CONTENTS

Preface .................................................. 1

D. Faenzi, A one-day tour of representations and invariant of quivers .......................... 3

L. Manivel, Prehomogeneous spaces and projective geometry ................................. 35

G. Ottaviani, Five lectures on projective invariants .............................................. 119
Preface

This volume contains the proceedings of the school “School (and Workshop) on Invariant Theory and Projective Geometry”, held in Povo–Trento (Italy) from September 17 to September 22, 2012.

The meeting was jointly supported by CIRM – Fondazione Bruno Kessler, GNSAGA–INdAM, GRIFGA, Dipartimento di Matematica–Università degli Studi di Trento, Dipartimento di Scienze Matematiche–Politecnico di Torino, Dipartimento di Matematica–Politecnico di Milano and Dipartimento di Matematica–Università degli Studi di Roma “Tor Vergata” as part of the joint project “Spazi di Moduli e teoria di Lie” cofinanced by Italian MIUR.

The papers contained in the present volume are the expanded version of the lectures delivered by the main speakers, L. Manivel, G. Ottaviani and D. Faenzi, during the school. They amply illustrate the different aspects and facets of invariant theory and its application to algebraic geometry and commutative algebra. The three papers have been refereed by peer reviewers.

The object of the lectures by L. Manivel is the study of prehomogeneous spaces, i.e. vector spaces endowed with a linear action of an algebraic group, such that there exists a dense orbit. The study of the projective invariants and their applications to algebraic geometry and commutative algebra is the main aim of the notes by G. Ottaviani. Finally the notes of D. Faenzi deal with representations and invariants of quivers.

We heartily thank the authors for having agreed to our proposal to contribute their papers to the present volume, the referees for their essential contribution and generous efforts, the participants in the School and Workshop, and finally Professor M. Badiale, the Executive Editor of the ‘Rendiconti del Seminario Matematico. Università e Politecnico di Torino’, for suggesting the possibility to edit this volume and helping us to realize this goal.

The editors
Velleda Baldoni,
Dipartimento di Matematica, Università degli Studi di Roma “Tor Vergata”
via della Ricerca Scientica 1, 00133 Roma, Italia
e–mail: baldoni@mat.uniroma2.it

Gianfranco Casnati,
Dipartimento di Scienze Matematiche, Politecnico di Torino
corso Duca degli Abruzzi 24, 10129 Torino, Italia
e–mail: gianfranco.casnati@polito.it

Federica Galluzzi,
Dipartimento di Matematica, Università degli Studi di Torino
via Carlo Alberto 10, 10123 Torino, Italia
e–mail: federica.galluzzi@unito.it

Roberto Notari,
Dipartimento di Matematica, Politecnico di Milano
piazza Leonardo da Vinci 32, 20133 Milano, Italia
e–mail: roberto.notari@polimi.it

Francesco Vaccarino,
Dipartimento di Scienze Matematiche, Politecnico di Torino
corso Duca degli Abruzzi 24, 10129 Torino, Italia
e–mail: francesco.vaccarino@polito.it
D. Faenzi

A ONE-DAY TOUR OF REPRESENTATIONS AND INVARIANTS OF QUIVERS

Abstract. These notes are taken from an introductory lecture on representations and invariants of quivers that I gave at the School (and workshop) on invariant theory and projective geometry, Trento, September 2012. All the results mentioned in this draft have appeared before, and are surveyed here unfortunately without proofs, or with only rough sketches of them. A great proportion of this material has been taken freely from one of the many excellent sets of lecture notes available on the web. I borrowed most of the ideas from [28]. A good source of inspiration, and of challenging exercises, is available at Derksen’s webpage, [21]. Let me mention three more exhaustive lecture notes: those by Crawley-Boevey, [13, 14], those by Brion, [12] and by Ringel, [50]. A very nice example worked out in detail appears in [32].

We will sometimes appeal to some basic projective geometry, see [33] for notation and background.

Contents

1. Representations of quivers ................................................. 4
   1.1. Quivers ............................................................. 4
   1.2. The category of representations of a quiver .................... 5
   1.3. Path algebra ....................................................... 6
   1.4. Standard resolution ................................................ 7
   1.5. Dimension vectors, Euler and Cartan forms .................... 8
   1.6. Action by change of basis, invariants ........................... 9
2. Quivers of finite and infinite type, theorems of Gabriel and Kac ........... 12
   2.1. Roots ............................................................. 12
   2.2. Quivers of finite type: Gabriel’s theorem ....................... 16
   2.3. Tame quivers: Euclidean graphs ................................ 17
   2.4. Finite-tame-wild trichotomy ................................... 19
   2.5. Kac’s theorem ..................................................... 20
3. Derksen-Weyman-Schofield semi-invariants .................................. 21
4. Quivers with relations ..................................................... 22
   4.1. Tilting bundles .................................................... 22
   4.2. Homogeneous bundles ............................................ 23
5. Moduli spaces of quiver representations .................................. 26
   5.1. Stability of quiver representations ............................. 27
   5.2. Moduli space and ring of semi-invariants ....................... 27
   5.3. Schofield’s result on birational type of quiver moduli ........... 28
   5.4. More reading .................................................... 29
References ................................................................. 30

*Partially supported by ANR GEOLMI and INDAM
1. Representations of quivers

The main object of these notes are quivers and their representations. Instead of just a linear map of vector spaces, these consist of a collection of linear maps, indexed by a graph, whose vertices correspond to the vector spaces involved, and whose edges carry an arrow pointing in the direction of the map.

1.1. Quivers

A quiver is a finite directed graph. The finiteness hypothesis is sometimes omitted. We will comment on a couple of situations that require infinite quivers, however for us a quiver will be finite unless otherwise stated.

We write a quiver $Q$ as $Q = (Q_0, Q_1)$, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows. Each arrow $a \in Q_1$ has a head $ha \in Q_0$ and a tail $ta \in Q_0$. In other words, a quiver consists of a quadruple $(Q_0, Q_1, t, h)$ where $t, h$ are functions $Q_1 \rightarrow Q_0$. Many authors prefer to write $(s, t)$, for source and target, instead of $(t, h)$. We depict $a$ as an arrow pointing from $ta$ to $ha$.

**Example 1.** Let us consider some basic examples.

1. The 1-arrow quiver.

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\]

2. The loop quiver.

\[
\begin{array}{c}
\bullet
\end{array}
\]

3. The oriented straight quiver $\vec{A}_n$ (here $\vec{A}_4$):

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\downarrow \\
\bullet
\end{array}
\]

4. The $n$-Kronecker $\Theta_n$ quiver (here the 2-Kronecker quiver $\Theta_2$) with 2 vertices 1, 2 and $n$ arrows from 1 to 2.

\[
\begin{array}{c}
1 \\
\Downarrow \\
2
\end{array}
\]

The Kronecker quiver will play a prominent role in these notes.

5. The $n$-star quiver (here the 4-star quiver):

\[
\begin{array}{c}
1 \\
\Downarrow \\
2 \\
\downarrow \\
3 \\
\downarrow \\
4 \\
\Downarrow \\
5
\end{array}
\]
(6) The n-loop quiver (here the double loop quiver):

(7) A disconnected quiver.

1.2. The category of representations of a quiver

Let us fix a field \( k \). A \( k \)-representation \( V \) of a quiver \( Q \) (over \( k \)) is a collection of \( k \)-vector spaces \( (V(x))_{x \in Q_0} \) and a collection of \( k \)-linear maps \( (V(a))_{a \in Q_1} \).

If \( V \) and \( W \) are representations of \( Q \), a morphism \( \varphi : V \rightarrow W \) is a collection of \( k \)-linear maps \( \varphi(x) : V(x) \rightarrow W(x) \), one for each \( x \in Q_0 \), such that, for all \( a \in Q_1 \) the diagram commutes:

\[
\begin{array}{ccc}
V(ta) & \xrightarrow{\varphi(ta)} & W(ta) \\
v(a) & & v(a) \\
V(ha) & \xrightarrow{\varphi(ha)} & W(ha).
\end{array}
\]

We write \( \text{Hom}_Q(V,W) \) for the set of these morphisms. This is in fact a \( k \)-vector space, while \( \text{Hom}_Q(V,V) \) is a \( k \)-algebra.

In many cases, representations will be assumed to have finite dimension, i.e., the vector spaces \( V_x \) are finite-dimensional for all \( x \in Q_0 \). In this case we can speak of the \textit{dimension vector} \( \dim(V) \) of \( V \), i.e., the assignment \( x \mapsto \dim V(x) \), as an element of \( Q_0^\mathbb{N} \). When the quiver \( Q \) is not finite, usually representations are assumed to have finite support, where the support of a representation \( V \) is the subset of \( x \in Q_0 \) such that \( V_x \neq 0 \).

The category of representations of \( Q \) over \( k \) is denoted by \( \text{Rep}_k(Q) \), or simply by \( \text{Rep}(Q) \). This is a \( k \)-linear category, in the sense that, given representations \( U,W,V \) of \( Q \), composition of maps \( \text{Hom}_Q(V,W) \times \text{Hom}_Q(W,U) \rightarrow \text{Hom}_Q(V,U) \) is a \( k \)-bilinear map.

\textbf{Example 2.} Looking at Example 1, we see the following.

(1) A representation of the 1-arrow quiver (see (1) in Example 1) with dimension vector \((m,p)\) is a linear map from an \( m \)-dimensional vector space to a \( p \)-dimensional one. After fixing basis in these spaces, the representation is identified with a matrix \( M : k^m \rightarrow k^p \). Giving a morphism \( \varphi \) from a representation with dimension
vector \((m, p)\) to another representation, with dimension vector \((m', p')\) amounts to the datum of a commuting diagram:

\[
\begin{array}{c}
\begin{tikzpicture}
\draw[->] (0,0) -- (1,0);
\node at (0.5,0) {\(k^m\)};
\node at (1.5,0) {\(k^p\)};
\draw[->] (0,-1) -- (1,-1);
\node at (0.5,-1) {\(k^{m'}\)};
\node at (1.5,-1) {\(k^{p'}\)};
\end{tikzpicture}
\end{array}
\]

(2) A representation of the loop quiver (see (2) in Example 1) with dimension vector \(m\) is an endomorphism of a vector space over \(k\) of dimension \(m\).

(3) For the Kronecker quiver \(\Theta_n\) (see (4) in Example 1) we can identify a representation of dimension vector \((m, p)\) with an \(n\)-tuple of \(m \times p\) matrices \(M_1, \ldots, M_n\).

A very useful way to think about this is to consider a projective space \(\mathbb{P}^{n-1}\) over \(k\), with ambient variables \(x_1, \ldots, x_n\), and to combine the matrices \(M_1, \ldots, M_n\) in:

\[
M = x_1 M_1 + \cdots + x_n M_n.
\]

The resulting object \(M\) is matrix of linear forms in \(n\) variables. Since each linear form is a global section of \(\mathcal{O}_{\mathbb{P}^{n-1}}(1)\), i.e., a map \(\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to \mathcal{O}_{\mathbb{P}^{n-1}}\), the matrix \(M\) can be written as:

\[
M : \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^m \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}^p.
\]

(4) A representation of the 4-star quiver (see (5) in Example 1) with dimension vector \((n_1, n_2, n_3, n_4, n_5)\) is the following datum. Set \(n = n_3\) and consider a vector space \(U_3\) of dimension \(n\), and vector spaces \(U_i\) of dimension \(n_i\): then \(V\) is given by 4 linear maps \(U_i \to U_3\). If all maps are injective, then, up to change of basis in the \(U_i\)'s, this amounts to a configuration of 4 linear subspaces of \(U_3\).

It is immediate to construct the kernel and cokernel of a morphism of representations \(V \to W\), and to check that \(\text{Rep}(Q)\) is an abelian category. We have the obvious notions of monomorphism and epimorphism of representations, i.e., of subrepresentation and quotient representation. A representation \(V\) of \(Q\) is irreducible (or simple) if \(V\) has no nontrivial subrepresentations. One also defines the direct sum representation in the obvious way, and \(V\) is said to be indecomposable if it is not isomorphic to a direct sum of two nontrivial representations.

### 1.3. Path algebra

The category of representations of a (finite) quiver \(Q\) over \(k\) is a category of modules over a certain algebra, called the path algebra, that we will now describe. So let us fix a quiver \(Q\) and a field \(k\). A path \(p\) in \(Q\) is a sequence \(p = a_1, \ldots, a_m\) with \(a_i\) in \(Q_1\) such that \(ha_i = a_{i-1}\) for all \(i\) (we write paths as compositions). We also write \(tp = ta_m\) and \(hp = ha_1\), and we say that \(p\) is a path from \(tp\) to \(hp\). Given two vertices \(x, y \in Q_0\), denote by \([x, y]\) the set of paths from \(x\) to \(y\). We write \(\langle [x, y] \rangle\) for the vector space freely

generated by elements of $[x,y]$. Given a vertex $x \in Q_0$, the trivial path based at $x$, denoted by $e_x$, is the path of length zero, with head and tail at $x$.

The path algebra $kQ$ is the associative algebra generated by paths from $x$ to $y$, for $x,y \in Q_0$. The unit $e$ of $kQ$ is the empty path, i.e., the product of all trivial paths based at $x \in Q_0$: $e = \prod_x e_x$. In other words, the set $\{e_x \mid x \in Q_0\}$ forms a family of orthogonal idempotents in $kQ$. Multiplication in $kQ$ is defined by:

$$a_1 \ldots a_m \times b_1 \ldots b_p \mapsto a_1 \ldots a_m b_1 \ldots b_p,$$

if $ta_m = hb_1$, or zero if $ta_m \neq hb_1$ (product by composition).

**Fact 1.** The category of left $kQ$-modules is equivalent to $\text{Rep}_k(Q)$.

To see why this is true, consider a $k$-representation $V$ of $Q$ and define the vector space $M = \bigoplus_{x \in Q_0} V(x)$. This is equipped with an action of $kQ$, defined first for trivial paths and length-1 paths, and extended to $kQ$ by linearity. For trivial paths, one sets $e_x \cdot v = v$ for all $x \in Q_0$ and $v \in V(x)$ and $e_x \cdot v = 0$ for $v \in V(y)$ with $x \neq y$. As for length-1 paths, for all $a \in Q_1$ one defines $a \cdot v = V(a)v$ if $v \in V(ta)$, and $a \cdot v = 0$ if $v \in V(y)$ with $y \neq ta$.

Conversely, with a $kQ$-module $M$ we associate the representation $V$ having $V(x) = e_x M$ for all $x \in Q_0$ and $V(a) : V(x) \to V(y)$ defined as the multiplication $e_y M \to e_y M$ when this makes sense, or zero otherwise. So, if $x = ta, y = ha$, and for all $v \in e_x M$, we get $V(a)v = a \cdot v \in e_y M$.

**Example 3.** The path algebra of the loop quiver is $k[t]$. For the $n$-loop quiver, we get the free associative algebra in $n$ variables. One sees also that $kQ$ is finite-dimensional if and only if $Q$ has no oriented cycles.

The path algebra of the straight quiver $\tilde{A}_n$ (see (3) in Example 1) is the algebra of lower triangular matrices of size $n$.

### 1.4. Standard resolution

Let us fix a field $k$ and our quiver $Q$. Given a vertex $x \in Q_0$, we define the standard representation $P_x$, by:

$$P_x(y) = \langle [x,y] \rangle, \quad P_x(a) : p \mapsto ap.$$

This representation corresponds to the submodule of $kQ$ spanned by the idempotent $e_x$. Let $V$ be a $k$-representation of $Q$. We define $P(V)$ and $\Omega(V)$ as:

$$P(V) = \bigoplus_{x \in Q_0} P_x \otimes_k V(x), \quad \Omega(V) = \bigoplus_{a \in Q_1} P_{ha} \otimes_k V(ta).$$

The representations $P(V)$ and $\Omega(V)$ fit together to give the so-called standard resolution of $V$, which is an exact sequence of the form:

$$0 \longrightarrow \Omega(V) \overset{\psi}{\longrightarrow} P(V) \overset{\phi}{\longrightarrow} V \longrightarrow 0.$$
Exercise: guess the maps $\Psi$ and $\Phi$.

Next, we would like to point out an important feature of the category $\text{Rep}(Q)$. Recall that an abelian category $\mathcal{A}$ is hereditary if, for all objects $A, B$ of $\mathcal{A}$, we have:

$$\text{Ext}^n_{\mathcal{A}}(A, B) = 0, \quad \text{when } n \geq 2.$$  

**FACT 2.** The category $\text{Rep}(Q)$ is an abelian $k$-linear hereditary category.

A proof of this fact can be obtained by the standard resolution, or from the fact that the algebra $\text{Hom}_Q(V, V)$ is local, see [13, 50]. Looking at the first method, given a representation $V$, one observes that the new representations $\Omega(V)$ and $P(V)$ are projective, i.e., by definition, they correspond to projective modules over $kQ$. Therefore, it is clear that $\text{Ext}^n_{\mathcal{Q}}(\Omega(V), W) = 0$ and $\text{Ext}^n_{\mathcal{Q}}(P(V), W) = 0$ for all $n \geq 1$ and all representations $W$ of $Q$. Now, by applying the functor $\text{Hom}_Q(\cdot, W)$ to the standard resolution, we obtain $\text{Ext}^{n}_{\mathcal{Q}}(V, W) = 0$ for all $n \geq 2$. The property just mentioned justifies the notation $\text{Ext}^{1}_{\mathcal{Q}}(V, W)$.

Moreover, this way we also get the *canonical exact sequence* attached to a pair of representations $V, W$. This is an exact sequence of the following form.

$$0 \to \text{Hom}_Q(V, W) \longrightarrow \bigoplus_{x \in Q_0} \text{Hom}_k(V(x), W(x)) \xrightarrow{d^V_W}$$

$$\bigoplus_{a \in Q_1} \text{Hom}_k(V(ta), W(ha)) \longrightarrow \text{Ext}^1_Q(V, W) \to 0.$$

The arrow $d^V_W$ is defined by:

$$d^V_W(q(x)_{x \in Q_0}) = (q(ha)V(a) - W(a) \circ q(ta))_{a \in Q_1}.$$

### 1.5. Dimension vectors, Euler and Cartan forms

Let us fix a quiver $Q$ and the field $k$. We already described the dimension vector of a $k$-representation of $Q$ as an element of $\mathbb{N}^{Q_0}$. Any assignment $\alpha : Q_0 \to \mathbb{N}$ will thus be called a *dimension vector* of $Q$. The set of dimension vectors of $Q$ is a subset of $\Gamma_Q = \mathbb{Z}^{Q_0}$.

**Euler form**

The Euler form (or Ringel form) is the bilinear map on $\Gamma_Q$ defined by:

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha), \quad \forall \alpha, \beta \in \Gamma_Q.$$

Let $(b_{i,j})$ be the matrix of $\langle \cdot, \cdot \rangle$ in the coordinate basis of $\Gamma_Q = \mathbb{Z}^{Q_0}$. Then:

$$b_{i,j} = \delta_{i,j} - \# \{a \in Q_1 \mid ta = i, ha = j \}.$$
Let $V, W$ be representations of $Q$ over $k$. We define the Euler characteristic:

$$\chi(V, W) = \dim_k \text{Hom}_Q(V, W) - \dim_k \text{Ext}_Q(V, W).$$

If $V, W$ have dimension vectors respectively $\alpha$ and $\beta$, then:

$$\chi(V, W) = \langle \alpha, \beta \rangle.$$

In particular we have:

$$\chi(V, V) = \dim_k \text{End}_Q(V) - \dim_k \text{Ext}_Q(V, V) = \langle \alpha, \alpha \rangle.$$

**Cartan form**

We can symmetrize the Euler form to obtain the Cartan (or Cartan-Tits) symmetric bilinear form on $Q^0$:

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle, \quad \forall \alpha, \beta \in \Gamma_Q.$$

This does not depend on the orientation of the arrow of $Q$. The matrix of $(c_{ij})_{i,j}$ of $(\cdot, \cdot)$ in the coordinate basis of $\Gamma_Q$ is called Cartan matrix.

**Example 4.** Consider the quiver $Q$:

$$\bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet$$

The underlying graph of $Q$ is $A_4$. The Euler and Cartan matrices of $Q$ read:

$$(c_{ij})_{1 \leq i,j \leq 4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad (b_{ij})_{1 \leq i,j \leq 4} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**1.6. Action by change of basis, invariants**

Let $Q$ be a fixed quiver and $k$ be a field. Let $\alpha \in \Gamma_Q$ be a dimension vector of $Q$. Once chosen $\alpha$, the representations of $Q$ of dimension vector $\alpha$ form a vector space, acted on by a product of general or special linear groups via change of basis. The invariants or semi-invariants relevant to this situation are the polynomial functions on this space, invariant under this group. Let us sketch this situation.

The space of all representations with fixed dimension vector

The $k$-vector space of all representations of dimension vector $\alpha$ is denoted by $\text{Rep}(Q, \alpha)$. By choosing a basis for all vector spaces $(V(x))_{x \in Q_0}$ we get an identification:

$$\text{Rep}(Q, \alpha) \simeq \prod_{x \in Q_1} \text{Hom}_k(k^{\alpha(x)}), k^{\alpha(x)})$$
So a representation $V \in \text{Rep}(Q, \alpha)$ can be seen as $V = (V(a))_{a \in Q_1}$ where $V(a)$ is a matrix with $\alpha(ta)$ columns and $\alpha(ha)$ rows.

The set of polynomial function $f$ on $\text{Rep}(Q, \alpha)$ is denoted by $k[\text{Rep}(Q, \alpha)]$. This is a polynomial ring in as many variables as $\dim_k(\text{Rep}(Q, \alpha)) = \sum_{a \in Q_1} \alpha(ta)\alpha(ha)$.

**Characters of the linear group of given dimension vector**

Given a dimension vector $\alpha$ of $Q$, the group $\text{GL}(Q, \alpha)$ is defined as:

$$\text{GL}(Q, \alpha) = \prod_{x \in Q_0} \text{GL}_{\alpha(x)}(k).$$

A character $\theta : \text{GL}(Q, \alpha) \to k^*$ is of the form:

$$\theta((g_x)_{x \in Q_0}) = \prod_{x \in Q_0} \det(g_x)^{\theta_x},$$

where each $\theta_x$ lies in $\mathbb{Z}$. So $\theta = (\theta_x)_{x \in Q_0}$ lies in $\Gamma = \Gamma_Q$.

We can also think of the space of characters $\mathcal{X}(\text{GL}(Q, \alpha))$ as $\Gamma^* = \text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$. Indeed, if $\sigma$ as an element of $\Gamma^*$, we define $\theta_\sigma$ by:

$$\theta_\sigma((g_x)_{x \in Q_0}) = \prod_{x \in Q_0} \det(g_x)^{\sigma(\delta_x)},$$

where the dimension vector $\delta_x$ is defined by $\delta_x(x) = 1$ and $\delta_x(y) = 0, \forall y \neq x$.

The general linear group $\text{GL}(Q, \alpha)$ acts by conjugation on $\text{Rep}(Q, \alpha)$, namely if $g \in \text{GL}(Q, \alpha)$ and $V \in \text{Rep}(Q, \alpha)$, then:

$$g.V = (g(ha) \circ V(a) \circ g(ta)^{-1})_{a \in Q_1}.$$

This action can be thought of as change of basis in the spaces $V(x)$. The kernel of the action $Z$ is a normal subgroup of $\text{GL}(Q, \alpha)$, consisting of transformations that act trivially on $\text{Rep}(Q, \alpha)$. Note that the 1-dimensional torus $k^*$ id is always contained in $Z$. We are naturally lead to consider the action of the group:

$$G(Q, \alpha) = \text{GL}(Q, \alpha)/Z,$$

and in many cases we have $G(Q, \alpha) = \text{GL}(Q, \alpha)/k^* \text{id}$.

The general linear group $\text{GL}(Q, \alpha)$ acts also on the ring of polynomial functions $k[\text{Rep}(Q, \alpha)]$ by the formula $g.f(V) = f(g^{-1}.V)$.

**Invariants and semi-invariants**

Let $G$ be a closed algebraic subgroup of $\text{GL}(Q, \alpha)$. 

DEFINITION 1. A G-invariant of \((Q, \alpha)\) is a polynomial function \(f\) over \(\text{Rep}(Q, \alpha)\) such that, for all \(g \in G\) we have \(g \cdot f = f\). We write \(\text{Rep}(Q, \alpha)\) for the ring of all invariants of \((Q, \alpha)\). If \(\theta\) is a character of \(\text{GL}(Q, \alpha)\), a semi-invariant of \((Q, \alpha)\) of weight \(\theta\) is a polynomial function \(f\) over \(\text{Rep}(Q, \alpha)\) such that:

\[ g \cdot f = \theta(g) f, \quad \forall g \in \text{GL}(Q, \alpha) \]

We denote by \(\text{SI}(Q, \alpha)_\theta\) the space of semi-invariants of \((Q, \alpha)\) of weight \(\theta\).

