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ON THE NEWTON-NELSON TYPE EQUATIONS ON VECTOR BUNDLES WITH CONNECTIONS

Abstract. An equation of Newton-Nelson type on the total space of vector bundle with a connection, whose right-hand side is generated by the curvature form, is described and investigated. An existence of solution theorem is obtained.

Introduction

In [5] (see also [6]) a certain second order differential equation on the total space of vector bundle with a connection was constructed and investigated. In some cases it was interpreted as an equation of motion of a classical particle in the classical gauge field. The form of this equation allowed one to apply the quantization procedure in the language of Nelson’s Stochastic Mechanics (see, e.g., [8, 9]). In [7] this procedure was realized for the vector bundles over Lorentz manifolds with complex fibers. The corresponding relativistic-type Newton-Nelson equation (the equation of motion in Stochastic Mechanics) was constructed and the existence of its solutions under some natural conditions was proved. The results of [7] were interpreted as the description of motion of a quantum particle in the gauge field.

In this paper we consider the analogous non-relativistic Newton-Nelson equation in the situation where the base of the bundle is a Riemannian manifold and the fiber is a real linear space. In this case some deeper results are obtained under some less restrictive conditions with respect to the case of [7].

We refer the reader to [2, 6] for the main facts of the geometry of manifolds and to [4, 6] for general facts of Stochastic Analysis on Manifolds.

1. Necessary facts from the Geometry of Manifolds

Recall that for every bundle $E$ over a manifold $M$, in each tangent space $T_{(m,x)}E$ to the total space $E$ there is a special sub-space $V_{(m,x)}$, called vertical, that consists of the vectors tangent to the fiber $E_m$ (called also vertical). In the case of principal or vector bundle, a connection $H$ on $E$ is a collection of sub-spaces in tangent spaces to $E$ such that $T_{(m,x)}E = H_{(m,x)} \oplus V_{(m,x)}$ at each $(m,x) \in E$ and this collection possesses some properties of smoothness and invariance (see, e.g., [6]).

Denote by $\mathcal{M}$ a Riemannian manifold with metric tensor $g(\cdot, \cdot)$. Let $\Pi : \mathcal{E} \to \mathcal{M}$ be a principal bundle over $\mathcal{M}$ with a structure group $G$. By $\mathfrak{g}$ we denote the Lie algebra of $G$. Let a connection $H$ with connection form $\theta$ and curvature form $\Phi = D\theta$ be given

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on \( \mathcal{E} \). Here \( D \) is the covariant differential (see, e.g., [2]). Recall that the 1-form \( \theta \) and the 2-form \( \Phi \) are equivariant and take values in the algebra \( g \) of \( G \) and that \( \Phi \) is horizontal (equals zero on vertical vectors).

We suppose \( G \) to be a subgroup of \( GL(k, \mathbb{R}) \) for a certain \( k \). Let \( \mathcal{F} \) be a \( k \)-dimensional real vector space, on which \( G \) acts from the left, and let on \( \mathcal{F} \) an inner product \( b(\cdot, \cdot) \), invariant with respect to the action of \( G \), be given. We suppose that a mapping \( e: \mathcal{F} \to g^* \) (where \( g^* \) is the co-algebra) having constant values on the orbits of \( G \), is given. This mapping is called charge.

Consider the vector bundle \( \pi: Q \to \mathcal{M} \) with standard fiber \( \mathcal{F} \), associated to \( \mathcal{E} \). We denote by \( Q_m \) the fiber at \( m \in \mathcal{M} \). Consider the factorization \( \lambda: \mathcal{E} \times \mathcal{F} \to Q \) that yields the bundle \( Q \) (see [2]). The tangent mapping \( T\lambda \) translates the connection \( H \) from the tangent spaces to \( \mathcal{E} \) to tangent spaces to \( Q \). This connection on \( Q \) is denoted by \( H^Q \).

Recall that the spaces of connection are the kernels of operator \( K^\pi: TQ \to Q \) called connector, that is constructed as follows. Consider the natural expansion of the tangent vector \( X \in T_{(m,q)}Q \) at \((m,q) \in Q \) into horizontal and vertical components \( X =HX + VX \), where \( HX \in H^\pi(m,q) \) and \( VX \in V_{(m,q)} \). Introduce the operator \( p: V_{(m,q)} \to Q_m \), the natural isomorphism of the linear tangent space \( V_{(m,q)} \to T_qQ_m \) to the fiber \( Q_m \) of \( Q \) onto the fiber (linear space) \( Q_m \). Then \( K^\pi X = pVX \).

