenstein and \(\dim (A/p) = \dim A\). One proves \((a)\) by using the "double annihilator theorem" for artinian local Gorenstein rings and the "multiplicity formula" for standard graded \(k\)-algebras. The remark on \((2.1)\) applies also to \((2.2)\). In case \(\dim X > 0\), it is no longer trivial that \(A/J\) is also Cohen-Macaulay, but is is quite well known \([PS]\).
Appendix: Gorenstein + Codimension 2 ⇒ Complete Intersection.

We now let $k$ be an arbitrary field and assume $X$ to be arithmetically Gorenstein, of codimension 2 in $P$. So $P$ is a height 2 homogeneous ideal in $K$ such that $A = K/P$ is Gorenstein, and in particular, Cohen-Macaulay.

**Theorem** (well known). $P$ is a complete intersection (i.e., generated by two elements).

**Proof.** By the usual argument - see remark on (2.1) - we may assume $n = 1$, i.e., $A$ is artinian. Let $(a, b)$ be defined by: $a = \min \{ t : P_t \neq 0 \}; A_t \neq \neq 0 \Rightarrow 0 \leq t \leq a + b - 2$. Well known (and trivial to prove) is: $H(A, t)$ is a nonincreasing function of $t$ for $t \geq a - 1$. This fact and (1.4b) show: $H(A, t) = H(A, a + b - 2 - t) = t + 1 (0 \leq t \leq a - 1); H(A, t) = a (a - 1 \leq t \leq b - 1)$. That is, in the language of [DGM], $P$ is a Hilbert-function-complete-intersection of type $(a, b)$; by abuse of notation, we say that $P$ is a $HCI (a, b)$. Suppose $P$ is not a complete intersection. Then by [DGM, (4.4)], there is a positive integer $d < a$, and a height 2 homogeneous ideal $J \supset P$, such that: $J$ is a $HCI (a - d, b - d); P = (DJ, G), D \in K_d, G \in K_{a+b-d}$. Since $J$ is a $HCI (a - d, b - d), J_t = K_t$ for $t \geq a + b - 2d - 1$. Let * denote the canonical map $K \to A$. So $D*J* = 0$; whence $0 \neq D*E A_d \cap ann A_{a+b-2d-1}$. This violates (1.4a) because $d + (a + b - 2d - 1) \leq a + b - 2$. Done.

REFERENCES


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*Lavoro pervenuto in redazione il 2/5/1984*