FACT 3. The ring of \(\text{SL}(Q, \alpha)\)-invariants functions over \(\text{Rep}(Q, \alpha)\) is the direct sum of the spaces of semi-invariant functions for \(\text{GL}(Q, \alpha)\):

\[ k[\text{Rep}(Q, \alpha)]^{\text{SL}(Q, \alpha)} = \bigoplus_{\theta \in \mathcal{X}(\text{GL}(Q, \alpha))} \text{SI}(Q, \alpha)_\theta. \]

To understand this fact, consider a semi-invariant \(f\) for \(\text{GL}(Q, \alpha)\), relative to a character \(\theta \in \mathcal{X}(\text{GL}(Q, \alpha))\). We know that there are integers \((\theta_i)_{i \in Q_0}\) such that, for all \((q_i)_{i \in Q_0}\), lying in \(\text{GL}(Q, \alpha)\), we have \(\theta(q) = \prod_{i \in Q_0} \det(q_i)^{\theta_i}\). So, for all \(h \in \text{SL}(Q, \alpha)\) we have \(\theta(h) = 1\), hence \(h \cdot f = f\) so that \(f\) is \(\text{SL}(Q, \alpha)\)-invariant. Conversely, one writes \(\text{GL}(Q, \alpha)\) as semi-direct product of \(\text{SL}(Q, \alpha)\) and \(H = \prod_{i \in Q_0} k^*\), and the characters of \(\text{GL}(Q, \alpha)\) are given by those of the algebraic torus \(H\). Then, once given a \(\text{SL}(Q, \alpha)\)-invariant function \(f\), we decompose it into eigenvectors for \(H\), hence into semi-invariants for \(\text{GL}(Q, \alpha)\).

EXAMPLE 5. Given an integer \(m\), an \(m\)-dimensional representation of the loop quiver \(Q\) is an endomorphism \(u\) of \(k^m\), so the algebra \(\text{Rep}(Q, m)\) is \(k[x_1, \ldots, x_m, m]\), with \(\text{GL}_m(k)\) acting by conjugation. According to Chevalley’s theorem, we have:

\[ \text{Rep}(Q, m)^{\text{GL}_m(m)} = k[e_1, \ldots, e_m], \]

where \(e_1, \ldots, e_m\) are the elementary symmetric functions of the eigenvalues of \(u\). We refer to [47] for an extremely interesting approach to invariant theory. Here \(e_i = \text{tr}(\lambda^i u)\), so \(e_i\) is a semi-invariant for \(\text{GL}_m(m)\) of weight \(i\).

EXAMPLE 6. Set \(Q = \mathbb{A}_2\) and let \(m, p\) and \(r \leq \min(m, p)\) be integers. A representation \(V\) of \(Q\) of dimension vector \(\alpha = (m, r, p)\) is a pair of maps:

\[ M \to R \to P; \quad \text{with } \dim(M) = m, \dim(R) = r, \dim(P) = p. \]

Thus the algebra \(B = \text{Rep}(Q, \alpha)\) is the symmetric algebra over the vector space \(\text{Hom}(M, R) \oplus \text{Hom}(R, P)\).

We consider the action of the subgroup \(G = \text{GL}(R) \cong \text{GL}_p(k)\) of \(\text{GL}(Q, \alpha)\) only. Fixing basis of \(M\) and \(P\), and given any two indexes \(i, j\) with \(1 \leq i \leq m\) and \(1 \leq j \leq p\), we have the \(G\)-invariant functions \([\cdot, \cdot]\) on \(B\) defined on \(\text{Hom}(M, R) \oplus \text{Hom}(R, P)\) by the \((j, i)\)-th coefficient of the composition matrix:

\[ (f, g) \mapsto [f, g] = (g \circ f)_{ji}. \]
Consider also a polynomial ring \( A = k[x_1, \ldots, x_{m,p}] \), and the \( m \times p \) matrix \( X \) of indeterminates. The ring \( k[\text{Hom}(M, P)] \) can be identified with \( A \). The First and Second Fundamental Theorems of invariant theory for \( \text{GL}_r(k) \) give:

i) a surjective ring homomorphism \( \phi: A \rightarrow B \) given by \( x_{i,j} \mapsto [y_{i,j}] \),

ii) an equality \( \ker(\phi) = (\wedge^{r+1}(X)) \).

This says that the GIT quotient \( \text{Hom}(M, R) \oplus \text{Hom}(R, P) / \text{GL}(R) \) is the affine subvariety of \( \text{Hom}(M, P) \) consisting of matrices of rank at most \( \dim(R) \), which is cut by all minors of order \( r + 1 \) of \( X \). We refer to [47].

2. Quivers of finite and infinite type, theorems of Gabriel and Kac

We fix a field \( k \) and a (finite) quiver \( \vec{Q} \). It is natural to ask if there are finitely or infinitely many indecomposable representations of \( \vec{Q} \) over \( k \).

**Definition 2.** If there are only finitely many isomorphism classes of indecomposable finite-dimensional representations of \( \vec{Q} \) over \( k \), then \( \vec{Q} \) is said to be of finite type (over \( k \)), or of finite representation type.

For instance, we take \( \vec{Q} = \vec{A}_2 \). For \( m, p > 0 \), a representation of dimension vector \((m, p)\), corresponding to a matrix \( k^m \rightarrow k^p \) or rank \( r \leq \min(m, p) \) is equivalent, up to changing basis in \( k^m \) and \( k^p \), to the direct sum of the identity \( I_r \) and of the zero matrix. Of course this matrix corresponds to a non-zero indecomposable representation if and only if \( r = 1 \), so \( \vec{A}_2 \) is of finite type. The only dimension vectors giving indecomposable representations are \((1, 0), (0, 1)\) and \((1, 1)\).

Similarly, we can look at the quiver \( \vec{Q} = \vec{A}_3 \). In this case, representations of \( \vec{Q} \) are given by maps \( M \xrightarrow{u} R \xrightarrow{v} P \). Besides the obvious dimension vectors \((1, 0, 0), (0, 1, 0), (0, 0, 1)\), we can build an indecomposable representation with dimension vector \((1, 1, 0)\) by setting \( M = R = k \) and \( u = \text{id}_k \), and for \((0, 1, 1)\) by setting \( R = P = k \) and \( v = \text{id}_k \). One more case arises taking the dimension vector \((1, 1, 1)\), so \( M = P = R = k \) and setting \( v = u = \text{id}_k \).

On the other hand, the loop quiver, for instance, is of infinite type, actually of tame type. Indeed the isomorphism classes of representations of this quiver are given by conjugacy classes of matrices, i.e. (at least over \( \overline{k} \)) by Jordan normal forms. Therefore, for any dimension \( m \), there is a one-dimensional family (parametrized by \( \lambda \in k \)) of non-isomorphic indecomposable representations whose normal form is:

\[
\begin{pmatrix}
\lambda & 1 \\
0 & \lambda \\
& \ddots & \ddots \\
& & 0 & \lambda \\
& & & 0 & \lambda
\end{pmatrix}
\]
2.1. Roots

Let us sketch briefly here the notion of Kac-Moody algebra and the related idea of root system. We refer to [36, 38] for a complete treatment, and to [53] for some lecture notes.

Kac-Moody algebras

For this part we work over the field \( \mathbb{C} \) of complex numbers. Kac-Moody algebras are an infinite-dimensional analogue of simple complex Lie algebras, and are associated with generalized Cartan matrices.

A square matrix \( C = (c_{i,j})_{i,j} \) of size \( n \) is a generalized Cartan matrix if:

i) on the diagonal we have \( c_{i,i} = 2 \);

ii) off the diagonal we have \(-c_{i,j} \in \mathbb{N} \) and \( c_{i,j} = 0 \iff c_{j,i} = 0 \).

There are two more conditions that one can add to \( C \). The first one is that \( C \) is indecomposable, which means that \( C \) is not equivalent under change of basis to a block matrix \( \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \), unless \( C_1 = C \). The second is that \( C \) is symmetrizable, namely that there are a symmetric matrix \( S = (s_{i,j})_{i,j} \) and a diagonal non-degenerate matrix \( D \), such that \( C = DS \). This decomposition is unique for \( C \) indecomposable if we assume that \( s_{i,j} \) are relatively prime integers and that \(-2s_{i,j} \in \mathbb{N} \).

Irreducible symmetric Cartan matrices are in one-to-one correspondence with connected graphs without loops. Indeed, with a graph without loops we associate the Cartan matrix as we have already did. The absence of loops is translated into the value 2 along the diagonal, connectedness corresponds to irreducibility, and the matrix is symmetric by definition. On the other hand, having an \( n \times n \) symmetric generalized Cartan matrix \( C = (c_{i,j})_{i,j} \), in order to construct the graph, we draw \( n \) vertices and we join the \( i \)-th and \( j \)-th vertices with \( |c_{i,j}| \) vertices.

With a generalized Cartan matrix \( C = (c_{i,j})_{i,j} \) of size \( n \), one also associates naturally a free abelian group \( \Lambda_C \) with \( n \) generators, denoted by \((\alpha_1, \ldots, \alpha_n)\). We should think of \( \Lambda_C \) as the set of dimension vectors of the graph associated with \( C \), or of any quiver with this underlying graph. For \( C \) symmetrizable, the matrix \( S \) gives a bilinear form on \( \Lambda_C \), denoted by \( (\cdot, \cdot) \). The matrix of the bilinear form is \( \frac{1}{2} C \).

**Definition 3.** The Kac-Moody algebra \( g \) associated with \( C \) is the unique \( \Lambda_C \)-graded Lie \( \mathbb{C} \)-algebra \( g = \oplus_{\alpha \in \Lambda_C} g_\alpha \) satisfying the following conditions:

i) for every \( \Lambda_C \)-graded ideal \( i \) of \( g \), we have \( i \cap g_0 = 0 \Rightarrow i = 0 \);

ii) the algebra \( g \) is generated by \( (e_i, f_i, h_i \mid i = 1, \ldots, n) \), where \( g_{\alpha_i} = \mathbb{C} e_i, g_{-\alpha_i} = \mathbb{C} f_i \) and \( (h_1, \ldots, h_n) \) is a basis of \( g_0 \);

iii) for all \( 1 \leq i, j \leq n \), we have the Lie algebra relations induced by \( C \):

\[
[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{i,j} h_i, \quad [h_i, e_j] = c_{i,j} e_j, \quad [h_i, f_j] = -c_{i,j} f_j.
\]
A root of $\mathfrak{g}$ is an element $\alpha \in \Lambda_C$ such that $\mathfrak{g}_\alpha \neq 0$. A root is positive if its expression in terms of the $\alpha_i$ involves only non-negative coefficients. In this context, a root $\alpha$ is simple if dim$(\mathfrak{g}_\alpha) = 1$: in particular $\alpha_1, \ldots, \alpha_n$ are simple. For $i = 1, \ldots, n$, one defines the reflection $r_i$ by $r_i(\alpha) = \alpha - q_i(\alpha)\alpha_i$, where $q_i$ is defined extending linearly the function $q_i(\alpha_j) = c_{i,j}$. The subgroup of $\text{GL}_m(\mathbb{C})$ generated by the $r_i$ is called the Weyl group $W$. This way we can define the real roots, namely the orbit of simple roots under $W$. All other roots are said to be imaginary. The terminology is justified by the fact that $\alpha$ is real if and only if $(\alpha, \alpha) > 0$ and imaginary if and only if $(\alpha, \alpha) \leq 0$.

We write $u \geq 0$ or $u > 0$ if all coefficients of $u$ are non-negative or positive. For an indecomposable generalized Cartan matrix $C$, one and only case occurs:

**Positive:** $C$ is non-degenerate and $Cu \geq 0$ implies $u = 0$ or $u > 0$. In this case $C$ is symmetrizable with $S$ positive definite, and $\mathfrak{g}$ is a simple Lie algebra.

**Zero:** rk$(C) = n - 1$, $Cu = 0$ for some $u > 0$, and $Cu \geq 0 \iff Cu = 0$. In this case $C$ is symmetrizable with $S$ positive semi-definite, and $\mathfrak{g}$ is an infinite-dimensional Lie algebra called affine Lie algebra: this is a central extension by $C$ of $\mathfrak{g}_e \otimes \mathbb{C}[t, t^{-1}]$, where $\mathfrak{g}_e$ is a finite-dimensional Lie algebra.

**Negative:** there is $u > 0$ with $Au < 0$, and $Au \geq 0$ together with $u \geq 0$ implies $u = 0$.

Recall that simple Lie algebras (the positive case) are classified by Dynkin diagrams. The algebras coming from the zero case are also classified completely, they come from Euclidean diagrams (see [10]): we will review part of them in a minute, in connection with quivers of tame representation type. The algebras of the negative case are a much more unexplored territory.

**Schur roots**

The indecomposable representations of a quiver $Q$ have some basic blocs, called Schurian representations, defined by the condition that their endomorphism algebra is constituted by homotheties. The dimension vectors of these representations are tightly connected with the positive roots of the Kac-Moody algebra associated with the graph underlying $Q$.

**DEFINITION 4.** A representation $V$ of $Q$ is called Schurian in case $\text{End}_Q(V) \simeq k$. In particular a Schurian representation is indecomposable. If $V$ has $\text{Ext}_Q(V, V) = 0$, then $V$ is said to be rigid. A dimension vector $\alpha$ is a Schur root if $\alpha = \dim(V)$, for a Schurian representation $V$. Moreover, $\alpha$ is called a real Schur root if there exists a unique representation $V \in \text{Rep}(Q, \alpha)$, up to isomorphism. On the other hand, $\alpha$ is called an imaginary Schur root if there exist infinitely many non-isomorphic representations $V \in \text{Rep}(Q, \alpha)$.

**REMARK 1.** Here are some more comments on Schur roots.

i) If $V$ is Schurian, then the tangent space at the point $[V]$ of the universal deformation
space of $V$ is identified with $\text{Ext}_Q(V,V)$. Therefore the dimension of this space is $1 - \langle \alpha, \alpha \rangle$.

ii) Asking that $\alpha$ is a Schur root is equivalent to ask that a general representation $V$ is of dimension vector $\alpha$ is indecomposable (here a claim about a “general” element of a given parameter space, means that the elements that do not satisfy the claim form a proper (Zariski) closed subset of the parameter space, i.e., the set of these elements is defined by a finite set of nontrivial polynomial equation).

For instance, if $Q = \vec{A}_3$, then the Schur roots are $(1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 1, 1)$ and $(1, 1, 1)$. Likewise, for any $n$, the Schur roots of the quiver $\vec{A}_n$ are the vectors with 0’s, and adjacent 1’s only. There are precisely the positive roots of the Lie algebra $\mathfrak{sl}_{n+1}$, whose Dynkin diagram is $\vec{A}_n$. This is not a coincidence, as we will see in a minute.

As another example, we see that, for the loop quiver, only 1 is a Schur root. Note that an indecomposable representation of dimension $m$ is given by matrices made of a Jordan block of size $m$. So these representations are parametrized by the eigenvalues of $M$, i.e. by the line $k$. However, this representation is not general: the general one will have $n$ distinct eigenvalues and will thus decompose as direct sum of $n$ indecomposable subrepresentations of dimension 1.

iii) We have said that a general representation $V$ of $Q$ having dimension vector $\alpha$ need not be indecomposable. However, it turns out that the dimension vectors $\alpha_1, \ldots, \alpha_n$ of the indecomposable summands of $V$ are determined by $\alpha$, so one writes $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_n$. The $\alpha_i$ are Schur roots. This is called the canonical decomposition of $\alpha$, see [27]. For instance for the loop quiver we have $n = 1 \oplus n$.

**EXAMPLE 7.** Let $n \geq 3$ and consider the Kronecker quiver $\Theta_n$ (see (4) of Example 1):

![Kronecker quivers](image)

Figure 1: Kronecker quivers $\Theta_3$ and $\Theta_4$.

Then, $\alpha = (m, p)$ is an imaginary Schur root if and only if:

$$\frac{m}{p} \in \left[ \frac{n - \sqrt{n^2 - 4}}{2}, \frac{n + \sqrt{n^2 - 4}}{2} \right].$$

On the other hand, one defines the generalized Fibonacci numbers:

$$a_k = \frac{(n + \sqrt{n^2 - 4})^k - (n - \sqrt{n^2 - 4})^k}{2^k \sqrt{n^2 - 4}}.$$
equivalently defined by the relations $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = ka_k - a_{k-1}$. It turns out that $\alpha = (m, p)$ is a real Schur root if and only if:

$$m = a_{k-1}, \quad p = a_k, \quad \exists k \geq 0.$$  

In terms of sheaves on $\mathbb{P}^{n-1}$, a representation $V$ of dimension vector $(m, p)$ amounts to a matrix $M : \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^m \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}$, and the datum of $V$ is equivalent to that of the sheaf $E = \text{cok}(M)$. Real Schur roots correspond to the case when $E$ is exceptional, i.e., $\text{Hom}_{\mathcal{O}_{\mathbb{P}^{n-1}}}(E, E) \simeq k$ and $\text{Ext}^{\geq 0}_{\mathcal{O}_{\mathbb{P}^{n-1}}}(E, E) = 0$. Imaginary Schur roots correspond to the case when $E$ is a simple bundle with positive-dimensional deformation space. See also [11] for the tightly related study of Fibonacci bundles.

2.2. Quivers of finite type: Gabriel’s theorem

A very nice classification of quivers of finite type is available. The story began with Gabriel’s theorem, see [30], and was then developed by a great number of authors. For more general fields (including non-algebraically closed fields), see [7, Section 4].

**Theorem 1** (Gabriel). A finite connected quiver $Q$ is of finite type if and only if its underlying undirected graph is a Dynkin graph of type $A, D, E$.

Moreover, in this case the indecomposable representations of $Q$ are in bijection with the positive roots of the associated Lie algebra.

Recall that $A_n$ has $n(n+1)/2$ positive roots, $D_n$ has $n(n-1)$ positive roots and $E_6, E_7, E_8$, have respectively 36, 63, 120 positive roots.

**Remark 2.** The fact of being of finite type does not depend on the orientation of arrows, nor does the set of indecomposable representations of a finite quiver. Indeed, the indecomposable representations of a quiver $Q$ of finite type are in bijection with the positive roots of the Lie algebra whose Dynkin graph is the undirected graph of $Q$.

After the original proof by Gabriel, [30,31], an argument using Coxeter functors was developed by Bernstein-Gelfand-Ponomarev in [8]. This allows to give a clean proof of one direction (perhaps the most difficult one) of Gabriel’s theorem, namely that quivers obtained by Dynkin graphs of type $A, D, E$ are of finite type.

An argument due to Tits allows us to understand the other direction of Gabriel’s theorem, namely that the underlying undirected graph of a finite connected quiver of finite type must be a simply laced Dynkin diagram. Indeed, let $V$ be an indecomposable representation of $Q$ and set $\alpha = \dim(V)$. Note that $\text{GL}(Q, \alpha)$ must act with a finite number of orbits on $\text{Rep}(Q, \alpha)$, otherwise there would be infinitely many non-isomorphic representations already with dimension vector $\alpha$. Now, if $V$ is Schurian then the stabilizer of the $\text{GL}(Q, \alpha)$-action on $\text{Rep}(Q, \alpha)$ is $k^*$. So we have the inequality:

$$\dim(\text{Rep}(Q, \alpha)) - \dim(\text{GL}(Q, \alpha) - 1) \leq 0,$$

which means:

$$\langle \alpha, \alpha \rangle \geq 1.$$
Therefore, the Cartan matrix of $Q$ is positive definite, and one is reduced to the classifications of simple Lie algebras, hence the graph must be of Dynkin type. Also, the graph must be simply-laced, for the loop quiver and the Kronecker quivers $\Theta_n$ with $n \geq 2$ are of infinite type as we have already seen.

One can also argue that, if $V$ is indecomposable and not Schurian, then we can find a subrepresentation $W$ of $V$ which is Schurian, and that satisfies $\text{Ext}_Q(W,W) \neq 0$. Since $\text{Ext}_Q(W,W)$ is the tangent space to the deformations of a Schurian representation $W$, and since this space is unobstructed ($\text{Rep}(Q)$ is hereditary), we get that $Q$ is of infinite type as soon as $V$ is not Schurian. To understand why $W$ exists, one notes that $V$ having non-trivial endomorphisms allows the construction of a nilpotent endomorphism $f$. Choose $f$ to be of minimal rank so that $f^2 = 0$. Then, choose an indecomposable summand $W$ of $\ker(f)$ intersecting non-trivially $\text{Im}(f)$. It is easy to check that $W$ has $\text{Ext}_Q(W,W) \neq 0$. If $W$ is again not Schurian, we get to the desired $W$ by iterating this process (see [12] for this proof).

2.3. Tame quivers: Euclidean graphs

We have just seen that quivers of finite type are classified. There is another class of quivers, which are of infinite type, but still exhibit a very moderate behavior, as far as the families of their representations are concerned. This is the class of quivers of tame type.

**Definition 5.** If $Q$ is of infinite type, and, for each dimension vector $\alpha$, all isomorphism classes of finite-dimensional indecomposable representations of $Q$ of dimension vector $\alpha$ form a finite number of families of dimension at most 1, then $Q$ is said to be of tame type.

**Example 8.** The Kronecker quiver $\Theta_2$ is of tame type over any algebraically closed field. We have said that a representation $V$ of $\Theta_2$ of dimension vector $(m, p)$ is a pair of matrices $M_1, M_2$, with $M_i : k^m \to k^p$, and that this amounts to a matrix of linear forms:

$$M = x_1 M_1 + x_2 M_2 : \mathcal{O}_{\mathbb{P}^1} (-1)^m \to \mathcal{O}_{\mathbb{P}^1}^p.$$  

These pencils are classified up to linear coordinate change on $k^m$ and $k^p$ according to their Kronecker-Weierstrass canonical form, going back to an observation of Weierstrass [54], taken up again by Kronecker in [40]. Let us sketch this here, and refer to [5, Chapter 19.1] or [7] for a comprehensive treatment. We write $M \simeq M' \boxplus M''$ if $M$ is equivalent to a block matrix having $M'$ and $M''$ on the diagonal and zero elsewhere. Given positive integers $u, v$, one defines:

$$C_u = \begin{pmatrix} x_1 & x_2 \\ x_2 & \ddots \\ \vdots & \ddots & \ddots & x_1 \\ x_2 & \end{pmatrix}, \quad B_v = \begin{pmatrix} x_1 & x_2 \\ x_2 & \ddots & \ddots \\ \vdots & \ddots & \ddots & x_1 \\ x_2 & \end{pmatrix}.$$
where $C_u$ has size $(u+1) \times u$ and $B_v$ has size $v \times (v+1)$. Also, given $a \in k$ and a positive integer $n$ one defines:

$$J_{a,n} = \begin{pmatrix} a & 1 \\ \vdots & \vdots \\ a & 1 \\ a & \end{pmatrix} \in k^{n \times n}, \quad \text{and:} \quad \mathfrak{J}_{a,n} = x_1 I_n + x_2 J_{a,n}.$$ 

Up to a change of variables in $\mathbb{P}^3$, we may assume that $\infty = (0:1) \in \mathbb{P}^3$ is not critical for $M$, i.e. that $M$ has no infinite elementary divisors.

**Proposition 1.** Up to possibly changing basis in $\mathbb{P}^1$, $M$ is equivalent to:

$$C_{u_1} \oplus \cdots \oplus C_{u_r} \oplus B_{v_1} \oplus \cdots \oplus B_{v_s} \oplus J_{1, a_1} \oplus \cdots \oplus J_{t, a_t} \oplus Z_{a_0, b_0},$$

for some integers $r, s, t, a_0, b_0$ and $u_i, v_i, n_i$, and some $a_1, \ldots, a_t \in k$, where $Z_{a_0, b_0}$ is the zero matrix of size $a_0 \times b_0$.

To have an insight on the Kronecker-Weierstrass normal form and of a matrix pencil, it is useful to note that:

$$\text{cok}(C_u) \simeq \mathcal{O}_{\mathbb{P}^1}(u), \quad \text{ker}(B_v) \simeq \mathcal{O}_{\mathbb{P}^1}(-v-1),$$

Note that these are all the indecomposable torsionfree sheaves on $\mathbb{P}^1$, if we allow $u, v \geq 0$. All of these sheaves are exceptional, and the dimension vectors $(u+1, u)$ and $(v, v+1)$ are real Schur roots.