On the manifold \( Q \) (the total space of bundle) we construct the Riemannian metric \( g^Q \) as follows: in the horizontal subspaces \( H^\pi \) we introduce it as the pull-back \( T^\pi g \), in the vertical subspaces \( V \) – as \( h \) and define that \( H^\pi \) are \( V \) orthogonal to each other.

We denote the projection of tangent bundle \( TM \) to \( \mathcal{M} \) by \( \tau: TM \to \mathcal{M} \) and by \( H^\tau \) the Levi-Civita connection of metric \( g \) on \( \mathcal{M} \). Its connector is denoted by \( K^\tau: T^2M \to TM \). The construction of \( K^\tau \) is quite analogous to that of \( K^\pi \) where \( Q \) is replaced by \( TM \) and \( TQ \) by \( T^2M = TT\mathcal{M} \).

Recall the standard construction of a connection on the total space of bundle \( Q \), based on the connections \( H^\pi \) and \( H^\tau \) (see, e.g., [3, 6]). The connector \( K^Q: T^2Q \to TQ \) of this connection has the form: \( K^Q = K^H + K^V \) where \( K^H: T^2Q \to H^\pi \) and \( K^V: T^2Q \to V \), and the latter connectors are introduced as: \( K^H = T\pi^{-1} \circ K^\pi \circ T^2\pi \) where \( T^2\pi = T(T\pi): T^2Q \to T^2\mathcal{M} \) and \( T\pi^{-1} \) is the linear isomorphism of tangent spaces to \( \mathcal{M} \) onto the spaces of connection \( H^\pi; K^V = p^{-1} \circ K^\pi \circ TK^\pi \).

Recall that \( \lambda \) is a one-to-one mapping of the standard fiber \( \mathcal{F} \) onto the fibers of bundle \( Q \), hence the charge \( e \) is well-defined on the entire \( Q \). Since \( T\lambda \) is also a one-to-one mapping of the connections and \( \Phi \) is equivariant, we can introduce the differential form \( \tilde{\Phi} \) on \( Q \) with values in \( g \) as follows. Consider \((m,q) = \lambda((m,p), f) \) for \((m,p) \in \mathcal{E} \) and \( f \in \mathcal{F} \). For \( X, Y \in T_{(m,q)}Q \) we denote by \( HX \) and \( HY \) their horizontal components. Then we define \( \tilde{\Phi}_{(m,q)}(X,Y) = \Phi_{(m,p)}(T\lambda^{-1}H\pi X, T\lambda^{-1}H\pi Y) \).

Denote by \( \Phi \) the coupling of elements of \( g \) and \( g^* \). Consider the vector \((m,q), X \) tangent to \( Q \) at \((m,q) \). It is clear that \( e((m,q)) \circ \Phi_{(m,q)}(\cdot, X) \) is an ordinary 1-form (i.e., differential form with values in real line). Denote by \( e((m,q)) \circ \Phi_{(m,q)}(\cdot, X) \) the tangent vector to the total space of \( Q \) physically equivalent to the form \( e((m,q)) \circ \tilde{\Phi}_{(m,q)}(\cdot, X) \) (i.e., obtained by lifting the indices with the use of Riemannian metric \( g^Q \)).
LEMMA 1 ([5]). The vector field \( e((m,q)) \Phi_{(m,q)}(\cdot, X) \) is horizontal, i.e., it belongs to the spaces of connection \( H^\pi \).

THEOREM 1 ([7]). Let \((m(t), q(t))\) be a smooth curve in \( Q \). Let \( X(t) \) be the parallel translation of the vector \( X \in T_{(m(t),q(t))}Q \) along \((m(t), q(t))\) with respect to \( H^0 \). (i) Both the horizontal \( HX(t) \) and vertical \( VX(t) \) components of \( X(t) \) are parallel along \((m(t), q(t))\) with respect to \( H^0 \). (ii) The parallel translation of horizontal vectors preserves constant the norms and scalar products with respect to \( g^0 \). (iii) The vector field \( T\pi X(t) \) is parallel along \( m(t) \) on \( M \) with respect to \( H^\pi \).

2. Mean derivatives on manifolds and vector bundles

Consider a stochastic process \( \xi(t) \) with values in \( M \), given on a certain probability space \((\Omega, \mathcal{F}, P)\). By \( \mathcal{B}_\mathbb{F} \) we denote the minimal \( \sigma \)-sub-algebra of \( \sigma \)-algebra \( \mathcal{F} \) generated by the pre-images of Borel sets in \( M \) under the mapping \( \xi(t) : \Omega \to M \) (the “present” or “now” of \( \xi(t) \)) and by \( E(\cdot | \mathcal{B}_\mathbb{F}) \) the conditional expectation with respect to \( \mathcal{B}_\mathbb{F} \).

