A SUMMARY OF RESULTS IN THE TOPOLOGICAL CLASSIFICATION OF PLANE ALGEBROID SINGULARITIES

In this note(1) we give first a brief survey of a number of papers, published during the last thirty years, which discuss from a topological point of view singularities of plane algebroid curve branches. In the second place we bring to light certain topological interpretations of both the multiplicity of intersection of two plane branches and of a certain class of plane quadratic transformation.

1. INTRODUCTION.

In 1928 K. BRAUNER [1] (2) published a paper which subsequently proved to be the starting point from which later writers were able to establish a purely topological classification of singularities in the plane. BRAUNER considers the plane of the complex variables $x = x_1 + ix_2$ and $y = y_1 + iy_2$ as the real four—dimensional Euclidean space $R^4$ in which the coordinates of a current point are $(x_1, x_2, y_1, y_2)$. That part of an algebroid singularity lying within a neighbourhood of its origin $O$, which we can also assume to be the origin of coordinates in the $(x, y)$—plane, is then a two—dimensional complex immersed in $R^4$. BRAUNER shows that if the boundary of a sufficiently small 4—cell, $|x|^2 + |y|^2 \leq$ constant, is projected stereographically onto ordinary 3—space, then in this space

---

(1) The material of this paper was the subject of the MSc Dissertation [9] and of a series of seminars delivered at the University of Turin during 1955.

(2) Numbers in square brackets refer to the bibliography at the end of the paper.
there arises a system of non-intersecting knots, projections of the 1-dimensional circuits which form the intersection of the boundary of the 4-cell with the 2-dimensional complex of the singularity. Any system of knots is known as a link and the order of a link is the number of individual knots which it contains. The order of the link obtained in this case is equal to the number of distinct branches of which the singularity is composed, each branch giving rise to a single component of the link.

Brauner gives a complete description of these tubeular knots (Schlauch-knoten) and indicates how their mutual relations could be investigated in the case of a link arising from a compound singularity. From this it emerges that two algebroid singularities, each consisting of a single branch, which are equivalent from the classical point of view (i.e. having the same multiplicity sequences) yield two isotopic knots.

The question is taken up again by E. Kähler in [5], where we find a description of the link arising in the general case of a compound singularity. It is but a step from here to show again that two compound singularities equivalent from the classical point of view (i.e. indistinguishable from one another by means of the multiplicity sequences and mutual intersection numbers of their constituent branches) give rise to two isotopic links.

In section (2), where we discuss these questions in greater detail in particular giving an explicit description of the tubular or Brauner knots, we have preferred to follow the paper of Kähler rather than that of Brauner. The essential difference between these papers is that whereas Brauner takes the 4-cell in the neighbourhood of the origin of the singularity to be the 4-sphere $|x|^2 + |y|^2 \leq$ constant, Kähler considers the 4-cell $|x| \leq$ constant, $|y| \leq$ constant. From the topological point of view there is of course no difference and in the sequel we are concerned in effect simply with the local topology at the origin.

In 1932 O. Zariski [13] and W. Burau [2] showed independently that two algebroid branches which are not similar give rise to two non-isotopic knots, and thus characterised the

---

(*) See for instance the Dissertation [9].

(1) We shall in future call such singularities similar.
multiplicity sequence of an algebroid branch from a purely topological point of view. Subsequently, Burau [3] extended this result to compound singularities. However, as we feel that the alternative proof of this extension given in [9] is not without its advantages we give an outline of it in section (3). This is based primarily on the fact that the multiplicity of intersection of two branches is equal to the linking coefficient of the two corresponding knots (6). This latter fact is a direct consequence of a remark made by S. Lefschetz [7] in connection with the definition of linking coefficients. We discuss this in section (4) where, however, we also give an indication of our original proof as it makes a direct appeal to the geometrical structure of the Braunier knots. This structure, interesting in itself, becomes essential to the ideas of section (5) in which, as an application, we obtain for a particular plane quadratic transformation (6) a simple geometrical interpretation which enables us to view in a new light a proof of the fact that every algebroid branch may be completely resolved by a finite sequence of such transformations.

2. THE BRAUNER TUBULAR LINKS.

In this section we give a brief outline of that part of the above quoted paper [5] of Kähler which leads to the result that two similar singularities give rise to two isotopic links. In doing this it has been found convenient to introduce for the Braunier knots a certain naturally defined symbolic notation which will be of use in the subsequent pages.

2.1 Suppose an algebroid singularity, whose origin is at the origin $O$ of non—homogeneous coordinates in the plane of the complex variables $x$ and $y$, to consist of $\mu$ distinct branches $P_v$, ($v = 1, 2, ..., \mu$). A rotation of axes, if necessary, will ensure that $x = 0$ is not a branch tangent and then the Puiseux

(3) O. Zariski alludes to this possibility in «Algebraic Surfaces», Chelsea (1948), page 13, and during the preparation of this paper my attention has been drawn both to a recent demonstration of the same result obtained by E. Martinelli [8] and to a paper of R. Caccioppoli [4]. These two papers are concerned with a topological proof of Bézout's theorem for curves in the plane which reduces to the calculation of linking coefficients.

(6) This transformation is particular in the sense that two of the base points of the associated net of conics coincide.
expansion of any one of these branches has the form

\[ y = a_1 x^m + a_2 x^{\frac{N_2}{m}} + a_3 x^{\frac{N_3}{m}} + \ldots \]  

(1)

where all the exponents have the same denominator \( m \) and

\[ 0 < m \leq N_1 < N_2 < N_3 < \ldots \]

In addition it is assumed that for all sufficiently small \( |x| \) these expansions are uniformly convergent.

If \( x = x_1 + ix_2 \) and \( y = y_1 + iy_2 \) where \( x_1, x_2, y_1, y_2 \) are real, then the \((x, y)\)-plane can be regarded as a real 4-dimensional Euclidean space in which the coordinates of a point may be taken to be \((x_1, x_2, y_1, y_2)\). In this space the curve branches will be 2-dimensional complexes which will again be denoted by \( P_v \) \((v = 1, 2, \ldots, \mu)\). In any specific instance it will be clear from the context in which sense \( P_v \) is being used.

On account of the uniform convergence of the Puiseux expansions it is possible to choose a cylindrical region \( \Sigma' \) given by \( |x| \leq r'' \), \( |y| \leq r' \), such that each of the \( \mu \) expansions (1) are valid on \( \Sigma' \), and for each of them \( |y| < r''' < r' \) whenever \( |x| \leq r'' \), and lastly, none of the \( P_v \) have any mutual intersections on \( \Sigma' \) other than at the origin. Let \( \Sigma \) be any region \( |x| \leq r, |y| \leq r' \), where \( r < r'' \); the value of \( r \) being, subject to the above inequality, at the disposal of future requirements.

