Summary. We prove the following theorem: The distance function $r$ on a Riemannian manifold $M$ is biharmonic if and only if $M$ is an 1-dimensional manifold or a 3-dimensional manifold with constant sectional curvature.

Introduction.

Let $x$ be a point in an arbitrary Riemannian manifold $M$, and let $U_x \subset M$ be an open set containing $x$. If $G$ is a function of class $C^\infty$ on $U_x - \{x\}$, in terms of polar geodesic coordinates at $x$ the Beltrami-Laplace operator on $G$ is given by the formula

\[ \Delta G = \frac{\partial^2 G}{\partial r^2} + \left( \frac{n-1}{r} + \frac{\partial \theta}{\partial r} \right) \frac{\partial G}{\partial r} + \tilde{\Delta} G, \]

where $\tilde{\Delta}$ denotes the Laplacian on the geodesic sphere $G_x(r) = \{y \in M; d(x, y) = r\}$ with center $x$ and radius $r$, and $\tilde{G}$ is the restriction of $G$ to $G_x(r)$ (see, for example, [6], also for more complete references).

By $\Delta^m G$ we shall denote the function obtained applying $m$ times to $G$ the operator $\Delta$, i.e.

\[ \Delta^2 G = \Delta(\Delta G), \quad \Delta^3 G = \Delta(\Delta^2 G), \ldots, \quad \Delta^m G = \Delta(\Delta^{m-1} G), \]

and we shall call $m$-harmonic or biharmonic of order $m$ a function $G$ satisfying the equation $\Delta^m G = 0$.

E. Almansi proved in [1] that in the Euclidean space $\mathbb{R}^n$ an $m$-harmonic
function can be, in general, represented by means of a linear combination of \( m \) functions which are linear in the coordinates \( x^i \) of the space, or equal to \( r^2 - a^2 \), where \( a \) is a constant and

\[
r^2 = \sum_{i=1}^{n} (x^i)^2.
\]

This theorem has interesting applications mainly in that part of the Elasticity theory which is related to the equation \( \Delta^2 = 0 \).

In a recent paper by A. Gray and T.J. Willmore [6], a Pizzetti’s formula, expressing a mean value property for \( C^\omega \) functions on \( \mathbb{R}^n \), has been extended, with suitable modifications, to \( C^\omega \) functions on an arbitrary Riemannian manifold.

In the same scheme of things, the author of this paper proved in [3] that the Almansi theorem maintains its validity for radial functions on those spaces on which the distance \( r \) is biharmonic. By straightforward calculation one can verify that, among Euclidean spaces and rank 1 symmetric spaces, \( \Delta^2 r = 0 \) holds only on \( \mathbb{R}^1, \mathbb{R}^3 \) and on \( S^3(\lambda) \) and \( H^3(\lambda) \), the 3-dimensional sphere and hyperbolic space with constant sectional curvature \( \lambda \). On these manifolds, the functions \( p \) which intervene in the Almansi theorem are linear in \( r \) and satisfy the following

**Property** (0.2). If \( q \) is a radial \((m-1)\)-harmonic function, and \( p = ar + b \), where \( a \) and \( b \) are constants, then

\[
\Delta^m(p \cdot q) = 0.
\]

In other words, if we multiply a radial function which is polyharmonic of order \( m - 1 \) by a function linear in \( r \), we obtain an \( m \)-harmonic function.

By means of this property, which plays a key role in the proof of the Almansi theorem as well in its generalization to Riemann manifolds, it is also possible to find the elementary polyharmonic functions of any order, whenever the elementary 1-harmonic functions are known.

The main purpose of this paper is to prove that in fact the only Riemannian manifolds characterized by the biharmonicity of the distance function are the 1-dimensional Riemannian manifolds and the 3-dimensional Riemannian manifolds with constant sectional curvature. The proof bases on power series expansions for the functions \( b = \Delta r \) and \( \tilde{\Delta} b \), the mean curvature of the geodesic sphere \( G_x(r) \) and the not radial part of its Laplacian, respectively. These expansions can be found in [5] and in [4].