On the other hand, $\text{cok}(\mathfrak{J}_{a,n}) \simeq \mathcal{O}_{\mathbb{P}^1}[-a, 1]$, the structure sheaf of the point $[-a, 1] \in \mathbb{P}^1$, counted with multiplicity $n$. This sheaf is indeed indecomposable and varies in the $\mathbb{P}^1$ parametrized by $a$.

We refer to [22,24,44] for the proof of the result analogous to Gabriel’s theorem, for quivers of tame type.

**Theorem 2.** Let $Q$ be a finite connected quiver. Then $Q$ is of tame type if and only if its underlying undirected graph is Euclidean of type $\widehat{A}, \widehat{D}, \widehat{E}$.

![Euclidean graphs of type $\widehat{A}_n$ and $\widehat{D}_n$.](image)

In this picture, we see the 3 exceptional simply-laced Euclidean graphs.
Representations of quivers

Similarly to the case of quivers of finite type, for tame quivers one is reduced to classify positive semidefinite (but not definite) generalized Cartan matrices.

2.4. Finite-tame-wild trichotomy

We have sketched the classification of quivers of finite and tame types. One can now ask what is the behavior of families of indecomposable representations over the remaining quivers. The first answer is that these families are large, in the sense that there is no integer bounding the dimension of families of indecomposable representations. The second answer will be provided by Kac’s theorem, that says that the dimension vectors of indecomposable representations are still controlled by the roots of a Lie algebra. A third answer will be the construction of the moduli space of representations: this allows to describe the set of representations, at least those that are semi-stable, as an algebraic variety.

Looking at the first of these topics, we say that a quiver $Q$ is of wild type (over $k$) if the category $\text{Rep}_k(Q)$ contains $\text{Rep}_k(L)$ as a full subcategory, where $L$ is the double loop quiver. We refer to [50, Lecture 6] (in those notes, wild is replaced by strictly wild). If $Q$ is of wild type, then there are arbitrarily large families of non-isomorphic finite-dimensional indecomposable representations.

FACT 4 (Finite-tame-wild trichotomy). A connected quiver $Q$ which is neither of finite nor of tame type is of wild type.

To understand this, one can construct explicitly a finite number of graphs admitting a dimension vector $\alpha$ with $(\alpha, \alpha) = -1$. These graphs include a 1-edge extension of the Euclidean graphs.
In this picture, we see the 3 simply-laced extended graphs.

![Extended graphs of type \(\widehat{E}_6, \widehat{E}_7, \widehat{E}_8\).](image)

There are three more minimal wild graphs:

![Wild graphs](image)

Any quiver whose generalized Cartan matrix is indefinite will contain one of these quivers. Finally one constructs for all these quivers \(Q\) a subcategory of \(\text{Rep}(Q)\) equivalent to that of the double loop quiver. This is achieved by considering some Ext-quivers (see [7, Section 4.1]), and using Ringel’s simplification process, [49].

### 2.5. Kac’s theorem

An analogue of Gabriel’s theorem for wild quivers is provided by fundamental work of Kac, [36, 37]. Let \(Q\) be a finite quiver, and let \(C\) be the generalized Cartan matrix associated to the underlying undirected graph of \(Q\). We denote by \(\mathfrak{g}\) the Kac-Moody algebra associated with \(C\).

**Theorem 3 (Kac).** Assume \(k\) is either finite, or algebraically closed, and that \(Q\) has no loops. Then we have the following.

1. **i)** If \(V\) is an indecomposable representation with \(\dim(V) = \alpha\), then \(\alpha\) is a positive root of \(\mathfrak{g}\).
2. **ii)** If \(\alpha\) is a real root of \(\mathfrak{g}\), then, up to isomorphism, there exists a unique indecomposable representation \(V\) with \(\dim(V) = \alpha\).
3. **iii)** If \(\alpha\) is an imaginary root of \(\mathfrak{g}\) and \(k\) is algebraically closed, then there are families of dimension \(\geq 1 - \langle \alpha, \alpha \rangle\) of non-isomorphic indecomposable representations \(V\) with \(\dim(V) = \alpha\). In particular \(Q\) is of infinite type.
Representations of quivers

3. Derksen-Weyman-Schofield semi-invariants

We will give an outline of the main result of [26]. Let Q be a quiver without oriented loops, and k be a field. Set $\Gamma = \Gamma_Q$. Let us also fix a dimension vector $\alpha$ of Q and a character $\gamma$ of $GL(Q, \alpha)$.

For any $\alpha \in \mathbb{N}^Q$ such that $\langle \alpha, \beta \rangle = 0$ and any pair or representations $V, W_Q$ over $k$ with dimension vectors respectively $\alpha, \beta$, we have a square matrix:

$$d^V_{\alpha, \beta} : \bigoplus_{x \in Q_0} Hom_k(V(x), W(x)) \longrightarrow \bigoplus_{a \in Q_1} Hom_k(V(ta), W(ha)).$$

We define the semi-invariant $c$ of the action of $GL(Q, \alpha) \times GL(Q, \beta)$ on $Rep(Q, \alpha) \times Rep(Q, \beta)$ by:

$$c(V, W) = \det(d^V_{\alpha, \beta}).$$

This is well-defined, up to a scalar which will be fixed once chosen a basis for the vector spaces $V(x)$ and $W(x)$ for all $x \in Q_0$. For fixed $V \in Rep(Q, \alpha)$, we get a semi-invariant $c^V$ of the action of $GL(Q, \beta)$ on $Rep(Q, \beta)$, also well-defined up to a scalar. The weight of $c^V$ is $\langle \alpha, - \rangle : \gamma \mapsto \langle \alpha, \gamma \rangle$.

**Theorem 4.** The ring $SI(Q, \beta)$ is a $k$-linear span of the semi-invariants $c^V$, where $V$ runs through all representations of $Q$ with $\langle \dim(V), \beta \rangle = 0$.

**Remark 4.** As an algebra, $SI(Q, \beta)$ is generated by $c^V$ such that $V$ is indecomposable. This follows from the fact that, given representations $V, W$ with $\langle \dim(V), \dim(W) \rangle = 0$, if we have an exact sequence of representations:

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

then:

$$\langle \dim(V'), \dim(W) \rangle = 0 \quad \Rightarrow \quad c^V(W) = c^{V'}(W)c^{V''}(W).$$
The proof of the theorem is delivered in three steps.

i) Reduce to the case that $Q$ has one sink and one source, plus vertices with zero-weight.

ii) Eliminate vertices $x$ with zero-weight, by replacing pairs $(a, b)$ of arrows having $ta = hb = x$ with a single arrow $c$ having $tc = tb$ and $hc = ha$, so that $c$ represents $ba$. By the fundamental theorems of invariant theory (see Example 6) invariants of matrices for GL are given by such compositions.

iii) Treat the case of the Kronecker $\Theta_m$ quiver, with weights $(1, -1)$.

4. Quivers with relations

In many cases of interest, one should consider not all the $k$-representations of a given quiver $Q$, but only those that satisfy some relations. This leads to introduce relations in the path algebra $kQ$, and one speaks of quiver with relations, or bound quiver. This has drastic effects on the category $\text{Rep}(Q)$, which is no longer hereditary. All moduli problems tend thus to be much more complicated, in particular the deformation space to a Schurian representation need no longer be smooth, for obstructions are present in general.

**Definition 6.** A relation $\rho$ of $Q$ is a linear combination $\rho = \sum \lambda_i p_i$ of paths $p_i$, all having same head and tail. Given a representation $V$ of $Q$, we can evaluate $\rho(V)$ by considering $\rho(V) = \sum \lambda_i p_i(V)$, where for each path $p = a_1 \cdots a_m$ we define $V(p) = V(a_1) \circ \cdots \circ V(a_m)$. Given a finite set $R$ of relations of $Q$, a $k$-representation $V$ of $(Q, R)$ is a $k$-representation $V$ of $Q$ such that $\rho(V) = 0$, for all $\rho \in R$. The category $\text{Rep}(Q, R)$ of $k$-representations of $(Q, R)$ is also an abelian $k$-linear category.

Given a relation $\rho$, we get a two-sided ideal $(\rho)$ of $kQ$. Likewise, $R$ determines an ideal $(R)$ of $kQ$.

**Fact 5.** The category of left modules over the quotient algebra $kQ/\langle R \rangle$ is equivalent to the category of representations with relations $\text{Rep}(Q)_R$.

For a proof, we refer to [3, Chapter III].

4.1. Tilting bundles

An interesting example of quiver with relations arises when looking at tilting bundles on a smooth projective variety $X$ over an algebraically closed field $k$. One considers the derived category $\mathcal{D}^b(X)$ of bounded complexes of coherent sheaves on $X$, and defines, for a vector bundle $\mathcal{F}$ on $X$, the algebra $B = \text{End}_X(\mathcal{F})$. This gives rise to a functor $\text{coh}(X) \to B - \text{mod}$ that sends $\mathcal{F}$ to $B$, and any other coherent sheaf $\mathcal{E}$ on $X$ to $\text{Hom}_X(\mathcal{F}, \mathcal{E})$, seen as a $B$-module. The associated derived functor is denoted by $\Phi_{\mathcal{F}}$: 
$\mathcal{D}(X) \to \mathcal{D}(B\text{-mod})$, and $\mathcal{I}$ is called tilting if $\Phi_{\mathcal{I}}$ is an equivalence, see [4]. Equivalently, this means that the endomorphism algebra of $\mathcal{I}$ has finite global dimension (i.e., the projective dimension of any module over this algebra is bounded by a fixed integer), that $\mathcal{I}$ has no higher self-extensions ($\text{Ext}_X^k(\mathcal{I}, \mathcal{I}) = 0$ for $k > 0$) and that the indecomposable direct summands $\mathcal{I}_1, \ldots, \mathcal{I}_t$ of $\mathcal{I}$ generate the derived category of $X$.

A quiver with relations arises when considering a vertex $o_i$ for each direct summand $T_i$, and a basis of the vector space $\text{Hom}_X(T_i, T_j)$ as the set of arrows $o_i \to o_j$. Relations are natural too: they are generated by linear combinations of paths corresponding to the kernel of the multiplication map $\text{Hom}_X(T_i, T_k) \otimes \text{Hom}_X(T_k, T_j) \to \text{Hom}_X(T_i, T_j)$, for all $i, j, k$.

We refer to [17] for a nice treatment of tilting bundles in relation with quiver representations. A fundamental example is the following.

**Example 9.** For any integer $n \geq 1$, the Beilinson bound quiver $\mathcal{B}_n$ for $\mathbb{P}^n$ has $n + 1$ vertices $o_0, \ldots, o_n$. From a vertex $o_i$ to a vertex $o_{i+1}$, we draw $n + 1$ arrows, labelled by $x_0, \ldots, x_n$, and by the vertex $i$. For instance, $\mathcal{B}_2$ looks:

![Figure 6: Beilinson quiver $\mathcal{B}_2$.](image)

The relations $R$ in the path algebra $k\mathcal{B}_n$ are generated by the products of the following form, for all $0 \leq h \leq n$ and all $i, j$:

$$p_{i,j} = x_{i+h} x_{j}^h x_{j} x_{i}^h.$$

The algebra $k\mathcal{B}_n$ is isomorphic to:

$$\text{End}_{\mathbb{P}^n}(\mathcal{I}), \quad \text{with } \mathcal{I} = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n}(i).$$

In fact, the quiver associated with $\mathcal{I}$, for $n = 2$ would look like:

![Figure 7: Beilinson quiver $\mathcal{B}_2$.](image)
However, this gives rise to the same quiver with representations. The theorem of Beilinson asserts that the derived category of bounded complexes of coherent sheaves on $\mathbb{P}^n$ is equivalent to the derived category of finitely generated modules over this algebra, see [6]. We refer for instance to [35] for an account of Beilinson’s theorem and several related topics.

4.2. Homogeneous bundles

An interesting application of quiver representations is the description of homogeneous vector bundles over rational homogeneous variety, we refer to [9, 46]. For this section we work over $k = \mathbb{C}$.

A rational homogeneous variety is a product of projective varieties acted on transitively by simple Lie groups. Each of these factors is then of the form $X = G/P$, with $G$ a simple affine algebraic group over $\mathbb{C}$, and $P$ a parabolic subgroup. Recall that $P$ is not reductive in general. In fact, one writes $P = LN$, where $L$ is reductive (called a Levi factor of $P$) and $N$ is a nilpotent group, non-trivial in general. In terms of Lie algebras, we have a parabolic subalgebra $\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of $G$, and a decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{l}$ is a reductive Lie algebra and $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}$.

These varieties are completely classified. Indeed, the choice of $G$ corresponds to the choice of a Dynkin diagram of type $A_n, B_n, C_n, D_n$ for some $n$, or $E_6, E_7, E_8$ or $F_4$ or $G_2$. Once this choice is made, the subgroup $P$ corresponds to the choice of a subset of vertices of the Dynkin graph. The Levi part $\mathfrak{l}$ is obtained by removing the chosen vertices from the Dynkin diagram of $\mathfrak{g}$, and taking the associated Lie algebra.

A particularly simple class of rational homogeneous varieties is that of cominuscule varieties, also called compact Hermitian symmetric varieties: the variety $X$ is of this kind, by definition, if $[n, n] = 0$. In the notation of [10], the possible pairs of Dynkin diagram and vertex (only one vertex must be chosen) are $(A_n, \alpha_j)$ (Grassmannians); $(B_n, \alpha_1), (D_n, \alpha_1)$ (quadrics); or $(D_n, \alpha_n), (D_n, \alpha_{n-1})$ (spinor varieties); or $(C_n, \alpha_n)$ (Lagrangian Grassmannians); or $(E_6, \alpha_1), (E_7, \alpha_6)$ (the Cayley plane) or the $E_7$ variety $(E_7, \alpha_7)$. In the last two cases, $X$ is said to be of exceptional type.

A vector bundle $E$ on $X$ is homogeneous if, for all $g \in G$, one has $g^*E \cong E$. The category of homogeneous bundles on $X$ is equivalent to the category of finite-dimensional representations of $P$, or of $\mathfrak{p}$. Indeed, starting with $E$ one restricts to $E$ to the point $e$ of $X$ corresponding to the unit of $G$, and gets a vector space with a $P$-action, and vice-versa starting with a $P$-module $M$ we get a homogeneous bundle by $M \times_P G$. For instance, $n$ corresponds to $\Omega_X$.

Given a homogeneous bundle $E$ and the associated representation $M$ of $P$, restricting the action to the Levi factor $L$ gives a splitting of $M$ into irreducible $L$-modules $M_i$. Letting $N$ act trivially on these factors and taking the associated bundles $F_i$ we get the graded $\text{gr}(E) = \oplus F_i$; in fact $E$ has a filtration (non-split in general) with successive quotients isomorphic to the $F_i$. The irreducible bundles $F_i$ are in bijection with the weights of $\mathfrak{g}$ which are dominant for the semi-simple part $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{p}$. If $P = P(\alpha_j)$,
such weights are of the form $\sum n_i \lambda_i$ with $n_i \geq 0$ for all $i \neq j$, where $(\lambda_1, \ldots, \lambda_m)$ are the fundamental weights of $G$.

However, the $P$-module structure of $M$, and hence the vector bundle $E$, are recovered from $F = \text{gr}(E)$ by the equivariant maps, induced from one another:

$$\theta : \Omega_X \otimes F \rightarrow F, \quad \theta_e : n \otimes (\oplus j M_j) \rightarrow \oplus j M_j.$$

In fact $\theta_e$ lifts to a map of $p$-modules if and only if $\theta_e \wedge \theta_e = 0$, and in this case the lift is unique ($\theta_e$ gives the so-called structure of Higgs module). Similarly, $(\text{gr}(E), \theta)$ is called a Higgs bundle if $\theta \wedge \theta = 0$.

It turns out that the representations of $P$ can be conveniently described by a bound quiver $(Q, R)$ associated with $X$. The vertices of $Q$ are indexed by the irreducible representations of $\mathfrak{g}$. Given two such representations and the associated weights $\lambda, \mu$, we draw an arrow from the corresponding vertices $x_\lambda \rightarrow x_\mu$ in $Q$ if and only if $\text{Ext}^1(E_\lambda, E_\mu)^G \neq 0$. It turns out that, when this space is non-zero, it is one-dimensional. The relations $R$ of $Q$, if $X$ is not of exceptional type, are generated by the following two basic kinds:

i) for all weights $\lambda, \mu, \nu$, and all diagrams:

$$\begin{array}{ccc}
E_{\lambda+\frac{a}{c}} & \xleftarrow{a} & E_{\lambda} \\
d & & b \\
E_{\lambda+\mu+\frac{c}{d}} & \xrightarrow{c} & E_{\lambda+\mu}
\end{array}$$

we have, if $\text{Ext}^2(E_\lambda, E_{\lambda+\mu+\frac{1}{c}})^G \neq 0$, a relation of the form $da - cb = 0$.

ii) for all weights as above, and all diagrams:

$$\begin{array}{ccc}
E_{\lambda} & \xrightarrow{b} & E_{\lambda+\frac{1}{c}} \\
 & & \\
E_{\lambda+\mu+\frac{c}{d}} & \xrightarrow{c} & E_{\lambda+\mu}
\end{array}$$

we have, if $\text{Ext}^2(E_\lambda, E_{\lambda+\mu+\frac{1}{c}})^G \neq 0$, a relation of the form $cb = 0$.

These relations are called commutativity of all squares. Note that $Q$ is an infinite quiver; however, we only need representations of $Q$ with finite support. Moreover, $Q$ is disconnected, has no loops, and two vertices are joined by one arrow at most. For instance, for $X = \mathbb{P}^2$, $Q$ has three connected components. The vertices of the component of $Q$ containing $\mathcal{O}_{\mathbb{P}^2}$ are in bijection with the pairs of integers $(i, j)$ with $i \geq 0$ and $j \leq 0$, and the point $(i, j)$ corresponds to $\text{Sym}^{i-j}(\mathcal{O}(i+2j))$, where $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2}(-1)$ is the rank-2
quotient bundle. The arrows go from \((i, j)\) to \((i + 1, j)\) and \((i, j - 1)\). This component begins as follows, and continues indefinitely in the lower right quadrant:

\[
\begin{array}{cccc}
\mathcal{O}_{\mathbb{P}^2} & \mathcal{D}(1) & \text{Sym}^2 \mathcal{D}(2) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{D}(-2) & \text{Sym}^2 \mathcal{D}(-1) & \text{Sym}^3 \mathcal{D}
\end{array}
\]

**Theorem 5.** Let \(X\) be a compact Hermitian symmetric space, not of exceptional type. The following categories are equivalent:

i) \(G\)-homogeneous bundles on \(X\);

ii) finite-dimensional representations of \(p\);

iii) Higgs bundles \((F, \theta)\) on \(X\);

iv) finite-dimensional representations of \((Q, R)\).

More results are available in this direction, let us mention two of them, from [46]. For instance, a combinatorial way to compute cohomology of a homogeneous bundle can be derived from the above result, with the aid of Borel-Bott-Weil theorem (cf. for instance [55] for more on cohomology of homogeneous bundles). Moreover, the equivalence of categories just mentioned carries over to define equivalent stability conditions, in terms of equivariant slope-semi-stability for homogeneous bundles, or in terms of GIT for representations of \((Q, R)\). Therefore, one can speak of moduli space of homogeneous bundles in terms of moduli spaces of representations of \((Q, R)\).

5. Moduli spaces of quiver representations

Let us fix a quiver \(Q\), and an algebraically closed field \(k\). We have seen that, once fixed a dimension vector \(\alpha\) of \(Q\), all representations of dimension vector \(\alpha\) of \(Q\) form a vector space, acted on by a product of general linear groups. One would like to think of the quotient space as variety of isomorphism classes of representations of dimension vector \(\alpha\), or as the moduli space of these representations. As usual in this situation, one has to throw away some representations in order to put a structure of variety on the quotient, so that geometric invariant theory comes into play to select an open subset where the quotient behaves well. We will sketch the construction here, following the fundamental paper of King, [39].

5.1. Stability of quiver representations

Let us fix a dimension vector \(\alpha\) and a character \(\theta\) of \(G(Q, \alpha) = \text{GL}(Q, \alpha)/k^* \text{id}\). The character \(\theta = (\theta_x)_{x \in Q_0}\) is just a character of \(\text{GL}(Q, \alpha)\), and it is non-trivial if and only
Representations of quivers

27

if not all \( \theta_i \) equal a fixed integer.

**Definition 7.** Let \( V \) be a representation of dimension vector \( \alpha \). Then \( V \) is \( \theta \)-semi-stable if there is \( f \in \text{SI}(Q, \alpha)_\theta \), with \( t > 0 \), such that \( f(V) \neq 0 \). If moreover the orbit \( G.V \) is closed of dimension \( \dim(G) \), then \( V \) is stable. It turns out that a semi-stable representation admits a filtration with stable factors. The direct sum of these factors is the graded object associated with such a filtration. Two semi-stable representations with filtrations giving rise to the same graded object are said to be \( S \)-equivalent.

We write \( \text{Rep}(Q, \alpha)^{ss} \) and \( \text{Rep}(Q, \alpha)^{s} \) for the open subsets of semi-stable and stable representations (with respect to \( \theta \)), respectively. The character \( \theta \) is generic if all \( \theta \)-semi-stable representations are \( \theta \)-stable.

Let \( \sigma \in \Gamma^* \) and assume \( \theta = \theta_\sigma \). We see that, if \( V \in \text{Rep}(Q, \alpha) \) is \( \theta \)-stable, then \( \sigma(\alpha) = 0 \); indeed for any \( \theta \neq \lambda \in k^* \), since \( \lambda \text{id} \) acts trivially on \( V \), the semi-invariant \( f \) satisfies:

\[
0 \neq f(V) = f(\lambda \text{id}V) = \theta(\lambda)f(V) = \lambda^{\sigma(\alpha)}f(V).
\]

**Fact 6.** The representation \( V \) is \( \theta \)-semi-stable (respectively, stable) if and only if, for all proper subrepresentations \( W \) of \( V \), we have \( \sigma(\dim(W)) \leq 0 \) (respectively, \( \sigma(\dim(W)) < 0 \)).

### 5.2. Moduli space and ring of semi-invariants

We have a quotient map:

\[
\text{Rep}(Q, \alpha)^{ss} \longrightarrow \text{Proj} \left( \bigoplus_{t \geq 0} \text{SI}(Q, \alpha)_{\theta t} \right).
\]

**Definition 8.** Let \( \alpha \) be a dimension vector of \( Q \) and \( \theta \) be a character of the group \( G(Q, \alpha) \). Then the moduli space \( \mathcal{M}_Q(\alpha, \theta) \) is defined as:

\[
\mathcal{M}_Q(\alpha, \theta) = \text{Rep}(Q, \alpha) \sslash G = \text{Proj} \left( \bigoplus_{t \geq 0} \text{SI}(Q, \alpha)_{\theta t} \right).
\]

This space parametrizes \( S \)-classes of \( G(Q, \alpha) \)-orbits of \( \theta \)-semi-stable representations of \( Q \) with dimension vector \( \alpha \). This space contains the open set \( \mathcal{M}_Q(\alpha, \theta)^{\text{ss}} \) of \( G(Q, \alpha) \)-orbits of \( \theta \)-stable representations of \( Q \) with dimension vector \( \alpha \). Here, two representations \( V, W \) of \( Q \) are in the same \( S \)-class if they both admit filtrations of subrepresentations having \( \theta \)-stable factors, in such a way that the two direct sums of these factors are isomorphic.

**Fact 7 (King).** The space \( \mathcal{M}_Q(\alpha, \theta) \) is a projective variety, and if \( \alpha \) is indivisible \( \mathcal{M}_Q(\alpha, \theta)^{\text{ss}} \) is a fine moduli space for families of \( \theta \)-stable representations. In this case, the set of generic characters is dense, and for any such character \( \theta \), \( \mathcal{M}_Q(\alpha, \theta)^{\text{ss}} = \mathcal{M}_Q(\alpha, \theta) \) is smooth.
Projectivity follows from the GIT statement that there is a proper map:

\[
\text{Proj} \left( \bigoplus_{t \geq 0} \text{SI}(Q, \alpha)_t \right) \longrightarrow \text{Spec} \left( \text{Rep}(Q, \alpha)^G \right),
\]

and from the fact that the second scheme is a single point. The fact that the moduli space is fine (i.e., that there exists a universal family of representations over it) follows from a descent argument of the universal sheaf to the quotient by \( G \). Smoothness over the stable locus follows from the fact that \( \text{Rep}(Q) \) is hereditary.