The boundary of \( \Sigma \) is the 3-dimensional complex \( \Sigma^* \), union of the two point sets

\[ Z : |x| = r, \quad |y| \leq r' \quad \text{and} \quad Z' : |x| \leq r, \quad |y| = r'. \]

Let the 1-dimensional circuit in which \( P_v \) intersects \( Z \) be denoted by \( \lambda_v \). (7)

Suppose now that we set \( x = t e^{\Phi}, y = t' e^{\Phi'} \) (with \( t, t', \Phi, \Phi' \)

(7) It is perhaps interesting to mention that Kähler shows that the fundamental group of the complex \( \Sigma^* \) arising from \( \Sigma \) by removal of those parts of the \( P_v \) \((v = 1, 2, \ldots, \mu)\) which it contains is isomorphic to the local fundamental group at the origin, that is, the fundamental group of the complex \( S^* \) arising from the boundary \( S \) of \( \Sigma \) by removal of those parts of the \( P^* \) which it contains (i.e. the \( \lambda_v \)).
real) and consider the transformation

\[
\begin{align*}
\xi + i\eta &= (R + t' \cos \Phi') e^{i\theta'} \\
\zeta &= t' \sin \Phi'
\end{align*}
\] (2)

This transformation defines a one—one mapping of \( Z \) (i.e. the region \( t = r, t' \leq r' \)) onto the boundary and interior of the torus \( T \),

\[
\begin{align*}
\xi + i\eta &= (R + r' \cos \Phi') e^{i\theta'} \\
\zeta &= r' \sin \Phi'
\end{align*}
\] (3)

in the Euclidean \((\xi, \eta, \zeta)\)—space. Similarly \( Z' \) can be mapped onto a torus \( T' \) by means of the transformation

\[
\begin{align*}
\xi' + i\eta' &= (R' + t \cos \Phi) e^{i\theta'} \\
\zeta' &= t \sin \Phi
\end{align*}
\] (R' > r)

The intersection \((t = r, t' = r')\) of \( Z \) and \( Z' \) corresponds in these mappings to the boundaries of \( T \) and \( T' \) respectively. In the induced mapping of the boundary of \( T \) onto that of \( T' \) there is a one—one correspondence between the meridians of \( T \) and the parallels of \( T' \), and between the parallels of \( T \) and the meridians of \( T' \). Thus \( T' \) can be mapped onto the closure of the complement of \( T \) with respect to the \((\xi, \eta, \zeta)\)—space in such a way that corresponding points of the boundaries of \( T \) and \( T' \) are identified. In doing this the \((\xi, \eta, \zeta)\)—space must be regarded as being closed by a point at infinity so as to form a 3—sphere, but this has no effect upon the following considerations. To the 1—dimensional circuits \( \lambda \) will correspond in the \((\xi, \eta, \zeta)\)—space certain knotted curves \( \overline{\lambda} \) which lie within the torus \( T \), and it is to these that we now turn our attention.

Suppose the branch \( P \) has the Puiseux expansion

\[
y = a_1 x^{m_1} + a_2 x^{m_2} + a_3 x^{m_3} + \ldots
\]

where \( m \leq N_1 < N_2 < N_3 < \ldots \). By means of the algorithm

\[
N_1 = m' n_1 ; \quad N_2 = m'' n_2 ; \quad N_3 = m''' n_3 ; \quad \ldots
\]

\[
m = m' m_1 ; \quad m' = m'' m_2 ; \quad m'' = m''' m_3 ; \quad \ldots
\]

where \((m_1, n_1) = 1 ; \quad (m_2, n_2) = 1 ; \quad (m_3, n_3) = 1 ; \quad \ldots \) (4)
the expansion (1) can be thrown into the form

\[ y = a_1 \omega^{m_1} + a_2 \omega^{m_2} + a_3 \omega^{m_3 m_1} + \ldots \]  

(5)

the above inequalities being equivalent to

\[ m_1 \leq n_1; \quad n_1 m_2 < n_2; \quad n_2 m_3 < n_3; \quad n_3 m_4 < n_4; \quad \ldots \]  

(6)

For sufficiently small values of \( r \) above, Kähler shows that the curve \( \lambda \) arising from this branch is isotopic to each of the knots \( A_K \) with \( K \geq K_0 \), where \( K_0 \) is the smallest integer for which \( m_1 m_2 \ldots m_{K_0} = m \), and where \( A_k \) is defined inductively as follows (8). \( A_1 \) is a curve winding \( m_1 \) times round \( \varrho_0 \) in the direction of the parallels and \( n_1 \) times round \( \varrho_0 \) in the direction of the meridians, were \( \varrho_0 \) is a torus lying inside \( T \) with equations of the form (3) in which \( r' \) has been replaced by a smaller number, say \( r_0 < r' \); and \( A_k \) lies on the non—singular tubular surface \( \varrho_{k-1} \) whose axis is \( A_{k-1} \); \( A_k \) winding \( m_k \) times round \( \varrho_{k-1} \) in the direction of the parallels and \( n_k \) times round \( \varrho_{k-1} \) in the direction of the meridians (9). The sense of winding of each of these knots round their respective tubular surfaces is always the same, in fact negative. That is to say, \( A_k \) has the appearance of a left hand screw on the surface \( \varrho_{k-1} \). It will be convenient to denote the knot \( A_k \) defined above by the symbol \( \{(m_1, n_1); (m_2, n_2); \ldots; (m_k, n_k)\} \) (10). It should be observed that this array of numbers suffices not only to describe the knot \( A_k \) but also to give the exponents of the first \( k \) terms of the expansion (1).

2.2 We now define the canonical form of a given array as that array obtained from the given one by deleting all pairs of bracketed numbers of which the first member is equal to

(\footnote{(8) The significance of \( K_0 \) is that \( N_{K_0}/m \) is the last characteristic exponent in the expansion (1).} \footnote{(9) By tubular surface is to be understood a homeomorph of a torus. If we suppose this to be given by the equations (3) then the transform of the circle \( \xi^2 + \eta^2 = R^2; \ \zeta = 0 \) will be known as the axis of the tubular surface. A meridian of a tubular surface is a 1—dimensional circuit on the surface homologous to zero in the interior of the surface and having unit linking with the axis. On the other hand, those circuits homotopic in the interior of the surface to its axis but having zero linking with it are known as parallels.} \footnote{(10) It is to be noted that we regard these knots as being unoriented.})
unity. Thus, in the canonical form of an array all \( m_r \) are strictly greater than unity.