It would probably be interesting to determine the Riemannian manifolds
on which $\Delta^k r = 0$, for arbitrary $k$. It is easy to see that the dimension of such spaces must be odd and less or equal to $2k - 1$; but in order to find out more information, especially concerning their curvature, it may be necessary to compute successive terms in the power series expansions we are taking into consideration. Nevertheless, there are a few reasons to conjecture that, for $k \geq 3$, the Riemannian manifolds on which the distance function is $k$-harmonic are flat manifolds.

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1. Polyharmonic radial functions on rank 1 symmetric spaces.

Let $\widetilde{F}$ be a function which is $C^\infty$ in a deleted neighborhood $U_x - \{x\}$ of $x$, and let $\widetilde{F}$ be a radial function near $x$; in other words, let's assume that at points which are sufficiently near $x$, $\widetilde{F}$ only depends on their distance from $x$. Then, for such a point $y$ one has $\widetilde{F}(y) = F(r(y))$, where $F : \mathbb{R} \to \mathbb{R}$, and $r(y) = d(x, y)$. For such functions one has $\Delta \widetilde{F} = 0$, and the formula (0.1) for the Laplacian reduces to the more simple expression

$$\Delta \widetilde{F} = \frac{\partial^2 F}{\partial r^2} + \left( \frac{n - 1}{r} + \frac{1}{\theta} \frac{\partial \theta}{\partial r} \right) \frac{\partial F}{\partial r} .$$

In general, for every two different directions at $x$, (1.1) represents two differential equations which are also different. However, in the case of harmonic spaces, and hence, in particular, in the case of rank 1 symmetric spaces, the function $\theta$ is independent of the direction, and therefore (1.1) is in fact just one differential equation.

In the case of rank 1 symmetric spaces, in addition, of course, to Euclidean space $\mathbb{R}^n$, the explicit expression of $\theta$ is easy to find. Let the prime ' denote radial derivation. Then the respective Laplacians can be written

$\mathbb{R}^n$:

$$\Delta \widetilde{F} = F'' + \frac{n-1}{r} F' ;$$

$S^n(\lambda)$ (the sphere with constant sectional curvature $\lambda$):

$$\Delta \widetilde{F} = F'' + ((n-1)\sqrt{\lambda} \cot \sqrt{\lambda} r) F' ;$$

$\mathbb{C}P^n(\lambda)$ (the complex projective space with constant holomorphic sectional curvature $4\lambda$):

$$\Delta \widetilde{F} = F'' + ((n-1)\sqrt{\lambda} \cot \sqrt{\lambda} r) F' ;$$
\[ \Delta \tilde{F} = F'' + ((2n-1)\sqrt{\lambda} \cot \sqrt{\lambda} r - \sqrt{\lambda} \tan \sqrt{\lambda} r) F'; \]

\( \mathbb{Q}P^n(\lambda) \) (the quaternionic space with maximum sectional curvature \( 4\lambda \)):

\[ \Delta \tilde{F} = F'' + ((4n-1)\sqrt{\lambda} \cot \sqrt{\lambda} r - 3\sqrt{\lambda} \tan \sqrt{\lambda} r) F'; \]

\( \mathbb{C}P^2(\lambda) \) (the Cayley plane with maximum sectional curvature \( 4\lambda \)):

\[ \Delta \tilde{F} = F'' + (15\sqrt{\lambda} \cot \sqrt{\lambda} r - 7\sqrt{\lambda} \tan \sqrt{\lambda} r) F'. \]

The Laplacians for the dual symmetric spaces corresponding to \( S^n(\lambda) \), \( \mathbb{C}P^n(\lambda) \), \( \mathbb{Q}P^n(\lambda) \) and \( \mathbb{C}P^2(\lambda) \) are obtained in each case by replacing the trigonometric functions \( \tan \) and \( \cot \) by the hyperbolic functions \( \tanh \) and \( \coth \) respectively.