Recall that the conditional expectation of a random element \( \Theta \) with respect to \( \mathcal{B}_\mathbb{F} \) can be represented as \( \Theta(\xi(t)) \) where \( \Theta \) is the so-called regression introduced by the formula \( \Theta(m) = E(\Theta | \xi(t) = m) \) (see, e.g., [10]).

Specify a point in \( M \) and consider the normal chart \( U_m \) at this point with respect to the exponential mapping of Levi-Civita connection on \( M \). In \( U_m \) construct the following regressions

\[
Y^{U_m}(t, m') = \lim_{\Delta \downarrow 0} E_{\mathcal{B}_\mathbb{F}} \left( \frac{\xi(t + \Delta) - \xi(t)}{\Delta} \mid \xi(t) = m' \right).
\]

\[
U^{*}(t, m') = \lim_{\Delta \downarrow 0} E_{\mathcal{B}_\mathbb{F}} \left( \frac{\xi(t) - \xi(t - \Delta)}{\Delta} \mid \xi(t) = m' \right).
\]

Introduce \( X^0(t, m) = Y^{U_m}(t, m) \) and \( X^0(t, m) = Y^{U_m}(t, m) \). Note that \( X^0(t, m) \) and \( X^0(t, m) \) are vector fields on \( M \), i.e., under the coordinate changes they transform like cross-sections of the tangent bundle \( TM \).

Forward and backward mean derivatives of \( \xi(t) \) are defined by the formulae \( D\xi(t) = \xi^0(t, \xi(t)) \) and \( D\xi(t) = \xi^0(t, \xi(t)) \).

The vector \( v^\xi(t) = \frac{1}{\Delta}(D + D_\xi)\xi(t) \) is called the current velocity of \( \xi(t) \). From the properties of conditional expectation it follows that there exists a Borel measurable vector field (regression) \( \nu^\xi(t, m) \) on \( M \) such that \( v^\xi(t) = \nu^\xi(t, \xi(t)) \).

Introduce the increment \( \Delta\xi(t) \) of process \( \xi(t) \): \( \Delta\xi(t) = \xi(t + \Delta) - \xi(t) \) and the so called quadratic mean derivative \( D_2 \) (see [1, 6]) \( D_2\xi(t) = \lim_{\Delta \downarrow 0} E_{\mathcal{B}_\mathbb{F}} \left( \frac{\Delta\xi(t) \otimes \Delta\xi(t)}{\Delta} \mid \mathcal{B}_\mathbb{F} \right) \). If \( D_2\xi(t) \) exists, it takes values in \( (2, 0) \)-tensors.

Everywhere below we are dealing with processes, along which the parallel translation with respect to an appropriate connection is well-posed. Here we use \( \xi(t) \) and parallel translation with respect to the connection \( H^\pi \) and such an assumption is
true, for example, if $\xi(t)$ is an Itô process on $\mathcal{M}$, i.e., an Itô development of an Itô process in a certain tangent space to $\mathcal{M}$ as it is defined in [6]. Denote by $\Gamma_{t,s}$ the operator of such parallel translation along $\xi(t)$ of tangent vectors from the (random) point $\xi(s)$ of the process to the (random) point $\xi(t)$.

For a vector field $Z(t, m)$ on $\mathcal{M}$ the covariant forward and backward mean derivatives $DZ(t, \xi(t))$ and $D^*Z(t, \xi(t))$ are constructed by the formulae

$$
DZ(t, \xi(t)) = \lim_{\Delta t \to 0} E \left( \Gamma_{t,t+\Delta t} Z(t + \Delta t, \xi(t + \Delta t)) - Z(t, \xi(t)) \ | \ | \mathfrak{N} \right); \\
D^*Z(t, \xi(t)) = \lim_{\Delta t \to 0} E \left( Z(t, \xi(t)) - \Gamma_{t,t-\Delta t} Z(t - \Delta t, \xi(t - \Delta t)) \ | \ | \mathfrak{N} \right).
$$

From formulae (1), (2), (3) and (4) it evidently follows that $T\pi DZ(t, \xi(t)) = D\xi(t)$ and $T\pi D^*Z(t, \xi(t)) = D^*\xi(t)$.