It is not difficult to see that the two knots defined firstly by a given array and secondly by its canonical form are isotopic \((11)\). On the other hand, since a pair \((m_r, n_r)\) with \( m_r \neq 1 \) arises when and only when the corresponding term of the Puiseux expansion is characteristic, it follows that the canonical form of the array of \( A_K (K \geq K_0) \) determines and is determined by the set of characteristic exponents of the Puiseux expansion. The similarity type \((12)\) of a single branch is uniquely determined by the characteristic exponents, consequently, two branches (given by expansions of the form \((5)\) for which \((4)\) and \((6)\) are satisfied) which are similar give rise to two isotopic knots. On the other hand, Burau \([2]\) has shown that the knots corresponding to two dissimilar branches are not isotopic, having in fact different Alexander polynomials \((13)\).

Perhaps it should be pointed out that these Brauner tubular knots may be defined quite independently of the Puiseux expansions by means of arrays for which the relations \((4)\) and inequalities \((6)\) hold good, for from such an array a Puiseux expansion of the form \((5)\) may be derived.

2.3 Turning now to the case in which the singularity consists of two distinct branches whose Puiseux expansions are

\[
y = a_1 x^m + a_2 x^{m'} + a_3 x^{m''} + \ldots
\]

and

\[
y = b_1 x^p + b_2 x^{p'} + b_3 x^{p''} + \ldots
\]

we see that by means of the algorithms

\[
N_1 = m'n_1 ; \quad N_2 = m''n_2 ; \quad N_3 = m'''n_3 ; \quad \ldots
\]

\[
m = m'm_1 ; \quad m' = m''m_2 ; \quad m'' = m'''m_3 ; \quad \ldots
\]

\[
Q_1 = p'q_1 ; \quad Q_2 = p''q_2 ; \quad Q_3 = p'''q_3 ; \quad \ldots
\]

\[
p = p'p_1 ; \quad p' = p''p_2 ; \quad p'' = p'''p_3 ; \quad \ldots
\]

\((11)\) In effect this depends upon the fact that any \((1, n)\) knot on a tubular surface is isotopic to the axis of the surface.

\((12)\) See footnote \((d)\) on page 2.

\((13)\) For definitions of the Alexander polynomial see for instance Reidemeister \([10]\) or Torres \([11]\).
where \((m_1, n_1) = 1\); \((m_2, n_2) = 1\); \((m_3, n_3) = 1\); \(\ldots\) \(\tag{7}\)

and \((p_1, q_1) = 1\); \((p_2, q_2) = 1\); \((p_3, q_3) = 1\); \(\ldots\)

these expansions may be rewritten in the form

\[
y = a_1 x^{n_1} + a_2 x^{m_2 \cdot n_2} + a_3 x^{m_1 \cdot m_2 \cdot n_1} + \ldots
\]

\[
y = b_1 x^{q_1} + b_2 x^{p_2 \cdot q_2} + b_3 x^{p_1 \cdot p_2 \cdot q_1} + \ldots
\]

\(\tag{8}\)

and the inequalities \(m \leq N_1 < N_2 < N_3 < \ldots\)

\(p \leq Q_1 < Q_2 < Q_3 < \ldots\)

are equivalent to

\[
m_1 \leq n_1 \quad n_1m_2 < n_2 \quad n_2m_3 < n_4 \quad n_3m_1 < n_4 \quad \ldots
\]

\[
p_1 \leq q_1 \quad q_1p_2 < q_2 \quad q_2p_3 < q_3 \quad q_3p_4 < q_4 \quad \ldots
\]

\(\tag{9}\)

Suppose that these two expansions are identical up to and including the \(h\) th terms but that the \((h+1)\) th terms are different. Thus

\[a_k = b_k \quad \text{and} \quad n_k p_k = m_k q_k\]

for \(1 \leq k \leq h\)

but not both of \(a_{h+1} = b_{h+1}\) and \(n_{h+1} p_{h+1} = m_{h+1} q_{h+1}\) hold simultaneously. Further, let \(K_0\) now denote an integer with the following properties

(i) \(\prod_{k=1}^{K} m_k = m\) and \(\prod_{k=1}^{K} p_k = p\) whenever \(K \geq K_0\)

(ii) \(K_0 > h + 1\)

As in the case of a single branch it can now be shown (Cf. for example \([5]\) and \([9]\)) that provided \(r\) is chosen sufficiently small the two branches give rise to a link isotopic with the linked pair of knots \(A_K\) and \(B_K\) \((K \geq K_0)\) described
as follows. $A_K$ and $B_K$ as individual knots are built up in exactly the same way as before, their attached arrays being $\{(m_1, n_1); (m_2, n_2); \ldots; (m_K, n_K)\}$ and $\{\{(p_1, q_1); (p_2, q_2); \ldots; (p_K, q_K)\}\}$. If we now denote the sequence of tubular surfaces used in the construction of $A_K$ and $B_K$ by $\varphi_k$ and $\sigma_k$ respectively, it can be shown that $A_k$ and $B_k$ are identical for $k = 1, 2, \ldots, h$ but that $A_{h+1}$ and $B_{h+1}$ are distinct. Further if $n_{h+1} p_{h+1} > m_{h+1} q_{h+1}$ then $\varphi_h$ lies wholly within $\sigma_h$ ($\varphi_h$ and $\sigma_h$ having of course the same axis $A_h = B_h$), but if $n_{h+1} p_{h+1} = m_{h+1} q_{h+1}$ then $A_{h+1}$ and $B_{h+1}$ are isotopic, but not coincident, and it is immaterial from the topological point of view whether $\varphi_h$ is regarded as lying within or without $\sigma_h$.

It should be observed that the branches comprising a singularity are regarded as being unordered, and the component knots of a link are regarded as being unordered and unoriented.

2.4 We note at this point that a certain difficulty arises with our notation in that one or both of the expansions may terminate before the $K_0$ th term and in order to preserve the validity of the above statements it is necessary to make the following convention. If, for example, the former of the two expansions (8) terminates at the $H$ th term, where $H < K_0$, then formally we set $m_{H+1}, m_{H+2}, \ldots, m_K$, equal to unity and $n_{H+1}$ equal to infinity. The justification for this is illustrated by the following example.

Let two branches of a singularity be given by the parametric expansions

$$\begin{align*}
&\begin{cases}
x = t^2 \\
y = t^3
\end{cases} \quad \text{and} \quad \begin{cases}
x = t^4 \\
y = t^5 + t^3
\end{cases}
\end{align*}$$

The corresponding knots are respectively

$A_1 \mid (2,3) \mid$ \quad and \quad $B_2 \mid (2,3); (2,9) \mid$.