### 5.3. Schofield’s result on birational type of quiver moduli

One more interesting fact concerning moduli spaces is that, once fixed \( Q \) and \( \alpha \), all moduli spaces \( \mathcal{M}_Q(\alpha, \theta) \) are birational (when non-empty). The birational transformation between \( \mathcal{M}_Q(\alpha, \theta) \) and \( \mathcal{M}_Q(\alpha, \theta') \) is called “wall-crossing”. A very nice result on the birational type of the moduli spaces \( \mathcal{M}_Q(\alpha, \theta) \) is due to Schofield, [51]. We would like to review it here. First of all, we have that, for a given quiver \( Q \) and a fixed dimension vector \( \alpha \), there is a stable representation for a given character \( \theta \) if and only if \( \alpha \) is a Schur root.

**Theorem 6.** Let \( \alpha \) be Schur root for \( Q \), and let \( \theta \) be a character of \( \text{GL}(Q, \alpha) \) such that there exists a \( \theta \)-stable \( V \in \text{Rep}(Q, \alpha) \). Then \( \mathcal{M}_Q(\alpha, \theta) \) is birational to the moduli space \( (k^{m \times m})^p \sslash \text{GL}_m(k) \) of \( p \) matrices of size \( m \) up to simultaneous conjugacy, for suitable \( m, p \).

Note that the rationality, and even the stable rationality of the above moduli space of matrices up to conjugacy, is unknown in general.

The proof of this result is carried out along these lines:

i) One shows the result for representations of the Kronecker quiver \( Q = \Theta_n \), as follows: given a dimension vector \((a, b)\) for \( \Theta_n \), with \( a, b \) coprime, one constructs two representations \( V_1 \) and \( V_2 \) such that:

- for a general representation \( V \) of dimension vector \((ma, mb)\) of \( \Theta_n \), we have \( \dim \text{Hom}_Q(V_1, V) = \dim \text{Hom}_Q(V_2, V) = m \);
- for the same \( V \), we have \( \text{Ext}_Q(V_1, V) = \text{Ext}_Q(V_2, V) = 0 \);
- we have \( \dim_k \text{Hom}_Q(V_1, V_2) = 1 + p \), with \( p = 1 - \langle (a, b), (a, b) \rangle = 1 - a^2 - b^2 + nab \).

With this setup, we have that \( \text{Hom}_Q(V_1 \oplus V_2, -) \) carries a general representation \( V \) to a representation of dimension vector \((m, m)\) of \( \Theta_{p+1} \). Up to the action of change of basis, this gives a birational equivalence between the moduli space of \( \Theta_n \)-representation of dimension vector \((ma, mb)\) and \((k^{m \times m})^p \sslash \text{GL}_m(k) \).
ii) The general case is reduced to the case of a Kronecker quiver by showing that a Schur root $\alpha$ admits smaller Schur roots $\beta$ and $\gamma$ such that $\alpha = a\beta + b\gamma$. This allows reduction to a dimension vector $(a, b)$ of a quiver with only two vertices (and only loops at the first vertex or arrows from the first to the second vertex).

5.4. More reading

Here are some suggestions on further topics tightly related to the above material. There is a lot more directions that one could be interested in, this is only an extremely partial and an subjective point of view.

Representation type of algebras and varieties

In view of the separation of quivers into the three different kinds finite, tame or wild, according to their representation theory, one can ask about the representation type of more general algebras, see for instance [23,25,45] for some early work in this direction. The literature in this area is fairly vast.

For graded Cohen-Macaulay rings $R$, this question leads to study families of maximal Cohen-Macaulay (MCM) modules over $R$ (see for instance [2]). If $R$ is the homogeneous coordinate ring of a smooth projective variety $X$, these families can be seen as families of vector bundles over $X$, whose intermediate cohomology modules are all zero.

One result of great interest for algebraic geometers is the classification of varieties of finite type in this sense (there are only finitely many indecomposable MCM modules over their coordinate rings). These are: projective spaces, smooth quadrics, rational normal curves, the Veronese surface in $\mathbb{P}^5$, and the rational cubic scroll in $\mathbb{P}^4$, see [29] and references therein. Although the finite-tame-wild trichotomy is not exhaustive for all varieties, it still might be so for smooth varieties. As last remarks let us note that several varieties are known by now to be of wild representation type, see for instance [16] for Segre products. Quite interestingly, the main argument to prove such a statement is to construct families of vector bundles with few cohomology by using again the Kronecker quiver $\Theta_m$, plus Kac’s theorem to see that the induced bundles are simple.

Cohomological properties of quiver moduli

The Betti numbers of the moduli spaces $\mathcal{M}_Q(\alpha, \theta)$ have been computed in [48]. The method here goes through a Hall-algebra variation of the Harder-Narasimhan method used for vector bundles over projective curves, plus Weil conjectures to reduce to counting quiver representations over finite fields.

In a different spirit, a nice paper by Craw investigates the geometry and the derived category of the moduli spaces $\mathcal{M}_Q(\alpha, \theta)$ when $\alpha$ is indivisible, $\theta$ is generic and $Q$ has no loops. These moduli spaces are towers of Grassmann bundles, and are equipped with a natural tilting bundle, see [19].
**Toric varieties and quiver moduli**

Toric varieties are closely related to moduli spaces of representations of quivers. In particular, Craw and Smith proved that a projective toric variety is a fine moduli space of representations of an appropriate bound quiver, [20], see also [18] for a comprehensive treatment. The quivers $Q$ and dimension vectors $\alpha$ involved in this construction are so that $\alpha$ has all values equal to 0 or 1, hence the group $\text{GL}(Q, \alpha)$ is already an algebraic torus.

**Moduli of semistable sheaves as quiver moduli**

Let $X$ be a smooth projective variety over an algebraically closed field, and $H$ be an ample divisor class on $X$. The moduli space of $H$-semi-stable sheaves on $X$ with fixed Hilbert polynomial with respect to $H$ has been constructed relying on ideas of Maruyama and Gieseker, and turns out to be a projective variety (see [52], and [41] for the positive characteristic case).

A result due to Álvarez-Cónsul and King, [1] ties in this construction with the moduli spaces $\mathcal{M}_Q^{\alpha}(\cdot, \cdot)$. Indeed, given $X$, $L$ and the Hilbert polynomial $P(t)$, one can choose two integers $m, p$ such that for any coherent sheaf $E$ on $X$ with Hilbert polynomial $P(t)$, one has $P(m) = h^0(E(m))$, $P(p) = h^0(E(p))$ and:

$$\begin{align*}
H^0(X, E(m)) \otimes \mathcal{O}_X(-m) &\twoheadrightarrow E, \\
H^0(X, E(m)) \otimes H^0(X, \mathcal{O}_X((p-m))) &\twoheadrightarrow H^0(X, E(p)).
\end{align*}$$

Set $n = h^0(X, \mathcal{O}_X((p-m)))$. This way, the point $E$ of the moduli space is identified with a representation of $\Theta_n$ of dimension vector $(P(m), P(p))$. One more amazing manifestation of the ubiquitous Kronecker quiver!

**References**


AMS Subject Classification: 16G20

Faenzi, Daniele
Laboratoire de mathématiques et de leurs applications - UMR CNRS 5142
Université de Pau et des Pays de l’Adour
Avenue de l’Université - BP 576 - 64012 PAU Cedex - France
e-mail: daniele.faenzi@univ-pau.fr

Lavoro pervenuto in redazione il 18.06.2013.
L. Manivel

PREHOMOGENEOUS SPACES AND PROJECTIVE GEOMETRY

Abstract. These notes are intended as an introduction to the theory of prehomogeneous spaces, under the perspective of projective geometry. This is motivated by the fact that in the classification of irreducible prehomogeneous spaces (up to castling transforms) that was obtained by Sato and Kimura, most cases are of parabolic type. This means that they are related to certain homogeneous spaces of simple algebraic groups, and reflect their geometry. In particular we explain how the Tits-Freudenthal magic square can be understood in this setting.

Contents

1. Introduction ................................................................. 36
2. Complex semisimple Lie algebras and their representations ............... 38
   2.1. Simple Lie algebras ............................................... 38
   2.2. Representations of semisimple Lie algebras ...................... 47
   2.3. Linear algebraic groups ......................................... 55
3. Nilpotent orbits ......................................................... 64
   3.1. The nilpotent cone .............................................. 64
   3.2. Classification of nilpotent orbits ................................ 70
4. Prehomogeneous spaces, generalities .................................... 81
   4.1. Prehomogeneous vector spaces .................................. 81
   4.2. Regular invariants ............................................... 83
   4.3. Cremona transformations ....................................... 85
   4.4. Castling transforms ............................................ 87
   4.5. The case of tensor products .................................... 89
   4.6. Relations with projective duality ................................ 90
5. Prehomogeneous spaces of parabolic type ................................ 92
   5.1. Classification of $\mathbb{Z}$-gradings ............................ 92
   5.2. Parabolic prehomogeneous spaces ............................... 93
   5.3. Classification of orbits ........................................ 95
   5.4. The closure ordering .......................................... 98
   5.5. Desingularizations of orbit closures ........................... 99
   5.6. Classification of irreducible prehomogeneous vector spaces ........ 100
   5.7. $\mathbb{Z}_{m\mathbb{Z}}$-graded Lie algebras .......................... 102
6. The magic square and its geometry .................................... 103
   6.1. The exceptional group $G_2$ and the octonions .................. 103
   6.2. The magic square ............................................... 107
   6.3. The geometry of the magic rectangle ........................... 110
Conclusion ........................................................................... 114
References ........................................................................... 115
1. Introduction

Prehomogeneous vector spaces are vector spaces endowed with a linear action of an algebraic group, such that there exists a dense orbit. In these notes we will be mainly interested in the action of a reductive complex algebraic group $G$ over a complex vector space $V$ of finite dimension $n$. Moreover we will focus on such prehomogeneous vector spaces, and related objects, as providing a wealth of interesting examples for algebraic geometers. These examples can be of various kinds. Let us mention a few of them.

- Many prehomogeneous vector spaces come with a unique relative invariant, a homogeneous polynomial which is $G$-invariant, possibly up to scalars. Under mild hypothesis this polynomial is automatically homaloidal, which means that its partial derivatives define a birational endomorphism of the projective space $\mathbb{P}^{n-1}$. Such birational endomorphisms are a priori difficult to construct. Prehomogeneous vector spaces allow to get many of them, with particularly nice properties.

- Prehomogeneous vector spaces are often more than just prehomogeneous: they can have only finitely many $G$-orbits. The orbit closures, and their projectivizations, are interesting algebraic varieties, in general singular (the archetypal example is that of determinantal varieties). They have a rich geometry, including from the point of view of singularities. Notably, they have nice resolutions of singularities, and sometimes nice non commutative resolutions of singularities, in the sense of Van den Bergh. A systematic exploration of these issues has been begun in [33, 34, 55].

- Many irreducible prehomogeneous vector spaces are parabolic, in which case they have finitely many orbits which can be classified in the spirit, and in connection with, nilpotent orbits of semisimple Lie algebras. Nilpotent orbits are important examples of symplectic varieties, being endowed with the Kostant-Kirillov-Souriau form. Their closures are nice examples of symplectic singularities. Moreover, we could try to compactify certain nilpotent orbits in such a way that the symplectic structure extends to the boundary, so as to construct compact holomorphic symplectic manifolds – a particularly mysterious and fascinating class of varieties.

- Beautiful examples of homogeneous projective varieties connected with exceptional algebraic groups have been studied by Freudenthal and its followers, in connection with his efforts, and those of other people, notably Tits and Vinberg, to understand the exceptional groups as automorphism groups of certain types of geometries. A culminating point of these efforts has certainly been the construction of the Tits-Freudenthal magic square. On the geometric side of the story, the very nice properties of the varieties studied by Freudenthal are deeply related with the existence of certain prehomogeneous vector spaces. This includes the series of Severi varieties, rediscovered by Zak and Lazarsfeld in connection with the Hartshorne conjectures on low codimensional smooth subvarieties of projective spaces.
Although our motivations are certainly of geometric nature, an important part of the notes are devoted to the Lie theoretic aspects of the story. They are organized as follows.

The first chapter is a quick reminder of the classical theory of complex semisimple Lie algebras, their representations, the corresponding algebraic groups and their homogeneous spaces. The material of this chapter can be found, with more details, in many good textbooks. We have decided to include it in the notes not only to fix notations but, more importantly, to provide the algebraic geometer who would not be familiar enough with Lie theory, with a brief compendium of what he should assimilate in order to access the next chapters.

The second chapter is devoted to nilpotent orbits, which is also a classical topic treated thoroughly in several textbooks. We insisted on two aspects of the theory. First, the geometric properties of nilpotent orbits and their closures: this includes the Kostant-Kirillov-Souriau form, the Springer resolution and its variants, the structure of the nilpotent cone and so on. Second, the classification problem and the two classical solutions provided by weighted Dynkin diagrams and Bala-Carter theory.

Our treatment of prehomogeneous vector spaces really begins in the third chapter, where the point of view is rather general. We discuss the existence of relative invariants and their properties, including the important connection with birational endomorphisms, or the question of computing their degrees, a problem for which projective duality can be of great help.

In the fourth chapter we focus on parabolic prehomogeneous vector spaces, which come from \(\mathbb{Z}\)-gradings of semisimple Lie algebras. We explain a method devised by Vinberg in order to classify all the orbits in parabolic prehomogeneous spaces, in connection with the classification of nilpotent orbits. We end the chapter with a version of the fundamental classification theorem of irreducible reductive prehomogeneous spaces, due to the monumental work of Sato and Kimura [46]. It turns out a posteriori that parabolic prehomogeneous spaces provide the most substantial chunk of the classification.

The fifth and final chapter is devoted to the Tits-Freudenthal magic square and its geometric aspects. We explain how to construct a Lie algebra, including all the exceptional ones, from a pair of normed algebras. From the minimal nilpotent orbits of the exceptional Lie algebras we explain how to produce series of prehomogeneous spaces and projective varieties with particularly nice properties. Remarkably, these series of varieties are closely connected with important conjectures, notably the LeBrun-Salamon conjecture which, despite several decided attempts, still remains out of reach.

These notes have been written for the school on Invariant Theory and Projective Geometry held at the Fondazione Bruno Kessler, in Trento, in september 2012. We warmly thank V. Baldoni, G. Casnati, C. Fontanari, F. Galluzzi, R. Notari and F. Vaccarino for the perfect organization.
2. Complex semisimple Lie algebras and their representations

2.1. Simple Lie algebras

In this lecture we briefly cover some standard material, the study and the classification of complex semisimple Lie algebras via their root systems. The reader is encouraged to consult [7, 21, 22, 47] for more details.

The Cartan-Killing classification

**Definition 1.** A Lie algebra is a vector space $\mathfrak{g}$ over a field $k$, endowed with a skew-symmetric bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, such that

$$\forall X, Y, Z \in \mathfrak{g}, \quad [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$ 

This identity is called the Jacobi identity.

One can try to understand finite groups by looking for normal subgroups, or commutative algebras by considering ideals. The same idea can be applied to Lie algebras.

**Definition 2.** An ideal $\mathfrak{i}$ in a Lie algebra $\mathfrak{g}$ is a subvector space such that $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$. In particular it is a Lie subalgebra, and the quotient space $\mathfrak{g}/\mathfrak{i}$ has an induced structure of Lie algebra.

If a Lie algebra has a proper ideal (proper means different from zero or the Lie algebra itself), it can be constructed as an extension of two smaller Lie algebras. This leads to the following definition, where a Lie algebra is noncommutative if its Lie bracket is not identically zero.

**Definition 3.** A simple Lie algebra is a noncommutative Lie algebra without any proper ideal.

Simple complex Lie algebras ($k = \mathbb{C}$) were classified at the end of the 19th century by W. Killing and E. Cartan. There are four (or rather three) infinite series $\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$ of classical Lie algebras and five exceptional Lie algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$.

Every simple complex Lie algebra is encoded in a Dynkin diagram. The Dynkin diagrams of the classical Lie algebras are as follows (the integer $n$ always denotes the number of vertices; the right-most column gives the dimension):
Prehomogeneous spaces and projective geometry

These algebras are familiar:

- \( \mathfrak{sl}_n \) is the Lie algebra of traceless matrices of size \( n \), the Lie bracket being given by the usual commutator \([X,Y] = XY - YX\);
- the \( n \)-th orthogonal Lie algebra \( \mathfrak{so}_n \subset \mathfrak{sl}_n \) is the subalgebra of skew-symmetric matrices;
- the \( n \)-th symplectic Lie algebra \( \mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n} \) is the subalgebra of matrices \( X \) such that \( tXJ + JX = 0 \), where \( J \) is any invertible skew-symmetric matrix; up to isomorphism, this algebra does not depend on \( J \).

The Dynkin diagrams of the exceptional Lie algebras are:

- \( G_2 \)
- \( F_4 \)
- \( E_6 \)
- \( E_7 \)
- \( E_8 \)

**The Killing form**

On any Lie algebra \( \mathfrak{g} \), one can define a symmetric bilinear form as follows. For any \( x \in \mathfrak{g} \), denote by \( \text{ad}(x) \) the operator on \( \mathfrak{g} \) defined by \( \text{ad}(x)(y) = [x,y] \), and let

\[
K(x,y) = \text{trace}(\text{ad}(x)\text{ad}(y)), \quad x,y \in \mathfrak{g}.
\]
This is the Killing form. Despite its name, its importance was stressed by E. Cartan. Its first relevant property is its invariance:

\[ K(x, [y,z]) = K([x,y], z), \quad x, y, z \in g. \]

The Killing form is able to detect important properties of the Lie algebra. For example, a Lie algebra is said to be \textit{semisimple} if it can be decomposed as a direct sum of simple Lie algebras.

\textbf{Proposition 1.} (Cartan’s criterion). A Lie algebra \( g \) is semisimple if and only if its Killing form is nondegenerate.

\textbf{Cartan subalgebras.} The main idea to classify simple complex Lie algebras is to diagonalize them with the help of a maximal abelian subalgebra whose elements are all semisimple. By \( x \in g \) semisimple we mean precisely that the operator \( \text{ad}(x) \), acting on \( g \), is diagonalizable. In the sequel \( g \) will be simple, and \( t \) will denote such a maximal subalgebra, which is called a \textit{Cartan subalgebra}.

\textbf{Example 1.} Check that \( sl_n \) is simple and that the subalgebra of diagonal matrices is a Cartan subalgebra.

Once we have a Cartan subalgebra \( t \), we can simultaneously diagonalize its adjoint action on \( g \). We get a decomposition

\[ g = \bigoplus_{\alpha \in t^*} g_{\alpha}, \quad g_{\alpha} = \{ x \in g, \ \text{ad}(h)(x) = \alpha(h)x \ \forall h \in t \}. \]

The Jacobi identity implies that

\[ [g_{\alpha}, g_{\beta}] \subset g_{\alpha+\beta}. \tag{1} \]

An easy consequence is that

\[ K(g_{\alpha}, g_{\beta}) = 0 \text{ whenever } \alpha + \beta \neq 0. \tag{2} \]

Since the Killing form is nondegenerate, this allows one to identify \( g_{-\alpha} \) with the dual of \( g_{\alpha} \). In particular, the restriction of the Killing form is nondegenerate on \( g_0 = C(t) = \{ x \in g, \ [x,t] = 0 \} \), the \textit{centralizer} of \( t \). One can prove that a Cartan subalgebra is its own centralizer, i.e., \( g_0 = C(t) = t \).

We can therefore write

\[ g = t \oplus \bigoplus_{\alpha \in \Phi} g_{\alpha}, \]

where \( \Phi \) is the set of nonzero linear forms \( \alpha \in t^* \) such that \( g_{\alpha} \neq 0 \). These are called the \textit{roots} of the pair \((g,t)\), and \( \Phi \subset t^* \) is the \textit{root system}.

Since the Killing form of \( g \) is nondegenerate on \( t \), we can identify \( t^* \) with \( t \). In particular, for each root \( \alpha \in t^* \) we have an element \( h_{\alpha} \in t \) defined by \( \beta(h_{\alpha}) = K(\alpha, \beta) \) for all \( \alpha, \beta \in \Phi \), where \( K \) is dual to the restriction of the Killing form on \( t \).
Properties of the root system

The key of the classification of simple Lie algebras will be to show that the root system $\Phi$ has very special properties.

**Proposition 2.** $\Phi$ generates $t^\ast$ and is symmetric with respect to the origin. Moreover, for all $\alpha \in \Phi$,

1. if $x \in g_\alpha$ and $y \in g_{-\alpha}$, then $[x,y] = K(x,y)h_\alpha$,

2. $\alpha(h_\alpha) \neq 0$,

3. $\dim g_\alpha = 1$.

This last property, and the fact that $[g_\alpha, g_\beta] \subset g_{\alpha + \beta}$, already give very precise information about the Lie algebra $g$. Indeed, we can choose for each root $\alpha$ a generator $x_\alpha$ of $g_\alpha$, and the Lie bracket is then given by

$$[h,h'] = 0 \quad \text{for } h,h' \in t,$$

$$[h,x_\alpha] = \alpha(h)x_\alpha \quad \text{for } h \in t,$$

$$[x_\alpha,x_{-\alpha}] = K(x_\alpha,x_{-\alpha})h_\alpha,$$

$$[x_\alpha,x_{\beta}] = C_{\alpha,\beta}x_{\alpha + \beta}, \quad \text{when } \alpha + \beta \neq 0,$$

for some structure constants $C_{\alpha,\beta}$ to be determined. To classify the simple Lie algebras, it is therefore necessary to classify the root systems first. To do this we need a deeper understanding of their properties.

A key point is the following observation. Choose a root $\alpha$, and then $X_\alpha \in g_\alpha$ and $X_{-\alpha} \in g_{-\alpha}$ such that $K(X_\alpha,X_{-\alpha}) = 2/\alpha(h_\alpha)$. If $H_\alpha = [X_\alpha,X_{-\alpha}]$, we get that $\alpha(H_\alpha) = 2$, and $H_\alpha$ is called the coroot of $\alpha$. We deduce the relations $[H_\alpha,X_\alpha] = 2X_\alpha$ and $[H_\alpha,X_{-\alpha}] = -2X_{-\alpha}$, which are precisely the commutation relations of the generators of $\mathfrak{sl}_2$.

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This means that $g^{(\alpha)} = g_\alpha \oplus g_{-\alpha} \oplus [g_\alpha, g_{-\alpha}]$ is isomorphic to $\mathfrak{sl}_2$. Moreover, for each root $\beta$, the space

$$g^{(\alpha)}_\beta = \bigoplus_{k \in \mathbb{Z}} g_{\beta + k\alpha}$$

is preserved by the adjoint action of $g^{(\alpha)}$, hence inherits the structure of a $\mathfrak{sl}_2$-module.

It is easy to construct $\mathfrak{sl}_2$-modules of arbitrary dimensions. For any integer $k \geq 0$, denote by $V_k$ the space of homogeneous polynomials of degree $k$ in two variables $u$ and $v$. We can define an action of $\mathfrak{sl}_2$ on $V_k$ by letting the generators $X,Y,H$ of $\mathfrak{sl}_2$ act as

$$X = v \frac{\partial}{\partial u}, \quad Y = u \frac{\partial}{\partial v}, \quad H = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}.$$
One can check that $V_k$ is an irreducible $\mathfrak{sl}_2$-module, which means that it does not contain any proper submodule. Note that $H$ acts on $\mathfrak{sl}_2$ by multiplication by $2i - k$, and these integers form a chain from $-k, -k + 2, \ldots$ to $k - 2, k$.

**Theorem 1.** Every finite dimensional $\mathfrak{sl}_2$-module is completely decomposable into a direct sum of irreducible modules. Every irreducible module of dimension $k + 1$ is isomorphic to $V_k$.

The information we have on $\mathfrak{sl}_2$-modules can now be applied to the action of $g'$ on $g'$. Since $H'$ acts on $g'$ by multiplication by $(H') - 2q = (H') - 2p$, that is $|H'| = q - p \in \mathbb{Z}$. Consequences:

1. For any roots $\alpha, \beta \in \Phi$, $s_\alpha(\beta) := \beta - (H') \alpha$ is again a root, since $-q \leq \beta(H') = p - q \leq p$.

2. If two roots are colinear, they are equal or opposite: indeed, if $\beta = t\alpha$, say $t > 1$, then $2 = \beta(H') = t\alpha(H')$, and since $\alpha(H')$ is an integer this forces $t = 2$, which is impossible.

Note also that for $h, h' \in t$, $K(h, h') = \sum_{\alpha \in \Phi} \alpha(h)\alpha(h')$, so that the Killing form is positive definite on a real form of $t$. These properties will be the basis of the definition of abstract reduced root systems.