Now consider a stochastic process $\eta(t)$ in the total space of bundle $\mathcal{Q}$ and introduce the process $\xi(t) = \pi \eta(t)$ on $\mathcal{M}$. Denote by $\Gamma^\eta_{t,s}$ the parallel translation of random vectors from the fiber $\mathcal{Q}_{\xi(s)}$ to the fiber $\mathcal{Q}_{\xi(s)}$ along $\xi(t)$ with respect to connection $H^\eta$. For $\eta(t)$ we introduce the covariant mean derivatives by formulae

$$
D\eta(t) = \lim_{\Delta t \to 0} E \left( \Gamma^\eta_{t,t+\Delta t} \eta(t + \Delta t) - \eta(t) \ | \ | \mathfrak{N} \right); \\
D^*\eta(t) = \lim_{\Delta t \to 0} E \left( \eta(t) - \Gamma^\eta_{t,t-\Delta t} \eta(t - \Delta t) \ | \ | \mathfrak{N} \right).
$$

(analogs of (3) and (4)). As above, $v^\eta(t) = \frac{1}{\Delta t} (D + D^*) \eta(t)$ is called the current velocity of $\eta(t)$.

In order to define the mean derivatives of a vector field along $\eta(t)$ on $\mathcal{Q}$ we use the parallel translation $\Gamma^\eta_{t,s}$ of vectors tangent to $\mathcal{Q}$ at $\eta(s)$, to vectors tangent to $\mathcal{Q}$ at $\eta(t)$ along $\eta(t)$ with respect to connection $H^\eta$. By analogy with formulae (3) and (4) for a vector field $Z(t, (m, q))$ on $\mathcal{Q}$ we introduce the covariant mean derivatives by formulae

$$
D^\eta Z(t, \eta(t)) = \lim_{\Delta t \to 0} E \left( \Gamma^\eta_{t,t+\Delta t} Z(t + \Delta t, \eta(t + \Delta t)) - Z(t, \eta(t)) \ | \ | \mathfrak{N} \right); \\
D^*_\eta Z(t, \eta(t)) = \lim_{\Delta t \to 0} E \left( Z(t, \eta(t)) - \Gamma^\eta_{t,t-\Delta t} Z(t - \Delta t, \eta(t - \Delta t)) \ | \ | \mathfrak{N} \right).
$$

**Lemma 2.** $\Gamma^\eta_{t,s}$ translates $H^\eta_{\eta(s)}$ onto $H^\eta_{\eta(t)}$ and $\nabla^\eta_{\eta(s)}$ onto $\nabla^\eta_{\eta(t)}$: the parallel translation of horizontal components preserves the norms and inner products with respect to $g^\eta$.

The assertion of Lemma 2 follows from Theorem 1 and from the fact that (see [3, 6]) that the parallel translation along random processes can be described as the limit
of parallel translations along the processes whose sample paths are piece-wise geodesic approximations of the sample paths of the process under consideration.

By symbols $\mathbf{D}^H$ and $\mathbf{D}^V$ we denote the derivatives introduced by formulae (7) and (8), respectively, for the horizontal components of vectors (i.e., taking values in $\mathbb{H}^n$). By symbols $\mathbf{D}^H$ and $\mathbf{D}^V$ we denote the derivatives for vertical components (i.e., taking values in $\mathbb{V}$). Thus, $\mathbf{D}^Q = \mathbf{D}^H + \mathbf{D}^V$ and $\mathbf{D}^Q = \mathbf{D}^H + \mathbf{D}^V$.

3. The Newton-Nelson equation on the total space of vector bundle

In the problem under consideration the Newton-Nelson equation takes the form

$$
(9) \quad \left\{ \begin{array}{l}
\frac{1}{2} (\mathbf{D}^Q \mathbf{D}_* + \mathbf{D}^Q \mathbf{D}) \eta(t) = e(\eta(t)) \bullet \Phi_{\eta(t)}(\cdot, \nu^H(t)) \\
\frac{1}{2} D_2 \xi(t) = \frac{h}{m} I
\end{array} \right.,
$$

where $\xi(t) = \pi \eta(t)$ (cf. [8, 9]).