The expansions differ for the first time at the second term, this term being absent from the first expansion. $A_1$ is a $(2,3)$ knot on the torus $\varphi_0$ (i.e. $A_1$ lies on $\varphi_0$ and winds round it twice in the direction of the parallels and three times in the direction of the meridians). Figure (i) shows a normalised
projection of $A_v$. $A_v$ and $B_v$ are identical and $B_2$ is a (2,9) knot on $o_v$. Normalised projections of $B_2$ and of the complete link are shown in figures (ii) and (iii) respectively. It is to be especially noted that $A_v$ lies within the surface $o_v$ on which $B_2$ lies.

Now if the second exponent in the expansion of the former of the two branches is regarded formally as being infinite, and the algorithm is carried one stage further, then instead of the knot $A_1 \langle(2,3)\rangle$ one obtains $A_2 \langle(2,3); (1,\infty)\rangle$. If on the other hand $A_1 \langle(2,3)\rangle$ is replaced by the knot $A_2 \langle(2,3); (1,k)\rangle$, $k$ being any positive integer, then $A_3$ is a $(1,k)$ knot on $o_v$ and is isotopic to $A_1$. Also, the link of $A_2$ and $B_2$ is isotopic to the link of $A_1$ and $B_2$; $A_1$, $A_2$ and $o_v$ all lying within $o_v$. We see that if the general criteria of section 2.3 are to remain valid then we must have $2k > 9$. From this it emerges that it is legitimate to replace $A_1$ by $A_2$ provided only that $k$ is chosen sufficiently large, hence the symbolic notation $A_2 \langle(2,3); (1,\infty)\rangle$. But it should be observed that this is purely conventional and that without this proviso the symbol in question does not define a knot in the usual sense.

In this instance $K$ was chosen equal to 2, had however the value of $K$ been chosen as 3 the corresponding knots would have had the symbolic representations

\[
A_3 \langle(2,3); (1,\infty); (1, \quad)\rangle
\]
\[
B_3 \langle(2,3); (2,9); (1, \infty)\rangle
\]

Here, in the algorithm, $n_3$ is indeterminate and its value is in fact of no consequence. The corresponding space in the array is left vacant.

2.5 Returning now to the general discussion; as in the case of a single branch we can again show (14), by first constructing a canonical form for an array which characterises these links, that two singularities (each consisting of two branches given by expansions of the form (8) with the conditions (7) and (9) satisfied) which are similar give rise to two isotopic links.

\(^{(14)}\) Details may be found in [9].
The statement that two singularities which are not similar give rise to two non-isotopic links is also true. The proof of this outlined in the next section is that given in [9].

The position in the general case of a singularity composed of several branches can now be visualised, although a complete description of the Brauner links of arbitrary multiplicity would be somewhat cumbersome and we shall not attempt to undertake it here. It suffices to note that these links have the peculiar property that they are completely determined by the character of the individual knots and of the mutual relations between the pairs of knots which they contain. As an immediate consequence of this and the preceding remarks we are able to state: The two links corresponding to two singularities, each composed of a finite number of branches, are isotopic or not according as the two singularities are or are not of the same similarity type.

3. THE UNIQUENESS THEOREM.

In order to show that two dissimilar singularities, each composed of two branches, give rise to two non-isotopic links, it suffices to show that the Alexander polynomial of a Brauner link of multiplicity two determines uniquely the similarity type of a corresponding singularity.

If the Alexander polynomial of the link with components $A$ and $B$ is $\Delta_{AB}(x, y)$ then it is known (G. Torres [11]) that

\[
\begin{align*}
\Delta_{AB}(1,1) &= v_{AB} \\
\Delta_{AB}(x,1) &= \Delta_A(x) \cdot \frac{(1 - x^{v_{AB}})}{(1-x)} \\
\Delta_{AB}(1, x) &= \Delta_B(x) \cdot \frac{(1 - x^{v_{AB}})}{(1-x)}
\end{align*}
\]

where $v_{AB}$ is the linking coefficient of $A$ and $B$, and $\Delta_A$ and $\Delta_B$ are the Alexander polynomials of $A$ and $B$ respectively. Now, as we shall prove in section (4), $v_{AB}$ is equal to the multiplicity of intersection of any two branches for which $A$ and $B$ are the associated knots. From this and the fact that $\Delta(x)$ determines uniquely the similarity type of a corresponding branch (section (2)) it follows at once that $\Delta(x, y)$ determines the similarity type of a corresponding singularity.
4. The Intersection Multiplicity of Two Branches.

We give in the following paragraphs two proofs of the fact that the multiplicity of intersection of two branches is equal to the linking coefficient of the two corresponding knots. The second, which is the proof given in [9], is perhaps more in line with the present discussion since, unlike the first, it depends directly upon the structure of the Brauner links which we have described above.

4.1 First Proof.

As remarked previously, that part of an algebroid singularity lying within a neighbourhood of the origin may be regarded as a two-dimensional complex immersed in a 4-cell $\Sigma_4$, this complex having no singularities on $\Sigma_4$ other than at the origin. The corresponding Brauner link is then effectively the system of 1-dimensional circuits forming the intersection of this 2-dimensional complex with the boundary $S_3$ of $\Sigma_4$. If a singularity consists of two branches and we denote the corresponding complexes by $a_2$ and $b_2$, then the algebraic multiplicity of intersection of the two branches at the origin is equal to the topological intersection number of the complexes $a_2$ and $b_2$ in $\Sigma_4$. Now S. Lefschetz has shown [7] that this last number is precisely equal to the linking coefficient in $S_3$ of the intersections $a_1$ and $b_1$ of $a_2$ and $b_2$ respectively with $S_3$, thus the theorem is proved (15).

4.2 Second Proof.

In the sequel we use the definition of the linking coefficient of two knots as given by Reidemeister in [10]. That is, given a normalised projection of two knots $A$ and $B$ we assign to each of the crossing points $D_k^{(A)}$ in which $A$ crosses over $B$ a characteristic $\varepsilon_k^{(A)}$ and to each of the crossing points $D_k^{(B)}$ in which $B$ crosses over $A$ a characteristic $\varepsilon_k^{(B)}$. It can be shown that $v_{AB} = |\Sigma_k^{(A)}| = |\Sigma_k^{(B)}|$ is an isotopy invariant of the link; it is known as the linking coefficient of $A$ and $B$. The numerical value of each of the characteristics $\varepsilon_k$ is always unity and in the projections we use for the Brauner links they

\footnote{I am indebted to Professor J. H. C. Whitehead who, in connection with this proof, drew my attention to the above quoted result of Lefschetz.}
all have the same sign, so that in our case $\sigma_{AB}$ is equal either to the number of points $D_k^{(b)}$, or alternatively, to half the total number of points $D_k^{(a)}$ and $D_k^{(b)}$. We propose to count these points, but first we shall give some further examples of the normalised projections which we use and hope that these will suffice to enable the general situation to be visualised.