On these spaces, radial biharmonic (and polyharmonic) functions have in general a rather complicated expression [3]. However, some of these spaces turn out to be, also from this viewpoint, quite privileged. If we express the Laplacian of the arbitrary function \( F \) in the form

\[ \Delta F = \frac{1}{\theta r^{n-1}} \frac{\partial}{\partial r} \left( \theta r^{n-1} \frac{\partial F}{\partial r} \right), \]

and use the 1-harmonic functions, we find that the biharmonic functions on \( S^3(\lambda) \) and \( H^3(\lambda) \) have the simple expression

\[ F = \frac{Ar}{2\sqrt{\lambda}} + \frac{B}{\lambda} r \cdot \cot \sqrt{\lambda} r + C \sqrt{\lambda} \cot \sqrt{\lambda} r + D, \]

and

\[ F = \frac{Ar}{2\sqrt{-\lambda}} + \frac{B}{\lambda} r \cdot \coth \sqrt{-\lambda} r + C \sqrt{-\lambda} \coth \sqrt{-\lambda} r + D \]

respectively, where \( A, B, C, D \) are some constants. It follows that, while the functions \( \cot \sqrt{\lambda} r \) and \( \coth \sqrt{-\lambda} r \) are obviously biharmonic being harmonic, the functions \( r, r \cdot \cot \sqrt{\lambda} r \) and \( r \cdot \coth \sqrt{-\lambda} r \) are biharmonic but not harmonic.

There are three reasons why these considerations are interesting.

1. Riemannian manifolds on which \( \Delta^2 r = 0 \) possess property (0.2), which was proved on \( \mathbb{R}^n \) by Almansi. 2. The Almansi decomposition theorem remains valid on Riemannian manifolds on which \( r \) is biharmonic [3]. 3. It is well known that on \( \mathbb{R}^2 \) and \( \mathbb{R}^n, n \neq 2 \), the elementary harmonic functions are \( \log r \) and \( 1/r^{n-2} \) respectively. From these functions and using the
functions \( p \) on \( \mathbb{R}^n \) satisfying property (0.2), it is possible to determine the elementary polyharmonic functions of any order. Analogously, on the Riemannian spaces on which \( \Delta^2 r = 0 \), allowing to obtain a \((k + 1)\)-harmonic function whenever a \( k \)-harmonic function is given, the same property permits us also to find the elementary harmonic functions of any order.

Applying twice the Laplacian to \( r \) on \( \mathbb{R}^n \), \( S^n(\lambda) \) and \( H^n(\lambda) \), we have

\[
\Delta^2 r = \frac{(n-1)(3-n)}{r^3},
\]

\[
\Delta^2 r = \lambda \sqrt{\lambda} (n-1)(3-n) \cot \frac{\sqrt{\lambda} r}{\sin^2 \sqrt{\lambda} r},
\]

\[
\Delta^2 r = \lambda \sqrt{-\lambda} (n-1)(3-n) \coth \frac{\sqrt{-\lambda} r}{\sinh^2 \sqrt{-\lambda} r}
\]

respectively. Thus the function \( r \) is biharmonic on \( \mathbb{R}^1 \), where it is also harmonic, and on \( \mathbb{R}^3, S^3(\lambda), H^3(\lambda) \). By straightforward calculation one can check that among Euclidean spaces and rank 1 symmetric spaces the function \( r \) possesses such a property only on the four mentioned spaces.

2. The question arises now whether other spaces can be found on which \( \Delta^2 r = 0 \). We shall prove the following

**Theorem 2.1.** The distance function \( r \) on a Riemannian manifold \( M \) is biharmonic if and only if \( M \) is an 1-dimensional manifold, or a 3-dimensional manifold with constant sectional curvature.

For the proof we shall use the power series expansions for the functions \( b = \Delta r \) and \( \Delta b \), the latter being the not radial part of the Laplacian of \( b \). These expansions hold on arbitrary Riemannian manifolds and can be found in [5] and in [4] respectively.

An elegant and well known method to determine these series expansions is based on special vector fields, the Jacobi fields, which appear in the variational theory of geodesics. We start recalling briefly of what this method consists (see, for example, [4], or [2], chapter 6).

Let \((M,g)\) be an \( n \)-dimensional Riemannian manifold. We denote by \( \mathfrak{X}(M) \) the Lie algebra of the \( C^\infty \) vector fields on \( M \), by \( \nabla \) the Riemannian connection and by \( R \) the curvature operator defined by

\[
R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y], \quad \text{where} \ X, Y \in \mathfrak{X}(M).
\]

Let \((x^1, \ldots, x^n)\) be a system of normal coordinates with origine \( x \in M \).
in a neighborhood $U_x$ of $x$, i.e. such that $x^i(x) = 0$, $i = 1, \ldots, n$. We recall that we always choose the radius of the geodesic sphere $G_x(r)$ sufficiently small in order that $\exp_x$ is defined in a ball of radius $r$ and center $x$. Further let $\xi \in T_xM$ be an unit vector, and denote by $\gamma(r)$ the geodesic tangent to $\xi$, i.e. $\gamma(r) = \exp_x(r\xi)$.