Expand the current velocity $\nu^H$ in the right-hand side of (9) into the sum of vertical and horizontal components: $\nu^H = \nu^H_\eta + \nu^V_\eta$, where $\nu^H_\eta \in \mathbb{H}^n$ and $\nu^V_\eta \in \mathbb{V}$. Since $\Phi_{\eta(t)}(\cdot, \cdot)$ is linear in both arguments, $\Phi_{\eta(t)}(\cdot, \nu^H_\eta) = \Phi_{\eta(t)}(\cdot, \nu^H_\eta) + \Phi_{\eta(t)}(\cdot, \nu^V_\eta)$. Then, since the form $\Phi$ is horizontal (see Lemma 1) we obtain that $\Phi_{\eta(t)}(\cdot, \nu^V_\eta) = 0$. Thus, the first equation of system (9) is equivalent to the following system:

$$
(10) \quad \frac{1}{2} (\mathbf{D}^H \mathbf{D}_* + \mathbf{D}^H \mathbf{D}) \eta(t) = e(\eta(t)) \bullet \Phi_{\eta(t)}(\cdot, \nu^H_\eta(t)),
$$

$$
(11) \quad \frac{1}{2} (\mathbf{D}^V \mathbf{D}_* + \mathbf{D}^V \mathbf{D}) \eta(t) = 0.
$$

For simplicity of presentation we denote $e(\eta(t)) \bullet \Phi_{\eta(t)}(\cdot, \nu^H_\eta(t))$ by $\alpha_{\eta(t)}(\cdot, \nu^H_\eta(t))$, where, by construction, $\alpha_{\eta(t)}(\cdot, \nu^H_\eta(t))$ is a linear operator in $\mathbb{H}^n_\eta ((1,1)$-tensor).

Introduce the horizontal $(1,2)$-tensor field $v^H T \alpha(\cdot, \cdot)$ on $\mathbb{Q}$. The vector $\text{tr} \nabla^H \alpha(\cdot, \cdot)$ is horizontal by construction.

THEOREM 2. Let for the tensor field $\alpha_{\eta(\cdot, \tau), \epsilon}(\cdot) \therefore$ there exist a constant $C > 0$ such that $\int^T_0 (||\alpha_{\eta(t), \epsilon}(\cdot)||^2 + ||\text{tr} \nabla^H \alpha_{\eta(t), \epsilon}(\cdot)||^2) dt < C$ for a certain $T > 0$ and every continuous curve $x(t)$ in $\mathbb{Q}$ given on $t \in [0, T]$. Here $\|\alpha_{\eta(t), \epsilon}(\cdot)\|$ is the operator norm (all the norms are generated by $g^Q$). Let also the connections $\mathbb{H}^n$ and $\mathbb{H}^n$ be stochastically complete (see [6]). Then for every point $(m, q) \in \mathbb{Q}$, every vector $p_0 \in \mathbb{H}^n_{(m, q)}$ and every time instant $t_0 \in (0, T)$ there exists a stochastic process $\eta(t)$ in $\mathbb{Q}$ such that: (i) it is well-defined on $[0, T]$; (ii) $\eta(0) = (m, q)$ and $D\eta(0) = p_0$; (iii) for all $t \in (t_0, T)$ the processes $\eta(t)$ and $\xi(t) = \pi \eta(t)$ satisfy (9); (iv) along $\eta(t)$ the charge $e(\eta(t))$ is constant.

Proof. For simplicity and without loss of generality we suppose that $\frac{h}{m} = 1$.

Consider on the space of continuous curves $C^0([0, T], \mathcal{T}_m \mathcal{M})$ the filtration $\mathcal{F}$, where for every $t \in [0, T]$ the $\sigma$-algebra $\mathcal{F}_t$ is generated by cylinder sets with bases
over [0, t]. Consider the Wiener measure \( \nu \) on the measure space \((C^0([0,T], \mathbb{R}^d), \mathcal{F}_T)\) and so the standard Wiener process \( W_m(t) \) in \( \mathbb{T}^m M \) as the coordinate process on the probability space \((C^0([0,T], \mathbb{R}^d), \mathcal{F}_T, \nu)\). Since \( \mathbb{H}^m \) is stochastically complete, the Itô development \( W^M(t) \) of \( W_m(t) \) with respect to \( \mathbb{H}^m \) on \( M \) is well-posed. Since \( \mathbb{H}^m \) is also stochastically complete, the horizontal lift \( W^Q(t) \) of \( W^M(t) \) onto \( Q \) with respect to \( \mathbb{H}^m \) with initial condition \((m, q)\) is also well-posed. A detailed description of the construction of processes \( W^M(t) \) and \( W^Q(t) \) can be found in [6].