We consider first the singularity with branches

\[
\begin{align*}
\begin{cases}
x = t^4 \\
y = t^6 + t^9
\end{cases}
\end{align*}
\quad \text{and} \quad
\begin{align*}
\begin{cases}
x = t^2 \\
y = t^3 + t^4
\end{cases}
\end{align*}
\]

In our previous notation the corresponding knots are

\[A \mid (2, 3); (2, 9)\] \quad \text{and} \quad \[B \mid (2, 3); (1, 4)\]

and, since $1.9 > 4.2$, $\varrho_1$ lies inside $\sigma_1$. See figure (iv). As another example consider the singularity with branches

\[
\begin{align*}
\begin{cases}
x = t^4 \\
y = t^6 + t^9
\end{cases}
\end{align*}
\quad \text{and} \quad
\begin{align*}
\begin{cases}
x = t^2 \\
y = 2t^3 + t^4
\end{cases}
\end{align*}
\]

The corresponding knots are again

\[A \mid (2, 3); (2, 9)\] \quad \text{and} \quad \[B \mid (2, 3); (1, 4)\]

but here, since $n_1p_1 = m_1q_1$ and $a_1 \neq b_1$, $\varrho_0$ and $\sigma_0$ are distinct concentric tori and $\varrho_1$ and $\sigma_1$ lie outside each other. In this case the projection can be drawn in two ways according as to which of the two tori $\varrho_0$ and $\sigma_0$ lies outside the other. As was remarked previously, it is irrelevant from the topological point of view which of the two alternatives is chosen. Figure (v) shows the case in which $\varrho_0$ lies inside $\sigma_0$ and figure (vi) shows the case in which $\sigma_0$ lies inside $\varrho_0$.

In figures (iii), (iv), (v) and (vi) all crossing points $D_k^{(a)}$ and $D_k^{(b)}$ are marked with a small dot and it is easily verified that the linking coefficient calculated in this way is in each case equal to the corresponding intersection multiplicity.

We pass on now to the general case and consider first of
all the link

\[ A_k \left( m_1, n_1 \right); \left( m_2, n_2 \right); \ldots; \left( m_{k-1}, n_{k-1} \right) \left( m_k, n_k \right) \] (16)

\[ B_k \left( p_1, q_1 \right); \left( p_2, q_2 \right); \ldots; \left( p_{k-1}, q_{k-1} \right) \left( p_k, q_k \right) \]

consisting of two knots on concentric tubes \( \sigma_{k-1} \) and \( \sigma_{k-1} \).

Suppose that \( n_k \leq m_k q_k \) and that \( \varphi_{k-1} \) lies wholly inside \( \sigma_{k-1} \).

If \( v_k^{(1)} \) denotes the linking coefficient of these knots, then \( v_k^{(1)} \)

is equal to the number of points at which \( B_k \) crosses over \( A_k \)

in a projection of the type illustrated above. By reference to a figure it will be seen that

\[ v_k^{(1)} = q_k m_k + p_k m_k T_{k-1} \]

where \( T_{k-1} \) is the number of self-crossing points of \( B_{k-1} \), the

penultimate knot in the constructional sequence leading to \( B_k^{(1)} \).

Again, by reference to a figure it will be seen that

\[ T_s = q_s (p_s - 1) + p_s^2 T_{s-1} \quad (1 < s < k) \]

\[ T_1 = q_1 (p_1 - 1) \]

(\( T_s \) being the number of self-crossing points of the knot \( B_s \) occurring in the step by step construction of \( B_k \) by means of tubular surfaces).

Consider secondly the link

\[ A_k^{(2)} \left( m_1, n_1 \right); \left( m_2, n_2 \right); \ldots; \left( m_{k-2}, n_{k-2} \right) \left( m_{k-1}, n_{k-1} \right); \left( m_k, n_k \right) \]

\[ B_k^{(2)} \left( p_1, q_1 \right); \left( p_2, q_2 \right); \ldots; \left( p_{k-2}, q_{k-2} \right) \left( p_{k-1}, q_{k-1} \right); \left( p_k, q_k \right) \]

and suppose that \( n_{k-1} \cdot p_{k-1} \leq m_{k-1} \cdot q_{k-1} \) and that \( \varphi_{k-2} \) lies wholly

within \( \sigma_{k-2} \). If \( v_k^{(2)} \) be the linking coefficient of these knots, then again by reference to a figure it will be seen that

\[ v_k^{(2)} = m_k p_k v_k^{(1)} \].

\( ^{(16)} \) The vertical stroke in the array

\[ \left\{ (m_1, n_1); (m_2, n_2); \ldots; (m_h, n_h) \right\} \]

\[ \left\{ (m_{h+1}, n_{h+1}); \ldots; (m_K, n_K) \right\} \]

\[ \left\{ (p_1, q_1); (p_2, q_2); \ldots; (p_h, q_h) \right\} \]

\[ \left\{ (p_{h+1}, q_{h+1}); \ldots; (p_K, q_K) \right\} \]

denotes that the two Puiseux expansions are identical up to and including the \( h \) th terms but that the \( (h + 1) \) th terms differ.
Consider now the general link

$$A^{(r)}_k \{(m_1, n_1); (m_2, n_2); \ldots; (m_{k-r}, n_{k-r})\} \quad \{(m_{k-r+1}, n_{k-r+1}); \ldots; (m_k, n_k)\}$$

$$B^{(r)}_k \{(p_1, q_1); (p_2, q_2); \ldots; (p_{k-r}, q_{k-r})\} \quad \{(p_{k-r+1}, q_{k-r+1}); \ldots; (p_k, q_k)\}$$

where $q_{k-r}$ is assumed to lie wholly within $\sigma_{k-r}$. The following reduction formula for the linking coefficient $\nu^{(r)}_k$ of these last two knots will be seen to be valid

$$\nu^{(r)}_k = m_k p_k \nu^{(r-1)}_{k-1}$$

Applying these results to the knots arising from the branches with expansions (8) which we assume to differ for the first time at the $(h+1)\text{th}$ terms and where we assume in addition that $n_{h+1} p_{h+1} < m_{h+1} q_{h+1}$ and that $q_h$ lies wholly within $\sigma_h$, we obtain the following expression for their linking coefficient, say $L$,

$$L = \nu^{(K-h)}_K = m_K p_K \nu^{(K-h-1)}_{K-1}$$

$$= m_K p_K m_{K-1} p_{K-1} \nu^{(K-h-2)}_{K-2}$$

$$= m_K p_K \ldots m_{h+2} p_{h+2} \nu^{(1)}_{h+1}$$

$$= m_K p_K \ldots m_{h+2} p_{h+2} (m_{h+1} q_{h+1} + m_{h+1} p_{h+1} T_h)$$

where, as above

$$T_s = q_s (p_s - 1) + p_s^2 T_{s-1} \quad (1 < s < h + 1)$$

$$T_1 = q_1 (p_1 - 1)$$

(11)