**Abstract root systems**

**Definition 4.** An abstract root system is a finite subset $\Phi$ of some finite dimensional Euclidian (real) vector space $V$, such that

1. $0 \notin \Phi$ and $\Phi$ generates $V$,

2. $\forall \alpha, \beta \in \Phi$, $c_{\alpha\beta} = 2\frac{\beta \cdot \alpha}{(\alpha, \alpha)} \in \mathbb{Z}$,

3. $\forall \alpha, \beta \in \Phi$, $s_\alpha(\beta) = \beta - c_{\alpha\beta}\alpha \in \Phi$,

4. If $\alpha, \beta \in \Phi$ are colinear, they are equal or opposite.

Note that the definition of $s_\alpha$ extends to $V$, and that the resulting map is just the orthogonal symmetry with respect to the hyperplane $\alpha$. In particular, $s_\alpha(\alpha) = -\alpha \in \Phi$.

The subgroup of the orthogonal group of $V$, generated by the reflections $s_\alpha$, $\alpha \in \Phi$, is a subgroup of the permutation group of $\Phi$. In particular, it is a finite group, called the Weyl group of the root system.

Of course, the root system of a simple complex Lie algebra will be an abstract root system. We have already proved it, except that the root system is naturally embedded in a complex vector space (the dual $t^*$ of a Cartan subalgebra), not in a real
Euclidean space. But the real span of the root system is a real subspace of $t^*$ whose real dimension is the complex dimension of $t$, and the restriction of the Killing form provides it with a Euclidean structure.

REMARK 1. The intersection of an abstract root system with any subspace of $V$ is again an abstract root system in the subspace that it generates.

REMARK 2. If we let $\tilde{\alpha} = \alpha / \langle \alpha, \alpha \rangle$, we get another abstract root system in $V$, with the same Weyl group. This is the dual root system of $\Phi$.

The dimension of $V$ is called the rank of $\Phi$. Up to scale, there is only one abstract root system of rank one, denoted $A_1$. In rank two, observe that if $\alpha, \beta$ are not parallel, $c_{\alpha\beta}c_{\beta\alpha} = 4\cos^2(\alpha, \beta) \in \{0, 1, 2, 3\}$. This implies that $c_{\alpha\beta}$ or $c_{\beta\alpha}$ is equal to $\pm 1$. Changing $\alpha$ in $\beta$ or $-\beta$ if necessary, we can suppose that $c_{\alpha\beta} = -1$. Then for each possible value of $c_{\alpha\beta} \in \{0, -1, -2, -3\}$, the angle between $\alpha$ and $\beta$, and (if $c_{\alpha\beta} \neq 0$) their relative lengths are fixed. This being true for any pair of roots, it is a simple exercise to check that any abstract root system of rank two is equivalent, up to isometry, to one of the four systems below, denoted $A_1 \times A_1$, $A_2$, $B_2$, or $C_2$.

The abstract root system $A_1 \times A_1$ is decomposable, i.e., it is a product of two proper systems. The other rank two systems are indecomposable. This is equivalent to the fact that the action of the Weyl group on the ambient space is irreducible.

In order to classify abstract root systems of higher rank, let $v$ be a linear form on $V$ that does not vanish on $\Phi$. This splits $\Phi$ into the disjoint union of $\Phi^+ = \{\alpha \in \Phi, v(\alpha) > 0\}$ and $\Phi^- = -\Phi^+$. Define a positive root $\alpha \in \Phi^+$ to be simple, if it cannot be written as the sum of two positive roots.

PROPOSITION 3. The set of simple roots $\Delta \subset \Phi^+$ is a basis of $V$. Moreover, each positive root is a linear combination of simple roots with non negative integer coefficients.

Now that we have a set of simple roots, we can define the Dynkin diagram $D(\Phi)$ as follows. This is an abstract graph whose vertices are in correspondence with the simple roots. If $\alpha, \beta \in \Delta$, we join the corresponding vertices by $c_{\alpha\beta}c_{\beta\alpha} \in \{0, 1, 2, 3\}$ edges; if $\alpha$ and $\beta$ do not have the same length, these edges are given arrows pointing to the smallest of the two simple roots.

Note that if $D(\Phi)$ is given, the angles and the respective lengths of the simple roots are known, hence the set of simple roots is determined up to isometry. Moreover, the integers $c_{\alpha\beta}$ can also be read off $D(\Phi)$, and this allows one to recover the full Weyl group. Applying the Weyl group to the simple roots, we get the whole root system back (one can show that the Weyl group acts transitively on the set of roots of the same length in an indecomposable root system, and that there can be at most two different lengths: when the roots do not have all the same length, we distinguish between short
roots and long roots). Conclusion: up to isometry, an abstract root system is completely determined by its Dynkin diagram.

\[ A_1 \times A_1, \quad A_2, \quad B_2, \quad G_2 \]

**Example 2.** Consider the Lie algebra \( \mathfrak{sl}_n \), and the Cartan subalgebra \( t \) of diagonal traceless matrices. Let \( \varepsilon_i(h) = h_i \), the \( i \)-th diagonal coefficient of \( h \in t \). The adjoint action of \( t \) on \( \mathfrak{sl}_n \) is diagonal in the basis given by a basis of \( t \) and the set of matrices \( e_{ij} \) with only one nonzero coefficient, a one at the intersection of the \( i \)-th line and the \( j \)-th column, where \( i \neq j \). The corresponding root is \( \varepsilon_i - \varepsilon_j \), and the Killing form is therefore given on \( t \) by

\[
K(h,h) = \sum_{i \neq j} (h_i - h_j)^2 = n \sum h_i^2.
\]

Choose the positive roots to be the \( \varepsilon_i - \varepsilon_j \) with \( i < j \). The simple roots are then the \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \), for \( 1 \leq i < n \), and we get that \( c_{\alpha_i \alpha_j} = -1 \) if \( |i - j| = 1 \) and 0 otherwise. The Dynkin diagram is thus a chain of length \( n - 1 \), denoted by \( A_{n-1} \).

The matrix \( (c_{\alpha \beta})_{\alpha, \beta \in \Delta} \) is called the Cartan matrix. It is an integer matrix, whose diagonal coefficients are equal to two and all other coefficients are nonpositive. Moreover, it has the important property that the closely related matrix \( (\langle \beta, \varepsilon \rangle c_{\alpha \beta})_{\alpha, \beta \in \Delta} \), is symmetric and positive definite, since its is just the matrix of the Killing form (or rather its dual) in the basis of simple roots.
EXAMPLE 3. Consider the matrix

\[
C = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

This is the Cartan matrix of the root system \( \Phi \subset \mathbb{R}^8 \),

\[
\Phi = \{ \pm \varepsilon_i \pm \varepsilon_j, i \neq j \} \cup \{ \frac{1}{2} \{ \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \varepsilon_7 \pm \varepsilon_8 \},
\]

where, for the roots in the second set, there is always an even number of minus signs. The first set has cardinality \( 4 \binom{8}{2} = 112 \), and the second one \( \sum_{0 \leq k \leq 4} \binom{8}{2k} = 128 \), hence \( \# \Phi = 240 \). Choosing the linear form \( v = N \varepsilon_8 - \varepsilon_1^2 - 2\varepsilon_2^2 - 3\varepsilon_3^2 - 4\varepsilon_4^2 - 5\varepsilon_5^2 - 6\varepsilon_6^2 \), with \( N \) large enough, we get the following sets of positive and simple roots:

\[
\Phi^+ = \{ \pm \varepsilon_i - \varepsilon_j, i < j \leq 8 \} \cup \{ \pm \varepsilon_i + \varepsilon_8, i < 8 \}
\]

\[
\cup \{ \frac{1}{2} \{ \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \varepsilon_7 \pm \varepsilon_8 \},
\]

\[
\Delta = \{ -\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4 - \varepsilon_5, \varepsilon_5 - \varepsilon_6, \varepsilon_6 - \varepsilon_7,
\]

\[
\frac{1}{2} \{ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8 \} \}
\]

It remains to compute the scalar products of the simple roots to obtain the Dynkin diagram

\[
\begin{array}{c}
E_8 \\
\end{array}
\]

Taking hyperplane sections one can obtain abstract root systems of type \( E_7 \), and then \( E_6 \). The Dynkin diagram of \( E_6 \) has a twofold symmetry which can be used to deduce a root system of type \( F_4 \) by folding. Similarly, the root system \( G_2 \), which we have already described, can be obtained by folding a root system of type \( D_4 \), the only indecomposable root system with a threefold symmetry.

The idea is the following: suppose that the Dynkin diagram of a simple Lie \( g \) has a symmetry, which means that, once we have fixed a Cartan subalgebra \( g \) and a set \( \Delta \) of simple roots, there exists a bijection \( s \) of \( \Delta \), preserving the Cartan integers. We extend this bijection to an automorphism of \( g \). To do this, one has first to show that \( g \) has a Chevalley basis, i.e., a system of vectors \( X_\alpha \in g_\alpha \) such that \( [X_\alpha, X_{-\alpha}] = H_\alpha \), and such that the linear map sending \( X_\alpha \) to \( X_{-\alpha} \), and equal to \(-Id\) on \( \mathfrak{h} \), is a Lie algebra automorphism of \( g \). Then we let

\[
s(X_\alpha) = X_{s(\alpha)}, \quad s(H_\alpha) = H_{s(\alpha)}, \quad \forall \alpha \in \Phi,
\]

and check that this defines a Lie algebra automorphism of \( g \).
Let $\tilde{\mathfrak{g}}$ denote the invariant subalgebra of $\mathfrak{g}$, and $\tilde{\mathfrak{h}}$ denote the invariant subalgebra of $\mathfrak{h}$. By restriction, we have a map $r : \mathfrak{h}^* \to \tilde{\mathfrak{h}}^*$, and we can let $\tilde{\Delta} = r(\Delta)$. Then $\tilde{\Delta} \simeq \Delta/\langle s \rangle$, the automorphism group generated by $s$. Moreover, one proves that $\tilde{\mathfrak{g}}$ is semisimple, with Cartan subalgebra $\tilde{\mathfrak{h}}$ and set of simple roots $\tilde{\Delta}$.

One can then deduce the Dynkin diagram of $\tilde{\mathfrak{g}}$ from that of $\mathfrak{g}$. For twofold symmetries, we get the following possibilities for $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$:

The properties of Cartan matrices are fundamental for the classification of root systems: once they have been observed, the classification is just a cumbersome exercise of Euclidian geometry. There are many variants of solutions, but the main point is that, since the restriction of a scalar product remains a scalar product, any subdiagram of an admissible diagram (i.e., a Dynkin diagram of an irreducible root system) must be admissible. Then one exhibits enough simple nonadmissible diagrams to reduce the possibilities for the admissible diagrams. For example, the two diagrams

are not admissible, and this implies that an admissible diagram has at most triple bonds, each vertex having valency at most three. The upshot is the following theorem due to Killing (circa 1887 – but Killing found two systems of type $F_4$, that Cartan recognized later to be equivalent).

**Theorem 2.** (Classification of root systems). An indecomposable Dynkin diagram must be of type A, B, C, D, E, F or G.

This is not the end of the story. Two statements have yet to be proven:

1. Every indecomposable abstract root system is the root system of a simple complex Lie algebra.
2. A simple complex Lie algebra has a uniquely defined Dynkin diagram and, up to isomorphism, is completely defined by this diagram.
The first point is an existence theorem. The case of Dynkin diagrams of type $A$, $B, C, D$ is easy, since they correspond to the classical Lie algebras. The exceptional Lie algebras $g_2, f_4, e_6, e_7$ and $e_8$ can be constructed explicitly in many different ways – we will see how in the last chapter of these lectures. One can also analyze in more details the structure constants $C_{ij}$ and deduce an abstract existence theorem; this was done by Chevalley and Tits. Another method, due to Serre, consists in defining the Lie algebra by generators and relations directly from the Cartan matrix.

The second point is a unicity theorem. The construction of the Dynkin diagram of a Lie algebra depends on several choices. First, one has to prove that the Cartan subalgebras of a semisimple Lie algebra $g$ are conjugate under automorphisms of $g$. Second, one shows that the Weyl group acts transitively on the sets of simple roots of the root system. Conversely, the fact that the structure of $g$ is completely determined by the Dynkin diagram, or by the root system, follows from a careful analysis of the structure constants.

Note that there are a few redundancies between the classical Dynkin diagrams. By the unicity theorem we have just mentioned, they detect some isomorphisms between certain small dimensional complex Lie algebras:

- $B_1 = A_1 \Rightarrow \mathfrak{so}_3 \cong \mathfrak{sl}_2$
- $D_2 = A_1 \times A_1 \Rightarrow \mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$
- $B_2 = C_2 \Rightarrow \mathfrak{so}_5 \cong \mathfrak{sp}_4$
- $D_3 = A_3 \Rightarrow \mathfrak{so}_6 \cong \mathfrak{sl}_4$

### 2.2. Representations of semisimple Lie algebras

We have already described the representations of the basic Lie algebra $\mathfrak{sl}_2$. We will see how to describe all the representations of any complex semisimple Lie algebra.

#### Basic definitions

**Definition 5.** A representation of a Lie algebra $\mathfrak{g}$ is a vector space $V$, with a morphism of Lie algebras $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$. The vector space $V$ is said to be a $\mathfrak{g}$-module.

Here $\mathfrak{gl}(V)$ stands for the Lie algebra of linear endomorphisms of $V$, whose Lie bracket is just the usual commutator. We will only consider finite dimensional representation.

**Definition 6.** A $\mathfrak{g}$-module $V$ is irreducible if it does not contain any proper submodule. It is indecomposable if it cannot be written as the direct sum of two proper submodules.

A basic result to be used later on is:
PROP 4. (Schur’s lemma). Let $u : U \rightarrow V$ be a map between irreducible $g$-modules, commuting with the action of $g$. Then $u$ is zero, or an isomorphism.

Proof. The kernel of $g$ is a $g$-submodule of $U$, so it must be all of $U$ and $u = 0$, or $u$ is injective. But the image of $V$ is also a $g$-submodule, so it must be all of $V$ and $u$ is an isomorphism. \hfill \Box

It can happen that a $g$-module is indecomposable without being irreducible, but not when $g$ is semisimple.

THEOREM 3. If $g$ is a semisimple complex Lie algebra, every finite dimensional $g$-module is completely reducible, i.e., can be decomposed into a direct sum of irreducible submodules.

This theorem was first proved by H. Weyl, who used the relation to Lie groups: a complex semisimple Lie algebra comes from a complex semisimple Lie group, which can be proved to be the complexification of a compact Lie group. This allows one to reduce the problem to representations of compact groups, which behave, as Hurwitz pointed out, very much like representations of finite groups. But then the result was known to follow from averaging methods. This is the famous unitary trick of H. Weyl.

Weights

From now on we suppose that $g$ is simple and we fix a Cartan subalgebra $t$. If $V$ is a representation of $g$, we can analyse its structure just in the same way that we analyzed the structure of $g$: we diagonalize the action of $t$ and define the weight spaces

$$V_\mu = \{ v \in V, \quad h.v = \mu(h)v \quad \forall h \in t \}, \quad \mu \in t^*.$$

We have that $V = \bigoplus_{\mu \in t^*} V_\mu$, and if $V_\mu \neq 0$, $\mu$ is a weight of the $g$-module $V$, with multiplicity $m_\mu$ equal to the dimension of $V_\mu$.

Recall that for each root $\alpha \in \Phi$, we defined in $g$ a subalgebra $g^{(\alpha)} \cong sl_2$, generated by a triple $(H_\alpha, X_\alpha, X^-_\alpha)$. Since $X_\alpha \in g_\alpha$, the Jacobi identity implies that $X_\alpha$ maps $V_\mu$ to $V_{\mu + \alpha}$. We can consider $V$ as a $g^{(\alpha)}$-module and apply what we know about $sl_2$-modules. First, $H_\alpha$ acts on $V_\mu$ by multiplication by $\mu(H_\alpha)$, which must therefore be an integer, say $k$. Second, if $v \in V_\mu$ is nonzero, the vectors $v, X^-_\alpha v, \ldots, X_{-\alpha}^k v$ are independent, in particular nonzero. Thus $\mu - i\alpha$ is also a weight of $V$ for $0 \leq i \leq k$, in particular $\mu - \alpha(H_\alpha)\alpha = s_\alpha(\mu)$ is a weight of $V$. Since the operator $X_{-\alpha}^k$ is an isomorphism, as we can easily check on the irreducible $sl_2$-modules, we conclude:

PROP 5. The weights of a $g$-module, and their multiplicities, are invariant under the Weyl group action.

DEF 7. Let $P \subset t^*$ be the set of all $\mu \in t^*$ such that $\mu(H_\alpha) \in \mathbb{Z} \forall \alpha \in \Phi$. This is a free $\mathbb{Z}$-module, a lattice, of rank the dimension of $t$, i.e., the rank of $g$; it is called the weight lattice.
If a set $\Delta$ of simple roots is fixed, a basis of $\mathcal{P}$ is given by the fundamental weights $\omega_\alpha, \alpha \in \Delta$, defined by the condition that $\omega_\alpha(H_\beta) = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$. The nonnegative integer linear combinations of the fundamental weights are the dominant weights, whose set we denote by $\mathcal{P}^+ \subset \mathcal{P}$.

Remark 3. The Cartan matrix gives the simple roots in terms of the fundamental weights, $\alpha = \sum_{\beta \in \Delta} c_{\alpha \beta} \omega_\beta$.

Note that the weights of the adjoint representation are 0 (with multiplicity equal to the rank of $g$), and the roots (with multiplicity one). In particular $\mathcal{P}$ contains the root lattice $Q$ generated by the roots.

Let us come back to the $g$-module $V$, and let us say that a weight $\lambda$ is a highest weight of $V$ if no weight of $V$ has the form $\lambda + \alpha$ for a positive root $\alpha \in \Phi^+$. If $v \in V_\lambda$ is nonzero, we claim that the subspace of $V$ generated by the vectors of the form $X_{-\alpha_p} \cdots X_{-\alpha_1} v$, for any choice of positive roots $\alpha_1, \ldots, \alpha_p \in \Phi^+$, is a $g$-submodule of $V$ (check this!). If $V$ is irreducible, this implies that (1) $\lambda$ has multiplicity one, (2) every weight of $V$ is of the form $\lambda - \sum_{\alpha \in \Delta} n_\alpha \alpha$ for some coefficients $n_\alpha \in \mathbb{Z}^+$. Since the set of weights is $W$-invariant, this implies that $\lambda$ must be dominant.

Theorem 4. An irreducible $g$-module is, up to isomorphism, uniquely determined by its highest weight $\lambda \in \mathcal{P}^+$; we denote it by $V_\lambda$.

Conversely, for each dominant weight $\lambda \in \mathcal{P}^+$, there exists an irreducible $g$-module $V_\lambda$ with highest weight $\lambda$.

We have precise information on the weights of $V_\lambda$.

Proposition 6. The set of weights of $V_\lambda$ is equal to the intersection of the convex hull of $W \lambda$, the set of images of $\lambda \in \mathcal{P}^+$ by the action of the Weyl group, with the translate $\lambda + Q$ of the root lattice.

Example 4. The adjoint representation of a simple complex Lie algebra $g$ is irreducible. This implies that among the positive roots, there is one root $\psi$ which is the highest weight, hence such that for any positive root $\beta$, the sum $\psi + \beta$ is not a root. It turns out that for all simple Lie algebras, except $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, the highest root is a fundamental weight. Said otherwise, the adjoint representation is fundamental.

The unicity assertion of Theorem 4 is easy to prove: if we have two irreducible modules $V$ and $V'$ with the same highest weight, we choose nonzero vectors $v$ and $v'$ in the corresponding weight spaces. Denote by $V'' \subset V \oplus V'$ the irreducible $g$-module generated by $(v, v')$. The restrictions to $V''$ of the projections to $V$ and $V'$ are certainly nonzero, so by Schur’s lemma they must be isomorphisms. Thus $V \cong V'' \cong V'$.

The existence assertion is much more delicate, and infinite dimensional modules are in general used to prove it. One way to avoid this (that was E. Cartan’s initial line of ideas), is to construct first the fundamental representations, i.e., the irreducible $g$-
modules whose highest weights are the fundamental ones. All other representations can be deduced from these if we observe that in the tensor product $V_\lambda \otimes V_\mu$ of two irreducible $g$-modules, the weight $\lambda + \mu$ is a highest weight. Therefore, the $g$-module generated by the tensor product $V_\lambda \otimes V_\mu$ of two highest weight vectors is an irreducible module $V_{\lambda+\mu}$ of highest weight $\lambda + \mu$. (This is called the Cartan product). Since any dominant weight is a sum of fundamental ones, this allows one to construct any irreducible $g$-module inside some tensor product of fundamental representations.

**Representations of slₙ**

Let us complete this program in the case of slₙ. An obvious representation is given by the natural action on $\mathbb{C}^n$, which is obviously irreducible. The weights are the $\epsilon_i$, $1 \leq i \leq n$, and with the choice of positive roots we made in the first lecture, the highest weight is $\omega_1 = \epsilon_1$, the first fundamental weight (check this!).

More generally, we can look at the induced action of $S_l$ on $\Lambda^k \mathbb{C}^n$, defined for $X \in \mathfrak{sl}_n$ by

$$X(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^k v_1 \wedge \cdots \wedge Xv_i \wedge \cdots \wedge v_k.$$ 

Here the weights are the sums $\epsilon_{i_1} + \cdots + \epsilon_{i_k}$, with $1 \leq i_1 < \cdots < i_k \leq n$, and the highest one is $\omega_k = \epsilon_1 + \cdots + \epsilon_k$, which is again a fundamental weight. We have thus identified the fundamental representations to be the wedge powers $\Lambda^k \mathbb{C}^n$, with $1 \leq k \leq n-1$, and we conclude that any irreducible $\mathfrak{sl}_n$-module can be obtained as a submodule of some tensor product $(\mathbb{C}^n)^{\otimes u_1} \otimes (\Lambda^2 \mathbb{C}^n)^{\otimes u_2} \otimes \cdots \otimes (\Lambda^{n-1} \mathbb{C}^n)^{\otimes u_{n-1}}$.

The representations of $\mathfrak{sl}_n$ were actually first constructed by I. Schur in a different way. The idea was to use the symmetric groups $S_k$. If $V \simeq \mathbb{C}^n$, $S_k$ acts on $V^\otimes k$ by permuting the factors: $\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$. If $A$ is any finite dimensional $S_k$-module, we can define

$$\text{Hom}_{S_k}(A, V^\otimes k) := \{ u \in \text{Hom}(A, V^\otimes k), \quad u(\alpha a) = \sigma u(a) \ \forall \sigma \in S_k \}.$$ 

The representations of $S_k$ are well-understood: they are completely reducible, and the irreducible representations are classified by partitions of $k$. Indeed, choose such a partition $\pi$, a nonincreasing sequence of positive integers $\pi_1 \geq \cdots \geq \pi_k$, with $\pi_1 + \cdots + \pi_k = k$. Such a partition is often represented by a Young diagram $D_\pi$ as follows: the lengths of the lines are the $\pi_i$'s, so in the example below $\pi = (4,3,1)$ is a partition of $k = 8$:

```
1 4 6 8
2 5 7
3
```

Number the boxes of the Young diagram $D_\pi$ from 1 to $k$, in an arbitrary way, to obtain a tableau $T$ of shape $\pi$. If $x_1, \ldots, x_k$ are indeterminates, we associate to the tableau $T$ the polynomial

$$P_T = \prod_{i<j} (x_i - x_j),$$
where the relation $i <_T j$ means that $i$ and $j$ appear in the same column of $D_\pi$, with $i$ above $j$. Of course, the group $\mathfrak{S}_k$ acts on the set of tableaux of shape $\pi$, as well as on the indeterminates $x_1, \ldots, x_k$, and $P_{\sigma T} = \sigma P_T$.

**Theorem 5.** The space of polynomials generated by the $P_T$’s, where $T$ describes the set of tableaux of shape $\pi$, is an irreducible $\mathfrak{S}_k$-module, denoted by $[\pi]$ and called a Specht module.

A basis of $[\pi]$ is given by the polynomials $P_T$, where $T$ describes the set of standard tableaux of shape $\pi$.

A tableau is standard when it is numbered by consecutive integers, starting from 1, is such a way that its labels increase from left to right on each line, and from top to bottom on each column.

Let us come back to our vector space $V$, and let $S_\pi V := \text{Hom}_{\mathfrak{S}_k}([\pi], V^\otimes k)$, the $\pi$-th Schur power of $V$. Note that $GL(V)$ acts on $V^\otimes k$ by $g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k$. Since this action commutes with the action of the symmetric group $\mathfrak{S}_k$, there is an induced action of $GL(V)$ on $S_\pi V$. This is also true for the Lie algebra $\mathfrak{sl}(V) \supset \mathfrak{sl}(V) \cong \mathfrak{sl}_n$.