If $\{e_i, i = 1, \ldots, n; e_1 = \xi\}$ is an orthonormal bases of $T_xM$, let $\{E_i\}$ be the orthonormal bases field obtained by parallel translation along $\gamma$ of the bases $\{e_i\}$.

A vector field $Y$ along $\gamma$ is a Jacobi vector field if and only if it satisfies the Jacobi equation

\[(2.2) \quad Y'' + R_{\xi Y} \xi = 0 ,\]

where the prime $'$ always denotes radial derivation. The $n - 1$ linearly independent Jacobi vector fields $\{Y_a, a = 2, \ldots, n\}$ determined by the initial conditions

\[Y_a(0) = 0 \quad Y'_a(0) = e_a , \quad a = 2, \ldots, n ,\]

are related to the normal coordinate vector fields $\partial/\partial x^a$ by the formula

\[(2.3) \quad Y_a(\gamma(r)) = r \frac{\partial}{\partial x^a} (\gamma(r)) .\]

The Jacobi vector fields are always tangent to the geodesic sphere $G_x(r)$, and so they are particularly adapted to study the geometry of $G_x(r)$.

If we put

\[Y_a(r) = (A_a^b E_b)(r) , \quad a, b = 2, \ldots, n ,\]

to every value of $r$ it corresponds an endomorphism $A(r)$ of the tangent space to $G_x(r)$ at $y = \exp_x(r\xi)$. The Jacobi equation induces the equation

\[(2.4) \quad A'' + R \circ A = 0 ,\]

with

\[A(0) = 0 \quad A'(0) = I ,\]

where $I$ is the identity and $R$ denotes the endomorphism of the tangent space to $G_x(r)$ defined by

\[R(r) = R_{\gamma'(r)} - \gamma'(r) .\]

Our interest for the endomorphisms $A(r)$ which are solution of the system (2.4) is due to the fact that the symmetric endomorphism

\[(2.5) \quad \sigma = A' A^{-1} \]
is nothing but the second fundamental form of the hypersurface $G_x(r)$ of $M$.

At the point $y = \exp_x(r\xi)$, the trace $b$ of $\sigma$ has the expression [4]

\begin{equation}
(2.6) \quad b = \frac{n - 1}{r} + \frac{\theta'}{\theta},
\end{equation}

where $\theta(y) = (\det g)^{1/2}(y) = r^{1-n}(\det A)(y)$. Hence the function $b$ represents the mean curvature of $G_x(r)$, and recalling the formula (0.1) giving the Laplacian on $M$, one observes at once that $b = \Delta r$.

From the power series expansions for the second fundamental form [4], [5], we obtain by contraction the expansion for the mean curvature of the geodesic sphere $G_x(r)$. If $y = \exp_x(r\xi)$, then

\begin{equation}
(2.7) \quad b(y) = \frac{n - 1}{r} - \frac{r}{3} \rho_{\xi\xi}(x) - \frac{r^2}{4} (\nabla_{\xi} \rho_{\xi\xi})(x) - \frac{r^3}{90} \left( 9 \nabla^2_{\xi\xi} \rho_{\xi\xi} + 2 \sum_{a,b=2}^{n} R_{\xi a \xi b} \right) (x) +
\end{equation}

\begin{equation}
+ \frac{r^4}{24} \left( -\frac{2}{3} \nabla^3_{\xi\xi\xi} \rho_{\xi\xi} - \frac{2}{3} \sum_{a,b=2}^{n} R_{\xi a \xi b} \nabla_{\xi} R_{\xi a \xi b} \right) (x) +
\end{equation}

\begin{equation}
+ \frac{r^5}{720} \left( -\frac{30}{7} \nabla^4_{\xi\xi\xi\xi} \rho_{\xi\xi\xi} - \frac{1}{7} \sum_{a,b=2}^{n} \left( 45 (\nabla^2_{\xi} R_{\xi a \xi b})^2 +
\end{equation}

\begin{equation}
+ 48 \nabla_{\xi\xi} R_{\xi a \xi b} R_{\xi a \xi b} - \frac{32}{21} \sum_{a,b,c=2}^{n} R_{\xi a \xi b} R_{\xi a \xi c} R_{\xi b \xi c} \right) (x) + O(r^6).
\end{equation}

Here $\rho$ and $R$ denote the Ricci tensor and the Riemannian curvature tensor of $M$ respectively.