Since $m_s = p_s \quad (0 < s < h + 1)$, (11) may be replaced by

$$T_s = q_s (m_s - 1) + m_s p_s T_{s-1} \quad (1 < s < h + 1)$$

$$T_1 = q_1 (m_1 - 1)$$

(12)

If the Puiseux expansions of the two branches are thrown
into the new forms

\[
y = a_0x^m + a_0x^{m-1} + a_0x^{m-2} + \ldots
\]

\[
y = b_0x^p + b_0x^{p-1} + b_0x^{p-2} + \ldots
\]

(13)

Then (see for instance [12] chapter 10) the total multiplicity of intersection of these two branches at the origin is the minimum of \( \lambda \) and \( \lambda' \) where

\[
\lambda = mq' + (m, n') q'' + (m, n', n'') q''' + \ldots + (m, n', n'', \ldots, n^{(h)}) q^{(h+1)}
\]

\[
\lambda' = pn' + (p, q') n'' + (p, q', q'') n''' + \ldots + (p, q', q'', \ldots, q^{(h)}) n^{(h+1)}
\]

If a branch, say the first of the two under consideration, has an expansion terminating at the \( h \) th term then \( n^{(h+1)} \) is set formally equal to infinity.

Now the assumption has been made that \( n_{h+1} p_{h+1} \geq m_{h+1} q_{h+1} \).

Since \( n_h = q_h \) this is equivalent to \( (n_{h+1} - m_{h+1} n_h) p_{h+1} \geq (q_{h+1} - p_{h+1} q_h) m_{h+1} \) and since \( m_s = p_s \) (1 \( \leq s \leq h \)) this is equivalent to \( (n_{h+1} - m_{h+1} n_h) p_{1} p_{2} \ldots p_{h+1} \geq (q_{h+1} - p_{h+1} q_h) m_1 m_2 \ldots m_{h+1} \).

Since, (comparing (8) with (13)) \( (n_{h+1} - m_{h+1} n_h) m = m_1 m_2 \ldots m_{h+1} n^{(h+1)} \) and \( (q_{h+1} - p_{h+1} q_h) p = p_1 p_2 \ldots p_{h+1} q^{(h+1)} \) the inequality is again seen to be equivalent to \( n^{(h+1)} p \geq mq^{(h+1)} \) or, what is the same thing, \( \lambda' \geq \lambda \). Therefore in the case under consideration the intersection multiplicity of the two branches is equal to \( \lambda \). Now \( \lambda \) can be rewritten in the form

\[
\lambda = mp \left\{ \frac{q'}{p} + \frac{1}{m_1} \cdot \frac{q'}{p} + \frac{1}{m_1 m_2} \cdot \frac{q''}{p} + \ldots + \frac{1}{m_1 m_2 \ldots m_h} \cdot \frac{q^{(h+1)}}{p} \right\}
\]

or

\[
\lambda = mp \left\{ \frac{q}{p} + \frac{1}{m_1} \left( \frac{q_2}{p_2} - \frac{q_1}{p_1} \right) + \frac{1}{m_1 m_2} \left( \frac{q_3}{p_3 p_2} - \frac{q_2}{p_2} \right) + \right.
\]

\[
+ \ldots + \frac{1}{m_1 m_2 \ldots m_h} \left( \frac{q_{h+1}}{p_{h+1} p_1 \ldots p_2} - \frac{q_h}{p_1 p_2} \right) \}
\]
or again

\[
\lambda = m p \left( \frac{m_1 - 1}{m_1} \cdot \frac{q_1}{\mu_1} + \frac{m_2 - 1}{m_1 m_2} \cdot \frac{q_2}{\mu_1 \mu_2} + \frac{m_3 - 1}{m_1 m_2 m_3} \cdot \frac{q_3}{\mu_1 \mu_2 \mu_3} + \ldots + \frac{m_h - 1}{m_1 m_2 \ldots m_h} \cdot \frac{q_h}{\mu_1 \mu_2 \ldots \mu_h} \right)
\]

If \( T_s (1 \leq s \leq h) \) has the same meaning as in (12) this can be rewritten once more to obtain

\[
\lambda = m p \left( \frac{m_h + 1}{m_1 m_2 \ldots m_{h+1}} \frac{p_{h+1} T_h + m_{h+1} q_{h+1}}{p_1 p_2 \ldots p_{h+1}} \right)
\]

Comparison of this with (10) shows, since \( m = m_1 m_2 \ldots m_K \) and \( p = p_1 p_2 \ldots p_K \), that \( \lambda = L \). Had it been assumed that \( n_{h+1} p_{h+1} \leq m_{h+1} q_{h+1} \) (instead of \( n_{h+1} p_{h+1} \geq m_{h+1} q_{h+1} \)) then it could have been shown in the same way that \( \lambda' \leq \lambda \) and \( L = \lambda' \). This completes the proof that the total intersection multiplicity of two branches is equal to the linking coefficient of the corresponding Brauner knots (17).

5. THE QUADRATIC TRANSFORMATIONS.

In this final section we shall discuss the special (18) quadratic transformation \( (x, y) \rightarrow (x, xy) \). This transformation is particularly suited to our purposes on account of the property that when used for resolving the multiple point of a single algebroid branch \( P \) whose origin \( O \) is assumed to coincide with the origin \( O \) of coordinates the next neighbouring infinitely near point \( O_1 \) on \( P \) again falls, after the resolution, at the origin \( O \) of coordinates provided that originally the branch tangent to \( P \) coincides with the \( x \)-axis (19). Thus in the course of resol-

---

(17) May we take this opportunity to mention that both the fundamental groups and the Alexander polynomials of Brauner links have been discussed by Burau [21], [3], and that a certain degree of simplification was obtained in the analogous treatment in [9] by making use of a suggestion made by Van Kampen in [6].

(19) This is special in the sense that two of the base points of the associated homoloidal net of conics coincide at the origin. An account of its elementary properties may be found for instance in *Algebraic Geometry* by J. G. Semple and L. Roth, Oxford (1949) pages 56-7.

(18) In fact the first neighbourhood of the origin is transformed into the \( y \)-axis, the neighbouring point in the direction of the \( x \)-axis being transformed into the origin.
ving a particular singularity it will in general be necessary, in order to ensure that the sequence of infinitely near points continues to fall at the origin \( O \) of coordinates, to rotate the axes so as to bring the branch tangent into coincidence with the axis of \( x \) before applying the next quadratic transformation.