**Theorem 6.** The Schur powers $S_\pi V$ are irreducible $\mathfrak{sl}(V)$-modules (or zero), and every irreducible $\mathfrak{sl}(V)$-module can be obtained that way.

**Theorem 7.** (Schur duality). For each $k > 0$, the canonical map

$$
\bigoplus_{\pi \text{ partition of } k} [\pi] \otimes S_\pi V \longrightarrow V^\otimes k
$$

is an isomorphism of $\mathfrak{S}_k \times GL(V)$-modules.

**Example 5.** If $\pi = (k)$, $D_\pi$ has only one line, and we have $P_T = 1$ for every tableau $T$ of shape $\pi$. Thus $[k]$ is just the trivial representation, and $S_\pi V$ is the space of symmetric tensors in $V^\otimes k$, the $k$-th symmetric power $S^k V$.

**Example 6.** If $\pi = (1^k)$, $D_\pi$ has only one column, and we have

$$
P_T = \pm \prod_{1 \leq i < j \leq k} (x_i - x_j)
$$

for every tableau $T$ of shape $\pi$. Hence $[1^k]$ is again one-dimensional but is not trivial: it is the sign representation of $\mathfrak{S}_k$. Thus $S_\pi V$ is the space of skew-symmetric tensors in $V^\otimes k$, the $k$-th wedge power $\wedge^k V$.

A vector in $S_\pi V = \text{Hom}_{\mathfrak{S}_k}([\pi], V^\otimes k)$ is defined by the image of a single generator $P_T$ of $[\pi]$, since the images of the other ones are then deduced from the $\mathfrak{S}_k$-action. For example, choose for $T$ the tableau numbered column after column, from top to bottom. If $\sigma$ is a transposition of two labels appearing in the same column of $T$, we
have \( \alpha P_T = -P_T \). Thus, the image of \( P_T \) must be skew-symmetric with respect to these two labels, and we recover the fact that \( S_\pi V \) is contained in a product of wedge powers, namely \( \wedge^{i_1} V \otimes \cdots \otimes \wedge^{i_s} V \), where the \( i_i \) denote the lengths of the columns of \( D_\pi \).

Let us now choose a basis \( v_1, \ldots, v_n \) of \( V \). Say that a tableau \( S \) numbered by integers between 1 and \( n \) is semistandard if its labels never decrease from left to right on each line, and increase from top to bottom on each column. One can prove that there exists a map \( \phi_S \in S_\pi V \) mapping \( P_T \) to the vector \( v_S \) defined as the tensor product of the \( v_{j_1} \wedge \cdots \wedge v_{j_l} \in \wedge^{l_i} V \subseteq V^{\otimes l_i} \).

**Proposition 7.** The maps \( \phi_S \) give a basis of \( S_\pi V \), whose dimension is therefore equal to the number of semistandard tableaux of shape \( \pi \), labeled by integers not greater than the dimension of \( V \).

A consequence is that \( S_\pi V \neq 0 \) if and only if the number \( l \) of parts of \( \pi \) does not exceed the dimension of \( V \).

We can also deduce the weights of \( S_\pi V \neq 0 \) as an \( \mathfrak{sl}(V) \)-module, since the Cartan subalgebra of matrices that are diagonal in the basis of \( V \) that we have chosen, has a diagonal action in the basis of \( S_\pi V \) provided by the \( \phi_S \). And the weight of \( \phi_S \) is obviously \( \omega_S = \sum S_i \varepsilon_i \), if we denote by \( S_i \) the number of labels equal to \( i \) in \( S \) (recall that the trace condition implies that \( \varepsilon_1 + \cdots + \varepsilon_n = 0 \)). The highest of these weights is obtained when \( S \) is the tableau numbered by \( i \)'s on the whole \( i \)-th line: we get

\[
\omega_S = \sum_i S_i \varepsilon_i = \sum_j (\pi_j - \pi_{j+1}) \omega_j.
\]

**Example 7.** Let \( n = 3 \) and \( \pi = (3,1) \), so that \( S_\pi \mathbb{C}^3 \) is the irreducible \( \mathfrak{sl}(3) \)-module with highest weight \( 2\omega_1 + \omega_2 \). It is straightforward to list the semistandard tableaux of shape \( \pi \) and to check that \( S_\pi \mathbb{C}^3 \) has dimension 15. Its weights are as indicated below, with a \( \circ \) for multiplicity one and a \( \bullet \) for multiplicity two.

The highest weight has 6 different images under the Weyl group action. On the boundary of their convex hulls, there are three other weights with multiplicity one. And there remain three other weights in the interior of the convex hull, with multiplicity two.
Representations of other classical Lie algebras

For the other classical Lie algebras we can identify the fundamental representations, which in principle allows the construction of all irreducible representations.

We begin with the symplectic algebra $\mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n}$. The natural representation on $\mathbb{C}^{2n}$ is of course irreducible, and fundamental. But the induced action on its wedge powers is not irreducible. Recall that $\mathfrak{sp}_{2n}$ is defined by the condition that $'XJ + JX = 0$, where $J$ denotes an invertible skew-symmetric matrix. Such a matrix can be interpreted as a nondegenerate skew-symmetric bilinear form, i.e., a symplectic form $\theta$ on $\mathbb{C}^{2n}$, and $X \in \mathfrak{sl}_{2n}$ belongs to $\mathfrak{sp}_{2n}$ if and only if

$$\theta(Xx, y) + \theta(x, XY) = 0 \quad \forall x, y \in \mathbb{C}^{2n}. $$

Being non-degenerate, the symplectic form $\theta \in \wedge^2 (\mathbb{C}^{2n})^*$ identifies $\mathbb{C}^{2n}$ with its dual. In particular, $\theta$ defines a dual tensor $\theta^* \in \wedge^2 \mathbb{C}^{2n}$, and the line generated by $\theta^*$ is a submodule of $\wedge^2 \mathbb{C}^{2n}$, which is therefore reducible. Nevertheless, define

$$\wedge^{(k)} \mathbb{C}^{2n} = \{ \psi \in \wedge^k \mathbb{C}^{2n}, \quad \psi \wedge (\theta^*)^{n-k+1} = 0 \}. $$

Proposition 8. For $1 \leq k \leq n$, $\wedge^{(k)} \mathbb{C}^{2n}$ is an irreducible $\mathfrak{sp}_{2n}$-module, and this is the complete list of fundamental representations of $\mathfrak{sp}_{2n}$.

As $\mathfrak{sp}_{2n}$-modules, the ordinary wedge powers $\wedge^k \mathbb{C}^{2n}$ decompose as

$$\wedge^k \mathbb{C}^{2n} = \bigoplus_{l \geq 0} \wedge^{(k-2l)} \mathbb{C}^{2n}. $$

The Dynkin diagram gives the following picture:
The case of the orthogonal Lie algebras $\mathfrak{so}_n$ as some peculiarities. Again we have the natural representation $C^n$, and its wedge powers $\wedge^k C^n$ can be proved to be irreducible, and fundamental, for $1 \leq k \leq \frac{n}{2} - 2$. Then something unexpected happens: $\wedge^{m-1} C^{2m}$ and $\wedge^m C^{2m+1}$ are irreducible but not fundamental, and $\wedge^m C^{2m}$ is not even irreducible! This phenomenon is due to the existence of the spinor representations.

**Remark 4.** Note the general fact that if we know the fundamental representation $V$ attached to a vertex which is at an end of the Dynkin diagram, we can construct other fundamental representations by taking wedge powers. If our vertex is the extremity of a chain of length $k$ in the Dynkin diagram, with only simple bonds, then one can show that the fundamental representations attached to the vertices of that chain, can be obtained as direct summands $V_l$ of the wedge powers $\wedge^l V$, $1 \leq l \leq k + 1$. This was observed by Cartan who used it to give the first construction of all fundamental representations of all complex simple Lie algebras.

**Weyl’s dimension formula**

Hermann Weyl was the first to obtain general formulas for the dimensions and multiplicities of an irreducible representation of a simple complex Lie algebra. To state his results, let us define the *character* of a $\mathfrak{g}$-module $V$ to be the formal sum

$$\chi(V) = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu \in \mathbb{Z}[\mathfrak{h}].$$

The $e^\mu$ are just formal symbols, but the exponential notation indicates that we can multiply them according to the rule $e^\mu e^\nu = e^{\mu + \nu}$. The character is then well-behaved under the natural operations on $\mathfrak{g}$-modules:

$$\chi(V \oplus V') = \chi(V) + \chi(V') \quad \text{and} \quad \chi(V \otimes V') = \chi(V) \chi(V').$$

If $w \in W$ is an element of the Weyl group, $w$ will permute the roots of $\mathfrak{g}$ and we can define its *length* to be

$$l(w) = \# \{ \alpha \in \Phi^+, \ w(\alpha) \in \Phi^- \}.$$ 

Let $\rho$ denote the sum of the fundamental weights, the smallest strictly dominant weight. One can show that $2\rho = \sum_{\alpha \in \Phi^+} \alpha$.

**Theorem 8.** (Weyl’s character formula). The character of an irreducible $\mathfrak{g}$-module $V_\lambda$ is given by the formula

$$\chi(V_\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda + \rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)}}.$$
**Corollary 1.** (Weyl’s dimension formula). The dimension of an irreducible \(g\)-module \(V_\lambda\) is given by the formula

\[
\dim V_\lambda = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.
\]

### 2.3. Linear algebraic groups

**Definition 8.** An affine algebraic group is a group \(G\), endowed with a structure of an affine variety over some field, such that the group law \(G \times G \to G\), mapping \((x, y)\) to \(xy^{-1}\), is algebraic.

An affine variety, say over the complex numbers, is fully determined by the algebra \(\mathbb{C}[G]\) of regular functions. The group multiplication induces a morphism

\[
\mathbb{C}[G] \to \mathbb{C}[G \times G] = \mathbb{C}[G] \otimes \mathbb{C}[G].
\]

Group operations are well-behaved algebraically: if \(\phi\) is any morphism of affine algebraic groups, not only its kernel, but also its image are affine algebraic groups. Another important property is that an affine algebraic group \(G\) has a unique irreducible component containing the identity element. This component \(G^0\) is a closed normal subgroup of finite index, and coincides with the connected component of the identity.

**Theorem 9.** Any affine algebraic group is isomorphic to a closed subgroup of \(GL_n(\mathbb{C})\) for some integer \(n\).

**Proof.** The algebra \(\mathbb{C}[G]\) is generated by some finite dimensional subspace \(V\), which can be chosen to be invariant by left translations (prove this!). Then we get a map \(\lambda\) from \(G\) to \(GL(V)\) by letting

\[
(\lambda(g)f)(x) = f(gx).
\]

This map is algebraic, a group morphism, and a closed embedding.

We could therefore have considered directly linear algebraic groups, which are defined as the closed subgroups of the \(GL_n\)’s.

**Algebraic groups and Lie algebras**

One of the possible algebraic definitions of the Lie algebra of an affine algebraic group \(G\) makes use of the algebra \(D(G)\) of derivations of \(\mathbb{C}[G]\), i.e., maps

\[
d : \mathbb{C}[G] \to \mathbb{C}[G], \quad df(g) = fd(g) + gd(f).
\]
Let $\lambda(g)$ denote the left translation by $g$ on $C[G]$, defined as in the proof of the previous Theorem. There is an induced action on $D(G)$ by $\lambda(g)(d) = \lambda(g) \circ d \circ \lambda(g^{-1})$, and we denote by $L(G)$ the space of left-invariant derivations, defined by the condition that $\lambda(g)(d) = d$ for all $g \in G$.

The commutator $[d,d'] = dd' - d'd$ endows $D(G)$ with a Lie algebra structure, and $L(G)$ is obviously a subalgebra. This structure is transfered to the tangent space of $G$ at the identity element $e$, by the map

$$\theta : L(G) \to T_eG \simeq \text{Der}(O_{G,e}, C) \quad \theta(d)(f) = (df)(e).$$

Here $O_{G,e}$ denotes the algebra of rational functions on $G$ that are defined at $e$. Locally around $e$ such a rational function can be written as a quotient of two regular functions, to which we can apply a derivation $d$ by the usual formula for the derivative of a quotient. Then we evaluate the resulting function at $e$.

**Proposition 9.** The map $\theta$ is a vector space isomorphism.

**Proof.** If $d \in T_eG$ and $f \in C[G]$, let $\delta^*(f)(x) = \delta(\lambda(x^{-1})f)$. This defines an inverse for $\theta$. \qed

We denote by $g$ the vector space $T_eG$ endowed with the resulting Lie algebra structure. If $\phi$ is any algebraic morphism between affine algebraic groups, it is a formal verification that the tangent map at the identity is a Lie algebra morphism.

**Example 8.** If $g \in G$, define the map $\text{Int}(g) : G \to G$ by $\text{Int}(g)(x) = gxg^{-1}$. The tangent map is denoted $\text{Ad}(g) : g \to g$, and since $\text{Ad}(g) \circ \text{Ad}(h) = \text{Ad}(gh)$, we get a morphism

$$\text{Ad} : G \to \text{Aut}(g) \subset GL(g)$$

called the adjoint representation. One can check that the tangent map to $\text{Ad}$ is the map $ad : g \to \text{Der}(g)$ defined by $ad(x)(y) = [x,y]$. Indeed, the Jacobi identity on $g$ is equivalent to the fact that the operators $ad(x)$ are derivations.

**Homogeneous spaces**

Let $H$ be some closed subgroup of an affine algebraic group $G$. We would like to endow the set $G/H$ of right $H$-cosets in $G$, with an algebraic structure. The resulting algebraic variety will be a homogeneous space, i.e., an algebraic variety with a transitive group action.

**Theorem 10.** (Chevalley). There exists an embedding $G \subset GL(V)$, and a line $L$ in the finite dimensional vector space $V$, such that

$$H = \{ g \in G, \ gL = L \}.$$

**Proof.** Let $I \subset C[G]$ denote the ideal of the closed subgroup $H$. Include a finite set of generators of $I$ into some finite dimensional subspace $B$ of $C[G]$, stable under left-translations, and let $A = B \cap I$. Then $G$ acts on $B$ – check that the stabilizer of $A$ is equal
to $H$. To prove the claim, there just remains to replace $A$ by the line $L = aA$ inside $V = aB$, where $a = \dim A$.

The $G$-orbit of $L$, i.e., the set $X = GL = \{gL, g \in G\} \subset V$, is then a quasi-projective variety with a $G$-action, with a base point $x = [L]$ whose stabilizer is equal to $H$, and such that the fibers of the map $G \to X, g \mapsto gx$, are exactly the $H$-cosets.

To call this space the quotient of $G$ by $H$, we need to prove that it is canonical in some sense. One can show that it has the following characteristic property: for each $G$-homogeneous space $Y$, with a base point $y$ whose stabilizer contains $H$, there exists a $G$-equivariant algebraic morphism $(X, x) \to (Y, y)$. A homogeneous space, like $X$, with this property is unique up to isomorphism (in characteristic zero), and this is what we call a quotient of $G$ by $H$.

The existence of quotients has a useful consequence: if $G$ is a connected affine algebraic group, there is a bijective correspondence between the closed connected subgroups of $G$ and their Lie algebras, which are subalgebras of $g = \text{Lie}(G)$. But beware that a Lie subalgebra of $g$ is not always the Lie algebra of a closed subgroup of $G$!

### Solvable groups

For any group $G$, one can define the derived subgroup $D^1(G) = [G, G]$ to be the normal subgroup generated by the commutators $[g, h] = g^{-1}hg$. If $G$ is an affine algebraic group, then $D(G)$ is a closed subgroup. More generally, we define the derived series inductively, by

$$D^{n+1}(G) = [D^n(G), D^n(G)].$$

**Definition 9.** The group $G$ is solvable if $D^n(G) = 1$ for some $n \geq 0$.

Solvable groups have a very nice geometric property:

**Theorem 11.** (Borel’s fixed point theorem). Let $G$ be a connected solvable affine algebraic group, and let $X$ be a complete $G$-variety. Then $X$ has a fixed point.

This has the following classical consequence. Suppose $G$ is a solvable closed connected subgroup of $GL(V)$. There is an induced action of $G$ on the set of complete flags of subspaces of $V$. Since this complete flag variety is projective, $G$ has a fixed point. Consequence:

**Corollary 2.** (Lie-Kolchin’s theorem). Let $G \subset GL(V)$ be a solvable closed connected subgroup. Then one can find a basis of $V$ in which $G$ consists of triangular matrices.

**Proof of Borel’s fixed point theorem.** Let $X(G)$ denote the set of fixed points of $G$ in $X$. We prove that $X(G) \neq \emptyset$ by induction on the dimension of $G$. 
The derived group \([G, G]\) is again a connected solvable affine algebraic group, of smaller dimension than \(G\). Therefore the fixed point set \(X([G, G])\) is a non-empty closed subset of \(X\). This is a complete variety, on which \(G\) acts in such a way that the stabilizer \(G_x\) of a point \(x\) always contains \([G, G]\). In particular, \(G_x\) is a closed normal subgroup, and this implies that the quotient space \(G/G_x\) is an affine algebraic group (this is the algebraic version of the classical fact that the quotient of any abstract group by a normal subgroup has an abstract group structure). In characteristic zero, \(G/G_x\) can be identified with the orbit \(Gx\) of \(x\), which is therefore affine.

But an orbit \(O\) is always constructible, so that its boundary \(\partial O = \overline{O} - O\) has strictly smaller dimension, and is of course \(G\)-stable. This implies in particular that an orbit of minimal dimension must be closed. In our particular situation, we can therefore choose \(x\) such that \(Gx\) is closed, hence complete. Being complete, affine, and connected since \(G\) is connected, it must be a point. Thus \(x \in X(G) \neq \emptyset\).

**Definition 10.** Let \(G\) be an affine algebraic group. A closed subgroup \(B\) is a Borel subgroup if it is a maximal connected solvable subgroup. A closed subgroup \(P\) is a parabolic subgroup if the quotient \(G/P\) is a projective variety.

**Theorem 12.** A closed subgroup of \(G\) is parabolic if and only if it contains a Borel subgroup. Moreover, Borel subgroups of \(G\) are all conjugate.

**Proof.** If \(P\) is parabolic in \(G\), a Borel subgroup \(B\) acts by left translations on the projective variety \(G/P\), and has a fixed point \(xP\) by Borel’s fixed point theorem. Then \(BxP \subset xP\), which implies that \(P\) contains the Borel subgroup \(x^{-1}Bx\) of \(G\).

Conversely, if a closed subgroup \(P\) contains a Borel subgroup \(B\), we have a surjective map \(G/B \to G/P\). If we can prove that \(G/B\) is projective, this will imply that \(G/P\) is complete, hence projective since we know by Chevalley’s theorem that it is quasi-projective. Embed \(G\) in some \(GL(V)\). Then \(G\) acts on the variety of complete flags of \(V\), and some orbit of this action, say \(G/H\), must be closed, hence projective. This means that \(H\) is parabolic, hence contains some conjugate \(x^{-1}Bx\) of \(B\). But by definition \(H\) is contained in the stabilizer of some complete flag of \(V\); this stabilizer is a group of triangular matrices, hence is solvable, so \(H\) itself is solvable. By the definition of Borel subgroups, we must have \(H^o = x^{-1}Bx\). Then the quotient map \(G/B \to G/\sqrt{H}x^{-1}\) is finite, and since \(G/\sqrt{H}x^{-1}\) is projective, this implies that \(G/B\) is projective.

So, Borel subgroups are parabolic. Applying the first part of the proof, we conclude that they are all conjugate.

**Semisimple algebraic groups**

**Definition 11.** If \(G\) is an affine algebraic group, define its radical \(R(G)\) to be the maximal connected solvable normal subgroup. \(G\) is semisimple if \(R(G) = 1\).

Note that the center \(Z(G)\) is a solvable normal subgroup, so that \(Z(G)^o \subset R(G)\).
In particular, a semisimple group has finite center.

**Proposition 10.** A connected affine algebraic group $G$ is semisimple if and only if its Lie algebra $\mathfrak{g} = \text{Lie}(G)$ is semisimple.

**Proof.** If $G$ is not semisimple, the last non zero term $N$ of the derived series $D^n(R(G))$ is a normal connected abelian subgroup. Its Lie algebra is an abelian ideal $\mathfrak{n}$ of $\mathfrak{g}$, which cannot be semisimple (apply Cartan’s criterion).

Conversely, suppose that $\mathfrak{g}$ is not semisimple, hence contains an abelian ideal $\mathfrak{n}$. The group $G$ acts on $\mathfrak{g}$ through the adjoint representation and we can let $H = C_G(\mathfrak{n})^0$, the connected component of the stabilizer of $\mathfrak{n}$. Then $\mathfrak{t} = \text{Lie}(H)$ is an ideal (by Jacobi).

Let $M = \{g \in G, \text{Ad}(g) \mathfrak{t} = \mathfrak{t}\}$. Its Lie algebra is

$$\mathfrak{m} = \{x \in \mathfrak{g}, \text{ad}(x)(\mathfrak{t}) \subset \mathfrak{t}\} = \mathfrak{g},$$

since $\mathfrak{t}$ is an ideal. Since $G$ is connected, this implies that $M = G$. Thus $H$ and $g^{-1}Hg$ have the same Lie algebra for all $g \in G$, hence they are equal, which means that $H$ is normal. Then its center $Z(H)$ is also normal, and its Lie algebra is the center of $\mathfrak{t}$, which contains $\mathfrak{n}$. This implies that $Z(H)^0$ is a non trivial connected abelian normal subgroup of $G$, so that $G$ is not semisimple. \hfill $\Box$

Another important property is that any semisimple Lie algebra is the Lie algebra of a semisimple affine algebraic group.

**Proposition 11.** Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $G = \text{Aut}(\mathfrak{g})^0$ is a connected semisimple affine algebraic group such that $\text{Lie}(G) = \mathfrak{g}$.

**Proof.** The group $G = \text{Aut}(\mathfrak{g})^0$ is clearly a closed subgroup of $GL(\mathfrak{g})$, and its Lie algebra is easily seen to coincide with the derivation algebra $\text{Der}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. So we just need to prove the following property:

**Lemma 1.** If $\mathfrak{g}$ is a semisimple Lie algebra, then every derivation is inner, i.e., $\text{Der}(\mathfrak{g}) = \mathfrak{g}$.

**Proof of the Lemma.** Recall that we have the adjoint map $\mathfrak{g} \to \text{Der}(\mathfrak{g})$, which is injective since a semisimple Lie algebra has trivial center. Now let $d$ be any non zero derivation. Letting $[x,d] = -d(x)$, we get a natural $\mathfrak{g}$-module structure on $\mathfrak{g} \oplus Cd$. Since $\mathfrak{g}$ is semisimple, this $\mathfrak{g}$-module is completely reducible, thus can be decomposed into the direct sum of $\mathfrak{g}$ with a line generated by some vector $d + x, x \in \mathfrak{g}$. But $\mathfrak{g}$ being semisimple, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and any one-dimensional representation is trivial. This means that for all $y \in \mathfrak{g}$, $[y,d] + [y,x] = 0$, i.e., $d(y) = [x,y]$ and $d = \text{ad}(x)$ is inner. \hfill $\Box$

This very good correspondence between semisimple groups and algebras being established, we would like to use what we learned about semisimple Lie algebras to understand the structure of semisimple groups.
Maximal tori

First we look for the counterparts of Cartan subalgebras, which will be maximal tori. A torus is a copy of \((\mathbb{G}_m)^k\), where \(\mathbb{G}_m \simeq \mathbb{C}^*\) denotes the multiplicative group. It has the property of being diagonalizable, which means that every (finite dimensional) representation can be diagonalized. Conversely, every connected diagonalizable group is a torus.

A difficult part of the theory is to prove that in any affine algebraic group, all maximal tori are conjugate. The fact that Borel subgroups are conjugate reduces to the case of solvable groups, which requires a careful analysis.

Once we have proved that, the following definition makes sense.

**DEFINITION 12.** The rank of an affine algebraic group is the dimension of its maximal tori.

The Lie algebra of any maximal torus \(T\) is a Cartan subalgebra \(t\): indeed, it is an abelian subalgebra, all of whose elements are semisimple, since a torus is diagonalizable, and the maximality is inherited from that of the torus.

Then we decompose the adjoint action of \(T\) on \(g\). If \(G\) is semisimple, this action can be diagonalized as

\[ g = t \oplus \bigoplus_{\alpha \in X^*(T)} g_{\alpha}. \]

The slight difference with the decomposition of section 1.1.2 is that the roots \(\alpha\) are not elements of \(t^*\), but of the character group \(X^*(T) = \text{Hom}(T, \mathbb{G}_m)\) of the maximal torus. Passing to the tangent map we recover the decomposition of the adjoint action of \(t\), the root space decomposition.