Since \( T \pi : \mathbb{H}^m \rightarrow T_m M \) is a linear isomorphism that defines the metric tensor \( g^\mathbb{H} = g^Q \) in \( \mathbb{H}^m \) by the pull back of \( g \) from \( T_m M \), we can translate the Wiener measure and the Wiener process from \( T_m M \) to \( \mathbb{H}^m \). Denote by \( W(t) \) the Wiener process obtained by this construction. This is a coordinate process on the space of continuous curves in \( \mathbb{H}^m \) with \( \sigma \)-algebra \( \mathcal{F}_t \) and Wiener measure.

For \( t_0 \geq 0 \) we introduce the real-valued function \( t_0(t) \) that equals \( \frac{1}{t_0} \) for \( t < t_0 \) and \( \frac{1}{t} \) for \( t \geq t_0 \). Its derivative \( t'_0(t) \) is equal to 0 for \( t < t_0 \) and to \( -\frac{1}{t^2} \) for \( t \geq t_0 \).

Now consider the following Itô equation in \( \mathbb{H}^m \):

\[
\beta(t) = \beta_0 + \frac{1}{2} \int_0^t \Gamma^\mathbb{H}_{0,s} tr \nabla^\mathbb{H} \alpha_{(s, W_0 q)} (\alpha, \cdot) ds + \int_0^t \Gamma^\mathbb{H}_{0,s} \alpha_{(s, W_0 q)} dW(s) - \frac{1}{2} \int_0^t t_0(s) \beta(s) ds - \frac{1}{2} \int_0^t t_0'(s) W(s) ds.
\]

(12)

Since equation (12) is linear in \( \beta \), it has a strong and strongly unique solution \( \beta(t) \). Since this solution is strong, it can be given on the space of continuous curves in \( \mathbb{H}^m \) equipped with Wiener measure. Consider the following density on the latter space of curves \( \theta(t) = \exp \left(-\frac{1}{2} \int_0^t \beta(s)^2 ds + \frac{1}{2} \int_0^t (\beta(s) \cdot dW(s)) \right) \). From the hypothesis and from Lemma 2 it follows that it is well-posed. Introduce the measure that has this density with respect to the Wiener measure. It is well-known that with the new measure the coordinate process takes the form \( \xi(t) = \int_0^t \beta(s) ds + w(t) \) where \( w(t) \) is a certain Wiener process adapted to \( \mathcal{F}_t \). Denote \( W^Q(t) \), considered with respect to the new measure, by the symbol \( \eta(t) \) and introduce the process \( \xi(t) = \pi \eta(t) \); \( \xi(t) \) is obtained from \( W^M(t) \) by the change of measure. Equation (12) turns into

\[
\beta(t) = \beta_0 + \frac{1}{2} \int_0^t \Gamma^\mathbb{H}_{0,s} tr \nabla^\mathbb{H} \alpha_{(s, W_0 q)} (\alpha, \cdot) ds + \int_0^t \Gamma^\mathbb{H}_{0,s} \alpha_{(s, W_0 q)} \beta(s) ds
\]

\[
+ \int_0^t \left( \Gamma^\mathbb{H}_{0,s} \alpha_{(s, W_0 q)} (\cdot) + \frac{1}{2} t_0(s) \right) dw(s) - \frac{1}{2} \int_0^t t_0(s) \beta(s) ds - \frac{1}{2} \int_0^t t_0'(s) \xi(s) ds.
\]

By construction, \( \eta(0) = (m, q) \) and \( D_\eta(t) = \beta_0 \). The process \( \eta(t) \) satisfies (11) also by construction. The fact that for \( t \in (t_0, T) \) the processes \( \eta(t) \) and \( \xi(t) = \pi \eta(t) \) satisfy (10) and that \( D_2 \xi(t) = I \) follows from the formulae for mean derivatives obtained in [6, Chapters 12 and 18].

Evidently \( \eta(t) \) is the horizontal lift of the process \( \xi(t) \) with respect to connection \( \mathbb{H}^m \) with the initial condition \((m, q)\). Recall that the horizontal lift \( \eta(t) \) of \( \xi(t) \) is a
parallel translation of \((m,q)\) along \(\xi(\cdot)\) with respect to \(H^\pi\). Hence, it can be presented in the form \((\xi(t), b_t(f))\) where \(b_t\) is the horizontal lift of \(\xi(t)\) to \(E\) with respect to connection \(H\) and \(f\) is a certain vector in the standard fiber \(F\). Thus, the sample paths of \(\eta(t)\) belong to an orbit of \(G\) and so the charge \(e\) is constant along \(\eta(t)\).

\[\square\]

References


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