On account of the possibility that at a certain stage in the resolution of a singularity the new branch tangent may coincide with the \( y \)-axis we must now admit into consideration branches

\[
y = a_1x^{\frac{N_1}{m}} + a_2x^{\frac{N_2}{m}} + a_3x^{\frac{N_3}{m}} + \ldots
\]

\( (N_1 < N_2 < N_3 < \ldots) \)

for which the inequality \( m \leq N_1 \) is no longer imposed.

Below, after first giving topological interpretations in terms of the Brauner knots for both the quadratic transformation and a rotation of axes, we go on to give an almost visual proof of the fact that every algebroid branch can be completely resolved by a finite succession of these transformations.

5.1 The topological transformations induced on the Brauner knots.

We shall now investigate the transformations induced on the Brauner knots by the quadratic transformation and rotation of axes in the \( (x, y) \)-plane. We turn our attention firstly to the quadratic transformation.

5.11 The Quadratic Transformation.

We start by considering the quadratic transformation

\[
\bar{x} = x; \quad \bar{y} = xy
\]

and regard it first of all as a transformation of the plane of the variables \( x \) and \( y \) into that of \( \bar{x} \) and \( \bar{y} \).

The region \( Z : |x| = r, \ |y| \leq r' \) is transformed by (14) into the region \( \bar{Z} : |\bar{x}| = r, \ |\bar{y}| \leq rr' \), and just as equations (2) define a one-one mapping of \( Z \) onto the boundary and interior of the torus \( T \) (see (3) of 2.1) the equations

\[
\begin{cases}
\xi + i\eta = (\overline{R} + it' \cos (\Phi + \Phi')) e^{i\varphi} \\
\zeta = it' \sin (\Phi + \Phi')
\end{cases}
\]

\( (\overline{R} > r r') \)
define a one-one mapping of $\mathbb{Z}$ onto the boundary and interior of the torus $\mathcal{T}$ given by the equations

\[
\begin{aligned}
\zeta &= r\eta \cos (\Phi + \Phi') \\
\bar{\zeta} &= r\eta \sin (\Phi + \Phi')
\end{aligned}
\]

Thus to the quadratic transformation (14) of the $(x, y)$—plane (we think for the moment of the $(x, y)$ and $(x, \bar{y})$—planes as being coincident) there corresponds a transformation of the $(\xi, \eta, \zeta)$—space in which the boundary and interior of $T$ are mapped topologically onto those of $\mathcal{T}$, and this, in such a way that if an $(m, n)$ curve on $T$ is homologous to $ma + nb$ on $T$, then an $(m, n)$ curve on $\mathcal{T}$ is homologous to $ma + (n-m)b$ on $\mathcal{T}$, where $a \to a$ and $b \to b$. Or, in other words, an $(m, n)$ curve on $T$ --- an $(m, n + m)$ curve on $\mathcal{T}$. This can be expressed more graphically as follows. To obtain $\mathcal{T}$ from $T$, cut $T$ open along a meridian circle (it is now a cylinder), holding one boundary circle fixed twist the other about the axis of $T$ through an angle of $2\pi$ and finally, re—identify the two boundary circles.

The Brauner knots lie within these tori and undergo analogous transformations. In fact, the knot

\[
\begin{aligned}
(m_1, n_1); (m_2, n_2); \ldots; (m_K, n_K) \quad &\text{is transformed into} \\
(m_1, n_1 + m_1); (m_2, n_2 + m_1 m_2); \ldots; (m_K, n_K + m_1 m_2 \ldots m_K)
\end{aligned}
\]

When we are concerned with the resolution of a singularity we regard (14) as a transformation from the plane of the variables $x$ and $y$ into that of $x$ and $y$, and in this case (15) is replaced by (20)

\[
\begin{aligned}
(m_1, n_1); (m_2, n_2); \ldots; (m_K, n_K) \quad &\to \\
(m_1, n_1 - m_1); (m_2, n_2 - m_1 m_2); \ldots; (m_K, n_K - m_1 m_2 \ldots m_K)
\end{aligned}
\]

(20) Here, the homeomorphism of the torus $T$ can again be described by the operations of cutting it open and giving it a twist through an angle of $2\pi$, the only difference being that in this case the twist is in the opposite sense and tends to «unwind» the Brauner knot.
5.12 The Rotations.

From the remarks made at the beginning of this section it is clear that we shall find it convenient, during the course of the resolution of a singularity, to rotate the axes (if necessary) so as to bring the branch tangent into coincidence with the \(x\)-axis before making the next quadratic transformation. This rotation \(^{21}\) can be of essentially two distinct types and we now discuss the question in greater detail.

If, after the application of a quadratic transformation, the branch tangent already coincides with the \(x\)-axis then no rotation is needed. If on the other hand the branch tangent coincides with neither of the coordinate axes the first pair \((m_1, n_1)\) of the associated array \(|(m_1, n_1); (m_2, n_2); \ldots; (m_K, n_K)|\) will have \(m_1 = n_1 = 1\). In this case, as can be seen by considering the Puiseux expansion of the branch \(^{22}\), to a rotation which carries the branch tangent into the \(x\)-axis there will correspond the deletion from the array of the first pair \((m_1, n_1)\). These rotations are of the first kind. The corresponding transformation of the Brauner knot can be written

\[
| (1, 1); (m_2, n_2); \ldots; (m_K, n_K) | \rightarrow | (m_2, n_2); \ldots; (m_K, n_K) | \quad (17)
\]

Finally we must consider the case in which, after the application of a quadratic transformation, the branch tangent coincides with the \(y\)-axis. Here, as can again be seen by considering the Puiseux expansion of the branch \(^{23}\), to a rotation which carries the branch tangent into the \(x\)-axis there will correspond the transformation of the Brauner knot

\[
\left\{ \begin{array}{c}
(m_1, n_1); (m_2, n_2); \ldots; (m_K, n_K) \\
(n_1, m_1); (m_2, n_2 - m_2(n_1 - n_1)); (m_3, n_3 - m_3m_2(n_1 - n_1)); \ldots \\
\ldots; (m_K, n_K - m_Km_2m_3 \ldots m_K(n_1 - n_1))
\end{array} \right. \quad (18)
\]

We shall call these rotations of the second kind.

\(^{21}\) By rotation we here mean a transformation of coordinates of the form

\[ x' = ax + by \quad \text{and} \quad y' = cx + dy \quad \text{with} \quad ad \neq bc. \]

\(^{22}\) It is sufficient to consider a rotation in which the \(y\)-axis remains fixed.