Root subgroups

The next step is to interpret the roots inside the group. At the Lie algebra level, to a root \(\alpha\) we can associate \(t_\alpha = \text{Ker}(\alpha) \subset t\), whose centralizer is \(t \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = t_\alpha \oplus \mathfrak{g}^{(\alpha)}\).

At the group level, beginning with \(\alpha \in X^*(T)\) we consider \(T_\alpha = (\text{Ker } \alpha)^o\). The Lie algebra of \(C_G(T_\alpha)^o / T_\alpha\) is then a copy of \(\mathfrak{sl}_2\). A more precise statement is the following:

**PROPOSITION 12.** A semisimple affine algebraic group of rank one is isomorphic either to \(SL_2(\mathbb{C})\) or to \(PSL_2(\mathbb{C}) = SL_2(\mathbb{C}) / \{\pm I\}\).

An immediate consequence is that the centralizer \(C_G(T_\alpha)^o\) has exactly two Borel subgroups \(B_\alpha\) and \(B_{-\alpha}\) containing \(T\), whose Lie algebras are \(t \oplus \mathfrak{g}_\alpha\) and \(t \oplus \mathfrak{g}_{-\alpha}\), respectively. The set \(U_\alpha\) of unipotent elements in \(B_\alpha\) is then a connected subgroup (use Lie-Koelchin’s theorem), and \(\text{Lie}(U_\alpha) = \mathfrak{g}_\alpha\).

One can prove that a one-dimensional connected affine algebraic group must be the multiplicative group \(\mathbb{G}_m\), or the additive group \(\mathbb{G}_a\). Only the latter is unipotent, thus \(U_\alpha \simeq \mathbb{G}_a\).
EXAMPLE 9. Let \( G = SL_n(\mathbb{C}) \). The subgroup \( T \) of diagonal matrices is a maximal torus, and the characters of \( T \) involved in the root space decomposition are given by \( \alpha(h) = h_i h_j^{-1} \) for some \( i \neq j \), where \( h_i \) denotes the \( i \)-th diagonal coefficient of \( h \in T \).

The corresponding root subgroup of \( G \) is

\[
U_{\alpha} = \{I + tE_{ij}, \; t \in \mathbb{C}\} \simeq \mathbb{G}_a,
\]

if \( E_{ij} \) denotes the matrix whose only non zero coefficient is a 1, at the intersection of the \( i \)-th line and \( j \)-th column.

Note that in the previous example,

\[
h(I + tE_{ij})h^{-1} = I + h_i h_j^{-1} t E_{ij} = I + \alpha(h)t E_{ij}.
\]

This is a general fact that the group isomorphisms \( x_{\alpha} : \mathbb{G}_a \rightarrow U_{\alpha} \) are such that

\[
hx_{\alpha}(t)^{-1} = x_{\alpha}(\alpha(h)t).
\]

If \( G \) is connected, it is generated by the root groups. We can also construct Borel subgroups from the root groups. For example, the group of triangular matrices is a Borel subgroup of \( SL_n(\mathbb{C}) \), and its Lie algebra is generated by the Lie algebra of the diagonal torus, and the positive root spaces. This suggests to recover a Borel subgroup \( B \), containing a maximal torus \( T \), from the Lie algebra

\[
b = \text{Lie}(B) = t^{i} \bigoplus_{\alpha \in \Phi(B)} \mathfrak{g}_\alpha.
\]

One can prove that \( \Phi(B) \) is a set of positive roots in the root system \( \Phi \) of \( \mathfrak{g} \).

PROPOSITION 13. Let \( \Phi(B) = \{\alpha_1, \ldots, \alpha_N\} \).

The map \( T \times U_{\alpha_1} \times \cdots \times U_{\alpha_N} \rightarrow B \) sending \( (h, u_1, \ldots, u_N) \) to the product \( hu_1 \cdots u_N \), is an isomorphism of algebraic varieties.

Proof. Since \( B \) is solvable, one can check that \( B = T \times B_u \), where \( B_u \) is the subgroup of unipotent elements. We consider the product map

\[
f : U_{\alpha_1} \times \cdots \times U_{\alpha_N} \rightarrow B_u.
\]

This is a \( T \)-equivariant map (recall that \( T \) acts on the root groups by conjugation), étale at the unit element since the tangent map at \( e \) is an isomorphism.

Let \( \lambda : \mathbb{G}_m \rightarrow T \) be a one parameter subgroup of \( T \), i.e., a group morphism from the multiplicative group to \( T \simeq (\mathbb{G}_m)^r \). If \( \alpha \in X^*(T) \) is a character, \( \alpha \circ \lambda \) is a homomorphism from \( \mathbb{G}_m \) to itself, so must be of the form \( t \mapsto t^k \) for some integer \( k = (\alpha, \lambda) \). In particular, \( \lambda \) defines a linear form on \( t^* \subset \Phi \). Since \( \Phi(B) \) is a set of positive roots in \( \Phi \), it is contained in some open half-space, so that we can find a one parameter subgroup \( \lambda \) such that \( (\alpha, \lambda) > 0 \) for all \( \alpha \in \Phi(B) \). Then

\[
\lambda(t). (x_{\alpha_1}(s), \ldots, x_{\alpha_N}(s)) = (\lambda(t)x_{\alpha_1}(s)\lambda(t)^{-1}, \ldots, \lambda(t)x_{\alpha_N}(s)\lambda(t)^{-1}) = (x_{\alpha_1}(t^{\alpha_1, \lambda}(s)), \ldots, x_{\alpha_N}(t^{\alpha_N, \lambda}(s))
\]
converges to $e$ when $t$ goes to zero. We conclude that every point is in the $T$-orbit of a point at which $f$ is étale, so by equivariance $f$ is étale everywhere. In particular, $f$ is open, and since its image is $T$-stable, $f$ must be surjective by the same argument as before. Finally, $f^{-1}(e)$ is finite, but also $T$-stable so consists in fixed points of $T$. But again for the same reasons, $e$ is the only such fixed point, so $f$ is étale of degree one, hence an isomorphism of algebraic varieties.

The same kind of arguments can be used to decompose the full group $G$ in terms of the Weyl group, which we can recover at the group level as the quotient by $T$ of its normalizer, $W = N_G(T)/T$.

**Theorem 13.** (Bruhat decomposition). There is a decomposition

$$G = \coprod_{w \in W} BwB$$

into double cosets, and for each $w \in W$, the product map

$$\left( \bigcap_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^+} U_{\alpha} \right) \times B \longrightarrow BwB$$

is an isomorphism of algebraic varieties.

**Corollary 3.** The product map $B_u \times B \longrightarrow G$ is an open immersion, and therefore $G$ is rational.

**Classification of homogeneous spaces**

Recall that an algebraic variety $X$ is homogeneous if some algebraic group acts transitively on $X$. The group needs not be affine: an elliptic curve $E$ is a projective algebraic group. More generally, an algebraic group which is a projective variety is called an abelian variety.

Over the complex numbers, an abelian variety of dimension $n$ is analytically isomorphic to a quotient $\mathbb{C}^n / \Gamma$, where $\Gamma \cong \mathbb{Z}^{2n}$ is a lattice.

**Theorem 14.** (Borel-Remmert). A homogeneous projective variety $X$ can be decomposed into the product of an abelian variety $A$, with a rational homogeneous variety $Y \cong G/P$, where $G$ is a connected semisimple affine algebraic group, and $P$ a parabolic subgroup.

Abelian varieties are extremely interesting objects but are not the subject of these lectures. We will only be concerned with projective varieties that are homogeneous under the action of an affine algebraic group. They are characterized among homogeneous projective varieties by the property of being rational. Their classification is equivalent to the classification of parabolic subgroups, to which we now turn.

Let $G$ denote a connected semisimple affine algebraic group, $B$ a Borel subgroup, $T$ a maximal torus of $G$ contained in $B$. From this data we deduce the root
system $\Phi \subset t^*$. The roots of $B$ are a set of positive roots $\Phi^+ = \Phi(B)$ in $\Phi$, inside which the indecomposable roots form the set $\Delta$ of simple roots.

**Definition 13.** For any subset $I$ of $\Delta$, we denote by $P_I$ the parabolic group generated by $B$ and the root subgroups $U_{-\alpha}, \alpha \notin I$.

**Theorem 15.** A parabolic subgroup of $G$ is conjugate to one of the $P_I$’s, and only one of them.

The Lie algebra $p_I$ of $P_I$ will be generated by $b$, whose roots we chose to be the positive ones, and the negative root spaces $g_{-\alpha}, \alpha \notin I$. We get

$$p_I = t \oplus \bigoplus_{\alpha \in \Phi(I)} g_\alpha,$$

where $\Phi(I)$ denotes the set of roots $\alpha$ such that $\langle \alpha, \omega_i \rangle \geq 0$ for $i \in I$. This Lie algebra naturally splits into the direct sum of the two subalgebras

$$l_I = t \oplus \bigoplus_{\alpha \in \Phi(I)_0} g_\alpha, \quad n_I = \bigoplus_{\alpha \in \Phi(I)_+} g_\alpha,$$

where $\Phi(I) = \Phi(I)_0 \bigcup \Phi(I)_+$ and $\Phi(I)_0$ is defined by the condition that $\langle \alpha, \omega_i \rangle = 0$ for $i \in I$. We know that $\Phi(I)_0$, being a linear section of $\Phi$, is again a root system, whose Dynkin diagram can be deduced from that of $g$ by keeping only the nodes of $I$. The commutator algebra

$$h_I = l_I \oplus \bigoplus_{\alpha \in \Phi(I)_0} g_\alpha, \quad t_I = \bigoplus_{i \in I} CH_\alpha,$$

is then semisimple with this Dynkin diagram, while $l_I$ is a direct sum of $h_I$ with its center, which is a complement of $l_I$ in $t$: this is an instance of a reductive Lie algebra. Moreover, $n_I$ is nilpotent, in the sense that the adjoint action of each of its element is nilpotent. At the group level, we have similarly $P_I = L_I N_I$, where $L_I$ is a reductive subgroup with Lie algebra $l_I$, $N_I$ a normal nilpotent subgroup with Lie algebra $n_I$, and the intersection of $L_I$ with $N_I$ is finite. This is called the Levi decomposition. The group $H_I$ is called the semisimple Levi factor. Note that $N_I$ is uniquely defined, but not $L_I$ and $H_I$.

**Remark 5.** A reductive Lie algebra can be characterized by the condition that its radical equals its center. At the level of representations, this is equivalent (in characteristic zero) to the fact that any finite-dimensional representation is completely reducible. In practice a nice criterion is the existence of a non-degenerate invariant form (not necessarily the Killing form).

For example, if $h$ is a semisimple element of a semisimple Lie algebra $g$, its centralizer $c_g(h)$ is always reductive (but not semisimple). Indeed, we may suppose that $h$ belongs to our preferred Cartan subalgebra $t$, and then

$$c_g(x) = t \oplus \bigoplus_{\alpha(h) = 0} g_\alpha.$$
The restriction to $c_g(h)$ of the Killing form of $g$ is then easily seen to be non degenerate.

Once we have fixed the Borel subgroup $B$ of $G$, the $P_i$’s are the only parabolic subgroups of $G$ containing $B$. Inclusion defines a natural ordering on this set of parabolic subgroups, from $B = P_0$ to $G = P_0$. Apart from $B$, the minimal parabolic subgroups are conjugate to the $Q_i = P_{\Delta - I_i}$, and the maximal parabolic subgroups are conjugate to the $P_i = P_\alpha$. Each inclusion induces a projection between rational homogeneous varieties:

\[
\begin{array}{ccc}
G/B & \searrow & \downarrow & \searrow & \downarrow & G/Q_r \\
G/Q_1 & \searrow & \downarrow & \searrow & \downarrow & G/Q_r \\
G/P_i & \searrow & \downarrow & \searrow & \downarrow & G/P_r \\
G/P_1 & \searrow & \downarrow & \searrow & \downarrow & G/P_r \\
\end{array}
\]

**Example 10.** For $G = SL_n(\mathbb{C})$, a Borel subgroup is the subgroup $B$ of matrices that are triangular in some fixed basis. Indeed, this is clearly a connected solvable subgroup of $G$, and we have seen that the quotient $G/B = \mathbb{P}_n$, the variety of complete flags, is projective. This implies that $B$ contains a Borel, and therefore is a Borel subgroup.

A subset $I \subset \Delta$ is a sequence of integers between 1 and $n - 1$, and the homogeneous variety $G/P_I$ is the variety of partial flags $\mathbb{P}_I$, the variety of flags of incident subspaces of $\mathbb{C}^n$ whose dimensions are given by the sequence $I$. In particular, the $G/P_I$ are the usual Grassmannians. The maps $G/P_I \rightarrow G/P_J$ exist for $J \subset I$, and are defined by forgetting some subspaces.

3. Nilpotent orbits

3.1. The nilpotent cone

Let $g$ be a semisimple complex Lie algebra, and $G = Aut(g)$ the corresponding adjoint group.

**The structure of the nilpotent cone**

**Definition 14.** An element $X \in g$ is called nilpotent (resp. semisimple) if $ad(X) \in End(g)$ is a nilpotent (resp. semisimple) operator.

**Basic facts.**

1. Any element $X \in g$ can be decomposed in a unique way as $X = X_s + X_n$ with $X_s$ semisimple, $X_n$ nilpotent, and $[X_s, X_n] = 0$. This is the Jordan decomposition.
2. Any nilpotent element $X \in \mathfrak{g}$ can be completed into a $\mathfrak{sl}_2$-triple $(Y, H, X)$ with $H$ semisimple, $Y$ nilpotent, and

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$ 

This is the Jacobson-Morozov theorem.

3. If $X$ is nilpotent, then it is $G$-conjugate to $tX$ for any non zero complex number $t$ (one can use conjugation by the one-parameter subgroup generated by $H$).

**Examples of $\mathfrak{sl}_2$-triples.**

1. Let $\alpha$ be any positive root. Choose a generator $X_\alpha$ of $\mathfrak{g}_\alpha$ and a generator $X_{-\alpha}$ of $\mathfrak{g}_{-\alpha}$. Let $H_\alpha = [X_\alpha, X_{-\alpha}] \in \mathfrak{h}$. One can check that $\alpha(H_\alpha) \neq 0$. Normalize $X_\alpha$ and $X_{-\alpha}$ in such a way that $\alpha(H_\alpha) = 2$. Then $(X_{-\alpha}, H_\alpha, X_\alpha)$ is a $\mathfrak{sl}_2$-triple. We have seen that these $\mathfrak{sl}_2$-triple play an essential role in the study of semisimple Lie algebras.

2. Keep the same notations. Let $X = X_{\alpha_1} + \cdots + X_{\alpha_r}$ be a sum of simple root vectors, where $r$ denotes the rank of $\mathfrak{g}$ and $\alpha_1, \ldots, \alpha_r$ are the simple roots. One can find $H \in \mathfrak{h}$ such that $\alpha_i(H) = 2$ for all $i$. Decompose it as $H = a_1 H_{\alpha_1} + \cdots + a_r H_{\alpha_r}$, and let $Y = a_1 X_{-\alpha_1} + \cdots + a_r X_{-\alpha_r}$. Then $(Y, H, X)$ is a $\mathfrak{sl}_2$-triple.

The last fact above explains the following terminology.

**Definition 15.** The nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ is the $G$-invariant set of all nilpotent elements.

In fact $\mathcal{N}$ is an affine algebraic subset of $\mathfrak{g}$, endowed with a natural scheme structure.

**Proposition 14.** The nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ is the common zero locus of the $G$-invariant polynomials on $\mathfrak{g}$ without constant term.

**Proof.** That such a polynomial has to vanish on any nilpotent element $X \in \mathfrak{g}$ follows from the fact that $X$ is $G$-conjugate to $tX$ for any $t \neq 0$. Conversely, if all the $G$-invariant polynomials without constant term vanish on an element $X \in \mathfrak{g}$, then the characteristic polynomial of $ad(X)$ is a monomial. Therefore $X$ is nilpotent.

What are the $G$-invariant polynomials on $\mathfrak{g}$? How can we write them down? The most important step towards an answer to these questions is provided by the Chevalley restriction theorem. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $H \subset G$ the maximal torus with Lie algebra $\mathfrak{h}$. Recall that the Weyl group $W = N_G(H)/H \simeq N_G(\mathfrak{h})/H$ acts on $\mathfrak{h}$.

**Theorem 16 (Chevalley’s restriction theorem).** The restriction map to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ defines an isomorphism between the algebra of $G$-invariant polynomials on $\mathfrak{g}$, and the algebra of $W$-invariant polynomials on $\mathfrak{h}$.
This is a great step forward because the Weyl group is a reflection group. This implies that $\mathbb{C}[h]^W$, the algebra of $W$-invariant polynomials on $h$, is a polynomial algebra. In other words, one can find $r$ algebraically independent homogeneous polynomials $f_1, \ldots, f_r$, such that

$$\mathbb{C}[h]^W = \mathbb{C}[f_1, \ldots, f_r].$$

Let $e_i = \deg(f_i)$ and $d_i = e_i - 1$. One can suppose that $d_1 \leq \cdots \leq d_r$. They are basic invariants of $g$ called the fundamental exponents. The polynomials $f_1, \ldots, f_r$ are not uniquely defined, but they can be constructed using finite group theory. Then one should be able to lift them to $G$-invariant polynomials $F_1, \ldots, F_r$ on $g$.

It is not difficult to see that the nilpotent cone has codimension $r$, so it must be cut out set-theoretically by $F_1, \ldots, F_r$. More precise results were obtained by Kostant.

Properties (Kostant).

1. The nilpotent cone $\mathcal{N} \subset g$ is the complete intersection of the $r$ invariant hypersurfaces $F_1 = 0, \ldots, F_r = 0$.

2. The fundamental exponents are symmetric, in the sense that the sum $d_{r+1-i} + d_i$ does not depend on $i$.

3. The sum of the fundamental exponents is equal to the number of positive roots of $g$.

Since this is a statement that we will use again and again, we include a

Proof of the Jacobson-Morozov theorem. Let $c(X) \subset g$ denote the centralizer of $X$. The invariance property of the Killing form implies that its orthogonal is exactly $[g,X]$. Note that if $Z$ belongs to $c(X)$, then $ad(X) \circ ad(Z)$ is nilpotent hence $K(X,Z) = 0$. Therefore there exists $H \in g$ such that $[H,X] = 2X$, and by the properties of the Jordan decomposition this remains true if we replace $H$ by its semisimple part. Otherwise said, we may suppose that $H$ is semisimple.

The relation $[H,X] = 2X$ implies that $ad(H)$ preserves $c(X)$, and we can diagonalize its action. If $[H,U] = tU$ with $t \neq 0$, then $K(H,U) = t^{-1}K([H,H],U) = t^{-1}K([H,H],U) = 0$. So there are two cases. Either $H$ is orthogonal to $c(X)$, or there exists $U$ such that $[X,U] = [H,U] = 0$ but $K(H,U) \neq 0$.

Let us consider the second case. The relation $K(H,U) \neq 0$ ensures that $U$ is not nilpotent, and using the Jordan decomposition we cannot suppose it is semisimple. Then recall that its centralizer $c(U)$ is reductive and contains $H$ and $X$. Then $X = [H,X]/2$ belongs to $[c(U),c(U)]$ which is a proper semisimple Lie algebra of $g$, and we can use induction.

Now we consider the first case. As we noticed above the orthogonal to $c(X)$ is $[X,g]$ so we can find $Y \in g$ such that $H = [X,Y]$. Let us diagonalize the action of $ad(H)$ on $g$. Since $H$ and $X$ are eigenvectors for the eigenvalues 0 and 2 respectively, the relation $H = [X,Y]$ remains true if we replace $Y$ by its component on the eigenspace corresponding to the eigenvalue $-2$. But then $[H,Y] = -2Y$ and we have our $sl_2$-triple. □
Symplectic structures on nilpotent orbits

Let $O \subset N$ be a $G$-orbit. We will see later on that there exist only finitely many such nilpotent orbits. Without knowing this we can prove that nilpotent orbits have the very special property of being symplectic manifolds.

First observe that $O$ being a $G$-orbit is a smooth locally closed subset of $\mathfrak{g}$. If $X \in O$, then the tangent space

$$T_X O = [\mathfrak{g}, X] \simeq \mathfrak{g}/c_g(X),$$

where $c_g(X)$ denotes the centralizer of $X$ in $\mathfrak{g}$.

**Definition 16.** The Kostant-Kirillov-Souriau skew-symmetric form on $O$ is defined at the point $X$ by

$$\omega_X(Y, Z) = K(X, [\bar{Y}, \bar{Z}])$$

if $Y = [X, \bar{Y}]$ and $Z = [X, \bar{Z}]$ belong to $T_X O \simeq [\mathfrak{g}, \mathfrak{g}]$. By the invariance property of the Killing form, this does not depend on the choice of $\bar{Y}$ and $\bar{Z}$.

**Proposition 15.** The Kostant-Kirillov-Souriau form is a holomorphic symplectic form on $O$.

**Proof.** We prove that it is non degenerate. Suppose that $Z = [X, \bar{Z}] \in T_X O$ belongs to the kernel of $\omega_X$. By the invariance of the Killing form, this means that $K(Z, \bar{Y}) = 0$ for any $\bar{Y} \in \mathfrak{g}$. Hence $Z = 0$ since the Killing form is non degenerate.

Note that we have not used that $X$ is nilpotent. Moreover the Killing form has only been used to identify $\mathfrak{g}$ with its dual. The correct generalization is that any coadjoint orbit is endowed with a natural symplectic form.

**Corollary 4.** Any nilpotent (or coadjoint) orbit has even dimension.

**Proof.** A skew-symmetric form can be non-degenerate only on a vector space of even dimension.

The Springer resolution

The archetypal symplectic variety is the total space of the cotangent bundle of a smooth variety $Z$. Indeed, consider the map $p : \Omega_Z \to Z$. There is a tautological one-form $\theta$ on $\Omega_Z$ defined by at the point $\Omega \in \Omega_Z$ by $p^* \Omega$. Then the two-form $\omega = d\theta$ is easily seen to be non-degenerate, and is obviously closed.

If $z_1, \ldots, z_n$ are local coordinates on $Z$ in a neighbourhood $U$ of some point $p$, then a one-form can be written over this neighbourhood as $\Omega = y_1 dx_1 + \cdots + y_n dx_n$. This yields local coordinates $(x, y)$ in $p^{-1}(U) \subset \Omega_Z$, in which $\omega = dy_1 \land dx_1 + \cdots + dy_n \land dx_n$.
Consider the full flag variety \( \mathcal{B} = G/B \), where \( B \) is a Borel subgroup. As homogeneous vector bundles on \( \mathcal{B} \) we have
\[
\mathcal{T}_\mathcal{B} = G \times^B \mathfrak{g}/\mathfrak{b}, \quad \Omega_B = G \times^B \mathfrak{n}.
\]
Indeed, the Killing form gives a \( B \)-invariant identification between \( (\mathfrak{g}/\mathfrak{b})^\vee \) and \( \mathfrak{n} \).

**Theorem 17 (Springer’s resolution).** The natural projection map
\[
\pi : \Omega_B = G \times^B \mathfrak{n} \rightarrow \mathfrak{g}
\]
is a \( G \)-equivariant symplectic resolution of singularities of the nilpotent cone.

**Proof.** Since \( \mathfrak{n} \) and its conjugates are contained in the nilpotent cone, the image of \( \pi \) is contained in \( \mathcal{N} \). Conversely, any nilpotent element \( X \) defines a one-dimensional solvable Lie algebra in \( \mathfrak{g} \), hence it must be contained in a Borel subalgebra. Up to conjugation this Borel subalgebra can be supposed to be \( \mathfrak{b} \), and then \( X \in \mathfrak{n} \) since the eigenvalues of \( \text{ad}(X) \) must be zero. This proves that the image of \( \pi \) is exactly \( \mathcal{N} \). In particular \( \mathcal{N} \) is irreducible. Moreover \( \pi \) is proper since it factorizes through \( G/B \times \mathfrak{g} \) and \( G/B \) is complete.

**Lemma 2.** The \( G \)-orbit of \( X = X_{\alpha_1} + \cdots + X_{\alpha_r} \) is dense in \( \mathcal{N} \).

This orbit is called the regular orbit and is denoted \( O_{\text{reg}} \).