In order to obtain the power series expansion for $\Delta b$, we shall also need the expansion for the function $\widetilde{\Delta} \rho_{\xi\xi}$, which can be found in [4].

If we take $e_i = \partial/\partial x^i(x)$, $i = 1, \ldots, n$, leaving for the rest the above hypothesis unchanged and making use of the covariant derivative on $G_x(r)$ we have

\begin{equation}
(2.8) \quad \widetilde{\Delta} \rho_{\xi\xi} = \frac{1}{r^2} \left( r - n \rho_{\xi\xi} \right) + O \left( \frac{1}{r} \right),
\end{equation}

where $r$ is the scalar curvature of $M$.

Now we are able to compute the expansion for the Laplacian of the mean curvature $b$, and to determine the Riemannian manifolds with biharmonic distance function.

**Proof of Theorem (2.1).** From (2.7) and (2.8) the power series expansion for $\Delta b$ stopped to the first two terms is
Hence, first of all the biharmonicity of the distance function implies that \( M \) must have dimension 1 or 3. Secondly, from vanishing of the coefficient of \( 1/r \) it follows \( \rho_{11} = \rho_{22} = \rho_{33} \) and \( \rho_{ij} = 0 \) for \( i \neq j \), \( i,j = 1,2,3 \). So, if \( \Delta b = 0 \) \( M \) is an Einstein space, a manifold with constant sectional curvature and a symmetric space.

3. Now we ask ourselves whether it is possible to determine all manifolds on which \( \Delta^3 r = 0 \), and, more in general, \( \Delta^k r = 0 \), for arbitrary \( k \).

It is easy to prove by induction that, for any \( k \), the first term of the expansion for \( \Delta^k r \) is:

\[
\Delta^k r = \frac{3 \cdot 5 \cdot \ldots \cdot (2k - 3) (n - 1) (3 - n) \ldots (2k - 1 - n)}{r^{2k - 1}}.
\]

The dimension of the manifolds on which the distance function is \( k \)-harmonic will be then odd and less or equal to \( 2k - 1 \).

On the other hand, on the sphere \( S^n(\lambda) \) one finds

\[
\Delta^3 r = 3(n - 1)(3 - n) \sqrt{\lambda} \cdot \lambda^2 \frac{\cot \sqrt{\lambda} r}{\sin^2 \sqrt{\lambda} r} \left\{ 4 + (5 - n) \left( 1 + 2 \cos^2 \sqrt{\lambda} r \right) \right\},
\]

therefore, in general, on \( S^5(\lambda) \) one has \( \Delta^3 r \neq 0 \).

Finally, taking account of (2.8) and (2.9) we obtain

\[
\Delta^3 r = \frac{3(n - 1)(3 - n)(5 - n)}{r^5} + \frac{1}{r^3} (n - 1) \{ 2r - 3(n - 1) \rho_{\xi\xi}(x) \} + O(r^0).
\]

So the condition \( \Delta^3 r = 0 \) implies that \( M \) must be a Ricci-flat Einstein manifold (\( \rho_{\xi\xi} = 0 \)) when \( n = \dim M > 3 \).

In order to give the complete classification for the Riemannian manifolds on which \( \Delta^k r = 0 \), \( k > 2 \), we probably need to calculate more terms in the expansions for \( h \) and \( \tilde{\Delta} b \). That seems to be not so easy. For the moment, the ideas just illustrated seem to lead to the following conjecture, suggested to me by L. Vanhecke

**Conjecture.** Let \( M \) be a Riemannian manifold of dimension \( n > 3 \). The distance function on \( M \) is \( k \)-harmonic, \( k > 2 \), if and only if \( M \) is a flat manifold of odd dimension less or equal to \( 2k - 1 \).
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