\(^{23}\) It is sufficient to consider the rotation which interchanges the \(x\) and \(y\) axes. The inversion of the Puiseux expansion is quite straightforward and need not detain us here.
The transformations (17) and (18), corresponding to rotations of the first and second kinds respectively, are both isotopies of the BRAUNER knots.

Before going on we draw attention to a point which should be mentioned in connection with the knot arising from a branch for which \( m > N_1 \). This inequality is equivalent to \( m_1 > n_1 \). Thus, although we shall be dealing with such knots in the following paragraphs, strictly speaking they are not BRAUNER knots under the definition given previously. The restriction \( m_1 \leq n_1 \), imposed originally, could be removed from the preceding work but there would then arise an additional complication in connection with the representation of the BRAUNER knots. To illustrate this it is sufficient to observe that even in the simplest case of a torus knot the representation \( (m, n) \) would no longer be unique since there would be in this case the equally good representation \( (n, m) \).

### 5.2 The resolution of singularities.

Suppose we are given an algebroid branch whose branch tangent is in coincidence with the \( x \)-axis and whose associated BRAUNER knot \( A_K \) has the canonical array

\[
A_K \langle (m_1, n_1); (m_2, n_2); \ldots; (m_K, n_K) \rangle \tag{19}
\]

If we apply the quadratic transformation (14) to this branch, the transformation being regarded as a resolution, the corresponding transform (see (16)) of \( A_K \) is

\[
\langle (m_1, n_1 - m_1); (m_2, n_2 - m_1 m_2); \ldots; (m_K, n_K - m_1 m_2 \ldots m_K) \rangle. \tag{20}
\]

We are concerning ourselves here with the transformation (14) considered as an instrument for the resolution of singularities. However we could equally well regard it as an instrument for their construction. In this case, after each quadratic transformation the branch tangent will coincide with the \( x \)-axis and we are at liberty to rotate the axes as we wish before making the next transformation. To the quadratic transformation will correspond the transformation (15) and to the rotations, which will again be of two essentially distinct types, will correspond the inverses of (17) and (18). (18) incidentally is an involutary operation and coincides with its inverse. We have here the means to interpret for the BRAUNER knots the concept of proximity relations between the points of an algebroid branch lying in successive neighbourhoods of its origin. Using the fact that the multiplicity of the first point in the multiplicity sequence of a branch whose BRAUNER knot has the array \( \{ (m_0, n_0); (m_0, n_0); \ldots; (m_n, n_n) \} \) is equal to \( m_1 m_2 \ldots m_n \) (in the case of a knot for which \( m_0 > n_0 \) the multiplicity in question is \( n_1 n_2 n_3 \ldots n_n \)) it is not difficult to verify that the multiplicity of a point is equal to the sum of the multiplicities of its proximate points.
Two things can happen, either the branch tangent remains distinct from the $y$–axis or it does not. In the first case, as a consequence of the inequalities (6) and the fact that (19) is canonical, (20) will also be canonical. We now rotate the axes, if necessary, so as to bring the new branch tangent into coincidence with the $x$–axis. In the first case above (20) is still the canonical form of the BRAUNER knot associated with the branch. In the second case (20) is replaced (see (18)) by

$$\{(n_1 - m_1, m_1); (m_2, n_2 + m_2 (m_1 - n_1)); ...; (m_K, n_K + m_K m_2 ... m_K (m_1 - n_1))\}$$

which will also be in canonical form.

We suppose now that we can apply the quadratic transformation (14) $k$ times to our original branch (always following it with a rotation of the first kind when necessary) before the new branch tangent comes into coincidence with the $y$–axis.

By applying (16) $k$ times in succession to (19) we obtain, for the knot associated with this final branch, the array

$$\{(m_1, n_1 - km_1); (m_2, n_2 - km_1 m_2); ...; (m_K, n_K - km_1 m_2 ... m_K)\}. \tag{21}$$

By our definition, $k$ is the largest integer for which $n_1 - km_1$ is positive \(^{(25)}\). Thus

$$m_1 > n_1 - km_1 > 0. \tag{22}$$

We note that as a consequence of the inequalities (6) $n_1 - km_1 > 0$ implies $n_v - km_1 m_2 ... m_{v-1} > 0$ ($v = 1, 2, ... , K$).

We now apply the deformation (18), which corresponds to an interchange of axes, to (21) and obtain

$$\{(n_1 - km_1, m_1); (m_2, n_2 + m_2 (m_1 - n_1)); ...; (m_K, n_K + m_K m_2 ... m_K (m_1 - n_1))\}.$$ 

For convenience we denote the knot defined by this array

\(^{(25)} n_1 - km_1 \) cannot be zero since $(m_1, n_1) = 1$ and $m_1 > 1$, this latter being a consequence of our assumption that (19) is canonical.
by $A_K'$ and rewrite it as

$$A_K' | (m_1', n_1'); (m_2', n_2'); ...; (m_K', n_K')$$

Further, we denote the complete sequence of above transformations by $Q$, thus

$$Q(A_K) = A_K'.$$

Now as a consequence of (22) we have

$$m_1' < m_1$$

and it is easily verified that in addition

$$m_1' < n_1'; n_1'm_2' < n_2'; n_2'm_3' < n_3'; ...; n_{K-1}' m_K' < n_K'$$

and $(m_v', n_v') = 1$ ($v = 1, 2, ..., K$). These being the analogues for $A_K'$ of the corresponding relations holding for $A_K$.

It is now clear that after a finite number, say $v$, of transformations $Q$ we shall obtain a knot

$$Q^v(A_K) = A_K^{(v)} | (m_1^{(v)}, n_1^{(v)}); (m_2^{(v)}, n_2^{(v)}); ...; (m_K^{(v)}, n_K^{(v)})$$

for which $m_1^{(v)} = 1$. We can then delete the pair $(m_1^{(v)}, n_1^{(v)})$ obtaining for $A_K^{(v)}$ a canonical array containing only $(K - 1)$ pairs of integers. We can therefore rename $A_K^{(v)}$ more suitably as $A_{K-1}$.

Applying repeatedly the entire process by which we have obtained $A_{K-1}$ from $A_K$ we obtain finally a knot $A_1 | (m, n)$ which in turn can be reduced to $A_1 | (1, n_0)$. This latter is isotopic to a circle, and our original branch will have been reduced to a simple branch passing through the origin of coordinates.

This completes a proof of the fact, which in terms of Brauner knots becomes almost intuitively obvious, that every algebroid branch may be completely resolved by means of a finite sequence of special quadratic transformations of type $(x, y) \rightarrow (x, xy)$. 


BIBLIOGRAPHY


Fig. (v)
Fig. (vi)