**Proof.** We have seen how to complete \( X \) into a \( \mathfrak{sl}_2 \)-triple \( \langle Y, H, X \rangle \) by choosing \( H \in \mathfrak{h} \) such that \( \alpha_i(H) = 2 \) for each \( i \). By \( \mathfrak{sl}_2 \)-theory, the kernel of \( \text{ad}(X) \) has its dimension equal to the number of irreducible components of \( \mathfrak{g} \) considered as a module over our \( \mathfrak{sl}_2 \)-triple. Moreover, this number it lower than or equal to (in fact equal to, since all the eigenvalues of \( H \) are even) the dimension of the kernel of \( \text{ad}(H) \). Since \( \alpha(H) \neq 0 \) form each root \( \alpha \), this kernel is simply \( \mathfrak{h} \) and its dimension is \( r \). This implies that \( \text{ad}(X) : \mathfrak{b} \rightarrow \mathfrak{n} \) must be surjective. This means that the \( B \)-orbit of \( X \) is dense in \( \mathfrak{n} \), and therefore the \( G \)-orbit of \( X \) is dense in \( \mathcal{N} \).

Now we can prove that \( \pi \) is birational. By the proof of the lemma, the source and the target of \( \pi \) have the same dimension. Hence the generic fiber is finite and we want to prove it is a single point. It is enough to prove it for the fiber of our preferred nilpotent element \( X = X_{\alpha_1} + \cdots + X_{\alpha_r} \). Since \([H,X] = 2X\) the fiber is preserved by the conjugate action of the one dimensional torus generated by \( H \). Since this group is connected it has to act trivially on each point of the fiber. This means that any conjugate \( n' \) of \( n \) containing \( X \) is preserved by \( H \). But then the algebra generated by \( H \) and \( n' \) is solvable, hence contained in a Borel subalgebra \( \mathfrak{b}' \). Since \( H \) is regular its centralizer \( \mathfrak{h} \) must be contained in \( \mathfrak{b}' \). To conclude, we use the fact that the Borel subalgebras that contain \( \mathfrak{h} \) are the conjugates of \( \mathfrak{b} \) by the Weyl group, which permutes their sets of simple roots simply transitively. Since \( n' \) contains \( X \), it contains all the positive simple roots, and therefore it has to coincide with \( \mathfrak{n} \).
There remains to prove that \( \pi \) is symplectic. Let us compute the pull-back by \( \pi \) of the Kostant-Kirillov-Souriau form on \( O \). Denote by \( n_- \) the sum of the negative root spaces in \( g \), and by \( N_- \subset G \) the corresponding connected subgroup. Then it follows from the Bruhat decomposition that \( G = N_- B \) and \( N_- \cap B = 1 \). In particular \( N_- \) can be identified with an open subset of \( G/B \) over which \( \Omega_B = G \times B \) is a locally trivial fibration, the natural map \( N_- \times n \to \Omega_B \) being an isomorphism.

Let \( y \in n \) belong to the regular orbit. This means that its \( B \)-orbit is open in \( n \), or equivalently that \( [b, y] = n \). A section of the projection map from \( \Omega_B \) to \( G/B \) at \( (e, y) \) is given by \( u \in N_- \mapsto (u, y) \). Hence a decomposition of \( T_{(e, y)} \Omega_B \) into the sum of the vertical tangent space \( n_- \) and the tangent space \( n_- \) of the section. The differential of \( \pi \) at \( (e, y) \) is given by \( d\pi(u) = u_+ + [y, u_-] \) if \( u = (u_+, u_-) \), with \( u_+ \in n \) and \( u_- \in n_- \). Therefore the pull-back \( \pi^* \omega \) of the Kostant-Kirillov-Souriau form can be computed as

\[
\pi^* \omega(u, v) = \omega (d\pi(u), d\pi(v)) = \omega(u_+ + [y, u_-], v_+ + [y, v_-]).
\]

Since \([b, y] = n\) we can write \( u_+ = [y, u_+] \) with \( u_+ \in b \). Then by definition of the Kostant-Kirillov-Souriau form we get

\[
\omega(u_+, [y, u_-]) = K(y, [u_+, v_-]) = K([y, u_+], v_-) = K(u_+, v_-).
\]

On the other hand, if we also write \( v_+ = [y, v_+] \) with \( v_+ \in b \), we get

\[
\omega(u_+, v_+) = \omega([y, u_+], [y, v_+]) = K(y, [u_+, v_+]) = 0
\]

because \( y \) and \([u_+, v_+] \) belong to \( n \). Finally, we obtain

\[
\pi^* \omega(u, v) = K(u_+, v_-) - K(v_+, u_-) + K([y, u_-, v_-]).
\]

This expression makes perfect sense even if \( y \) is not on the regular orbit, so that the pull-back of the Kostant-Kirillov-Souriau form by \( \pi \) extends to a skew-symmetric form which is defined everywhere. Moreover this skew-symmetric form is clearly non-degenerate. With a little more work we can check that \( \pi^* \omega \) coincides with the canonical symplectic form on the open subset \( p^{-1}(U) \), hence everywhere since this identification is compatible with the left action of \( G \) on \( G/B \).

**Remark 6.** A nilpotent element \( X \) belongs to \( O_{\text{reg}} \) if and only if its centralizer has dimension \( r \). Indeed, we may suppose that \( X \) is contained in the nilradical \( n \) of our Borel subalgebra \( b \). Its the centralizer of \( X \) has dimension \( r \), then the map \( ad(X) : b \to n \) must be surjective and again this implies that the \( B \)-orbit of \( X \) is dense in \( n \). But then, it has to meet the \( B \)-orbit of \( X_0 = X_{\alpha_1} + \cdots + X_{\alpha_r} \), which means that they are conjugate, so that \( X \in O_{\text{reg}} \).

Kostant proved that \( O_{\text{reg}} \) is exactly the smooth locus of the nilpotent cone \( \mathcal{N} \).

It is also worth mentioning that the projectivized nilpotent cone is an interesting projective variety. Recall that given an irreducible projective variety \( Z \subset PV \), one defines its dual variety \( Z^* \subset PV^* \) as the closure of the set of hyperplanes that are tangent to \( Z \) at some smooth point. We say that \( Z \subset PV \) is self-dual if it is projectively equivalent to \( Z^* \subset PV^* \). This phenomenon is quite exceptional.
PROPOSITION 16 (Popov). The projectivized nilpotent cone $\mathbb{P}\mathcal{N} \subset \mathbb{P}g$ is a self-dual projective variety.

Proof. We can identify $g$ and its dual using the Killing form, hence we consider the dual variety $\mathbb{P}\mathcal{N}^*$ as a subvariety of $\mathbb{P}g$, and $\mathcal{N}^* \subset g$ its affine cone.

Let $X = \sum_{i \in A} X_i$. We know that $X$ is regular nilpotent, hence $\mathcal{N} = \mathcal{O}X$. The tangent space to $\mathcal{N}$ at $X$ is $[g,X]$ and the space of hyperplanes containing it is $[g,X]^\perp$. Therefore $\mathcal{N}^*$ is the closure of $G.[g,X]^\perp$.

Since $X \in [g,X]^\perp$, as follows from the invariance of the Killing form, we deduce that $\mathcal{N} \subset \mathcal{N}^*$. On the other hand, $[g,X] \supset b$, hence $[g,X]^\perp \subset b^\perp = n$. Since $n \subset \mathcal{N}$, we deduce that $\mathcal{N}^* \subset \mathcal{N}$.

The minimal orbit

Since the adjoint representation is irreducible, its projectivization $\mathbb{P}g$ contains a unique closed orbit $X = G/P$. Since it is contained in the closure of every orbit in $\mathbb{P}g$, it must the projectivization of a nilpotent orbit $O_{\text{min}}$, contained in the closure of every non zero nilpotent orbit.

Note that any highest root vector defines a point in $X$, hence also any long root vector. If we denote the highest root by $\psi$, we can complete a highest root vector $X_\psi$ into a $\mathfrak{sl}_2$-triple $(X_\psi, H_\psi, X_\psi)$. The eigenspace decomposition of $ad(H_\psi)$ is

$$g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$$

with $g_2 = g_\psi$ and $g_{-2} = g_{-\psi}$. Moreover the tangent space to $X$ at the point $x_\psi$ defined by the line $g_\psi$ is

$$T_{x_\psi}X = [g_{-\psi}, g_\psi] \oplus g_1.$$

In particular the dimension of $X$ is one plus the number of positive roots $\alpha$ such that $\alpha(H_\psi) = 1$.

3.2. Classification of nilpotent orbits

The number of nilpotent orbits is finite

We have seen that any nilpotent element can be completed into a $\mathfrak{sl}_2$-triple. To what extent do these triples depend on their components? As much as they can, that is:

1. (Kostant) Any two $\mathfrak{sl}_2$-triples with the same positive nilpotent element are conjugate.

2. (Malcev) Any two $\mathfrak{sl}_2$-triples with the same semisimple element are conjugate.

This implies that we can encode any nilpotent orbit $O \subset g$ by the orbit of the semisimple elements that can occur in the corresponding $\mathfrak{sl}_2$-triples. This is quite convenient because semisimple orbits are much simpler to understand.
Proposition 17. The set of semisimple orbits is in bijection with $\mathfrak{h}/W$.

In more sophisticated terms, the fact is that the only closed orbits in $\mathfrak{g}$ are the semisimple ones. The claim is then a reformulation of Chevalley’s restriction theorem, restated as $\mathfrak{g}/G \simeq \mathfrak{h}/W$.

Proof. Any semisimple element is contained in a Cartan subalgebra, and all Cartan subalgebras are conjugate, hence any semisimple orbit meets our given $\mathfrak{h}$. There remains to show that $H$ and $H'$ in $\mathfrak{h}$ are $G$-conjugate if and only if they are $W$-conjugate. Suppose that $H = \text{Ad}(g)H'$. Then $\mathfrak{h}$ and $\text{Ad}(g)\mathfrak{h}$ are two Cartan subalgebras containing $H$, hence contained in $c_{\mathfrak{g}}(H)$. But this is a reductive algebra, inside which $\mathfrak{h}$ and $\text{Ad}(g)\mathfrak{h}$ are certainly two Cartan subalgebras. Hence they must be conjugate by an element $k$ in the adjoint group of $c_{\mathfrak{g}}(H)$. Then $\text{Ad}(kg) = \text{Ad}(k)\text{Ad}(g)\mathfrak{h} = \mathfrak{h}$ and $\text{Ad}(k)\text{Ad}(g)H' = \text{Ad}(k)H = H$. Thus $kg$ defines an element of $W$ mapping $H'$ to $H$.

If $X$ is nilpotent and we complete it into a sl$_2$-triple $(Y, H, X)$, we can suppose that $H \in \mathfrak{h}$. Observe that by sl$_2$-theory, $\alpha(H) \in \mathbb{Z}$ for any root $\alpha$. Moreover, using the action of the Weyl group we may suppose that $\alpha(H) \geq 0$ for any positive root $\alpha$. The orbit of $H$ is then completely characterized by the set of non negative integers $\alpha_1(H), \ldots, \alpha_r(H)$.

Proposition 18. One has $\alpha_i(H) \in \{0, 1, 2\}$ for all $i$.

Proof. By sl$_2$-theory, since $X_{\alpha}$ is an eigenvector of $\text{ad}(H)$ for the eigenvalue $\alpha_i(H)$, we know that $[Y, X_{\alpha}]$ is an eigenvector of $\text{ad}(H)$ for the eigenvalue $\alpha_i(H) - 2$. But $Y$ must be a combination of negative root vectors, and therefore $[Y, X_{\alpha}]$ must belong to $\mathfrak{h}$, since we can never obtain a positive root by adding a simple root to a negative one. So the corresponding eigenvalue cannot be positive, that is, $\alpha_i(H) - 2 \leq 0$ as claimed. But beware that it could also happen that $[Y, X_{\alpha}] = 0$. In that case $X_{\alpha}$ is a lowest weight vector for our sl$_2$-triple, and therefore the eigenvalue of $\text{ad}(H)$ must be non positive, that is $\alpha_i(H) \leq 0$, hence in fact $\alpha_i(H) = 0$.

As an immediate consequence, we obtain the important result:

Theorem 18. There exist only finitely many nilpotent orbits.

Corollary 5. The nilpotent cone is normal.

Proof. The singular locus of $\mathcal{N}$ is a union of finitely many nilpotent orbits, all of codimension at least two since they all have even dimensions. Therefore $\mathcal{N}$ is smooth in codimension one. Being a complete intersection, it is normal by Serre’s criterion.

Remark 7. Beware that a nilpotent orbit closure is not always normal, and that deciding whether it is or not is a difficult problem, still not completely solved for the exceptional Lie algebras.
The theorem also provides us with a concrete way to encode each nilpotent orbit, by a weighted Dynkin diagram: just give the weight $\alpha_i(H) \in \{0,1,2\}$ to the node corresponding to the simple root $\alpha_i$. For example the weighted Dynkin diagram corresponding to the regular orbit $O_{\text{reg}}$ has all its vertices weighted by 2.

**Example 11.** Nilpotent orbits in $\mathfrak{sl}_n$ are classified by their Jordan type, which is encoded by a partition of $n$. We will denote by $O_\lambda$ the nilpotent orbit corresponding to the partition $\lambda$. The regular orbit $O_{\text{reg}} = O_{\lambda}$ (a unique Jordan block), while the minimal orbit $O_{\text{min}} = O_{21\ldots 1}$. The regular orbit is represented by the matrix $X$ whose non-zero entries are ones above the diagonal. We complete it into a $\mathfrak{sl}_2$-triple $(Y,H,X)$ with

$$
H = \begin{pmatrix}
  h_1 & 0 & 0 & \cdots & \\
  0 & h_2 & 0 & \cdots \\
  0 & 0 & h_3 & \cdots \\
  \cdots & \cdots & \cdots & \cdots & h_{n-1} & 0 \\
  \cdots & \cdots & \cdots & \cdots & 0 & h_n
\end{pmatrix},
$$

$$
Y = \begin{pmatrix}
  y_1 & 0 & 0 & \cdots & \\
  0 & y_2 & 0 & \cdots \\
  \cdots & \cdots & \cdots & \cdots & 0 \\
  \cdots & \cdots & \cdots & \cdots & y_{n-1} & 0
\end{pmatrix},
$$

where $h_i = n + 1 - 2i$ and $y_i = h_1 + \cdots + h_i$. One can deduce a $\mathfrak{sl}_2$-triple corresponding to any nilpotent orbit by treating each Jordan block separately. The only point is that the semisimple element $H$ that one obtains this way does not verify the normalization condition $\alpha_i(H) = h_i - h_{i+1} \geq 0$. This is easily corrected simply by putting the diagonal entries of $H$ in non increasing order. One can then read off the resulting matrix the weighted Dynkin diagram of the orbit.

For $\mathfrak{sl}_6$ we get ten non zero orbits, whose weighted Dynkin diagrams are the following:

$O_6$ $\begin{array}{llllll}
  2 & 2 & 2 & 2 & 2 & 2 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{array}$  

$O_{321}$ $\begin{array}{llllll}
  1 & 1 & 0 & 0 & 1 & 1 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{array}$  

$O_{31}$ $\begin{array}{llllll}
  2 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{array}$  

$O_{21}$ $\begin{array}{llllll}
  2 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{array}$  

$O_{23}$ $\begin{array}{llllll}
  2 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{array}$  

$O_{231}$ $\begin{array}{llllll}
  1 & 1 & 0 & 0 & 1 & 1 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{array}$  

$O_{212}$ $\begin{array}{llllll}
  0 & 1 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{array}$  

$O_{213}$ $\begin{array}{llllll}
  1 & 0 & 0 & 0 & 1 & 1 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{array}$  

The major drawback of weighted Dynkin diagrams is that it is not clear how to characterize which, among the $3^r$ possible weighted Dynkin diagrams, really represent nilpotent orbits. This is why other, more effective methods have been developed, notably Bala-Carter theory that will be explained below.

**Richardson orbits**

The Springer resolution has shown how to obtain the nilpotent cone by collapsing a vector bundle over $G/B$. One can wonder if other orbit closures can be obtained from other $G$-homogeneous spaces $G/P$. 

Consider a parabolic subalgebra \( p \) of \( g \), and its Levi decomposition \( p = l \oplus u \). Since \( u \) is contained in the nilpotent cone, and since there are only finitely many nilpotent orbits, there exists a unique nilpotent orbit \( O_p \) meeting it along a dense subset. Such an orbit is a Richardson orbit. For example the regular orbit is Richardson.

**Proposition 19 (Richardson).** Let \( p \) be a parabolic subalgebra, \( O_p \) the associated Richardson orbit and \( P \subset G \) the corresponding parabolic subgroup. Then the natural map

\[
\pi_p : \Omega_{G/P} = G \times^P u \to g
\]

is a generically finite \( G \)-equivariant map over its image \( \bar{O}_p \).

**Example 12.** A parabolic subgroup \( P \) of \( SL_n \) is the stabilizer of a flag \( V^\bullet \) of subspaces \( 0 \subset V_1 \subset \cdots \subset V_m \subset \mathbb{C}^n \) of prescribed dimensions. The cotangent bundle of \( G/P \) can be identified with the space of pairs \((V^\bullet, x)\) where \( x \in \mathfrak{sl}_n \) is such that \( x(V_i) \subset V_{i-1} \) for all \( i \) (with the usual convention that \( V_0 = 0 \) and \( V_{m+1} = \mathbb{C}^n \)). We say that the flag \( V^\bullet \) is adapted to \( x \).

A flag that is canonically attached and adapted to \( x \) is the flag defined by \( V_i = \text{Ker}(x^i) \). Clearly it is the only flag adapted to \( x \) with the same dimensions. If \( p \) is the parabolic subalgebra defined by this flag, this implies that \( \pi_p : \Omega_{SL_n/P} \to \mathfrak{sl}_n \) is birational, hence a resolution of singularities of \( \bar{O}_p \). Moreover any nilpotent orbit in \( \mathfrak{sl}_n \) is a Richardson orbit.

Beware that the degree of \( \pi_p \) is not always equal to one. A generically finite proper and surjective morphism from a smooth variety to a singular one is sometimes called an alteration of singularities.

**Example 13.** The projective space \( \mathbb{P}^{2n-1} \) is acted on transitively by \( SL_{2n} \), but also by the symplectic group \( SP_{2n} \). In the symplectic setting a line \( L \subset \mathbb{C}^{2n} \) should be viewed as corresponding to the parabolic subgroup of \( SP_{2n} \) stabilizing the symplectic flag \( 0 \subset L \subset L^\perp \subset \mathbb{C}^{2n} \). Moreover \( \Omega_{\mathbb{P}^{2n-1}} \) is identified with the space of pairs \((L, x)\) with \( x \in \mathfrak{sp}_{2n} \) adapted to the symplectic flag attached to \( L \).

We claim that the projection map \( \Omega_{\mathbb{P}^{2n-1}} \to \mathfrak{sp}_{2n} \) is not birational over its image \( \bar{O} \). This can be seen as follows. We know that the projection map \( \Omega_{\mathbb{P}^{2n-1}} \to \mathfrak{sl}_{2n} \) is birational over its image \( \bar{O} \), where \( O \) is the minimal nilpotent orbit in \( \mathfrak{sl}_{2n} \). Denote by \( J \) the skew-symmetric matrix defining the symplectic form. One can check that the natural map

\[
p : \mathfrak{sl}_{2n} \to \mathfrak{sp}_{2n}, \quad y \mapsto y - J^{-1}y^t J
\]

sends \( \bar{O} \) to \( \bar{O}_p \). Since \( p(y) = p(-J^{-1}y^t J) \), the induced map \( p : \bar{O} \to \bar{O}_p \) has degree at least two – and in fact degree two. This implies that \( \Omega_{\mathbb{P}^{2n-1}} \to \mathfrak{sp}_{2n} \) has degree two over its image \( \bar{O}_p \).
Resolutions of singularities

A variant of the preceding construction allows to define a canonical resolution of singularities for any nilpotent orbit closure. We start with a nilpotent element $e$ in some nilpotent $O$. We complete it into a $\mathfrak{sl}_2$-triple $(e, h, f)$. Then $h$ defines a grading on $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, and we denote by $\mathfrak{p}$ the parabolic subalgebra $\mathfrak{g}_{\geq 0}$, with obvious notations.

**Lemma 3.** The parabolic algebra $\mathfrak{p}$ only depends on $e$, and not on $h$.

Let $P$ denote the parabolic subgroup of the adjoint group $G$, with Lie algebra $\mathfrak{p}$. Each subspace $\mathfrak{g}_{\geq i}$ of $\mathfrak{g}$ is a $P$-module. Consider the natural map

$$\varphi : G \times^P \mathfrak{g}_{\geq 2} \rightarrow \mathfrak{g}.$$ 

We know that $e \in \mathfrak{g}_{\geq 2}$ and that the map $Ad(e) : \mathfrak{p} \rightarrow \mathfrak{n}_2$ is surjective, so that the $P$-orbit of $e$ is dense in $\mathfrak{g}_2$. This implies that the image of $\varphi$ is exactly the closure of $O$. Moreover, by the previous lemma the preimage of $e$ is a unique point, hence $\varphi$ is birational. We have proved:

**Theorem 19.** The map $\varphi : G \times^P \mathfrak{g}_{\geq 2} \rightarrow \mathcal{O} \subset \mathfrak{g}$ is a $G$-equivariant resolution of singularities.

Using the same argument as for the Springer resolution of the nilpotent cone, one can check that the Kostant-Kirillov-Souriau symplectic form on $O$ extends to a global skew-symmetric form on $\varphi : G \times^P \mathfrak{g}_{\geq 2}$. This exactly means that:

**Corollary 6.** Every nilpotent orbit closure is a symplectic variety.

Beware that the extended skew-symmetric form will in general not remain non-degenerate on the boundary of the nilpotent orbit. When it does, the orbit closure is said to have symplectic singularities. Such singularities are quite rare. The closure of a Richardson orbit does admit symplectic singularities when its Springer type alteration of singularities has degree one. For example this is always the case for orbits in $\mathfrak{sl}_n$. It will also be true for even orbits, that we define as follows.

**Definition 17.** The orbit $O$ is even if $\mathfrak{g}_{\geq 1} = \mathfrak{g}_{\geq 2}$. Equivalently the weighted Dynkin diagram of $O$ has weights 0 or 2 only.

Since $\mathfrak{g}_{\geq 1}$ is the radical of $\mathfrak{p}$ (that we denoted by $u$ in the Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus u$, we get in this case that $\pi_\mathfrak{p} = \varphi$ is birational. We conclude:

**Proposition 20.** Every even orbit is Richardson and its closure admits a Springer type resolution of singularities, which is symplectic. In particular an even orbit closure has symplectic singularities.

**Remark 8.** Note that the normalization $\tilde{O}$ of $\mathcal{O}$ is Gorenstein with trivial canonical bundle. Indeed the mere existence of the Kostant-Kirillov-Souriau form im-
plies that the canonical bundle is trivial on the smooth locus, and the singular locus has codimension at least two since every nilpotent orbit has even dimension. When it is a symplectic resolution of singularities, the birational morphism \( q \) is thus crepant. In general, the fact that the pull-back by \( q \) of the Kostant-Kirillov-Souriau form extends to a global two-form implies that the push-forward by \( q \) of the canonical bundle is trivial. By a result of Flenner this ensures that \( \overline{O} \) always has rational singularities.

Finally, let us mention that by a relatively recent result of Kaledin, the singular locus of \( \overline{O} \) is exactly the complement of \( O \). Its codimension is at least two, but it can be bigger.

One can prove that a nilpotent orbit closure admitting a symplectic resolution must be Richardson. Moreover any symplectic resolution of a Richardson orbit is a Springer type alteration \( \pi_p \) [19], and this happens precisely when \( \pi_p \) has degree one.

**Mukai’s flops**

An interesting point is that two non conjugate parabolic subalgebras \( p \) and \( p' \) may define the same Richardson orbit.

**Proposition 21.** Suppose that two parabolic subalgebras \( p \) and \( p' \) have conjugate Levi subalgebras. Then they define the same Richardson orbit.

**Proof.** Let \( p = l \oplus u \) be the Levi decomposition. The closure of the corresponding Richardson orbit is \( G.u \). Let \( \mathfrak{z} \) be the center of \( l \). Let us suppose that \( p \) is standard, in which case \( \mathfrak{z} \subset \mathfrak{h} \) is contained in the Cartan subalgebra and \( u \subset \mathfrak{n} \). For any root space \( g_\alpha \) contained in \( u \) one can find \( Z \in \mathfrak{z} \) such that \( \alpha(Z) \neq 0 \), therefore \( [\mathfrak{z}, \mathfrak{g}] \) contains \( u \). But then the closure of \( G.\mathfrak{z} \) contains \( G.(\mathfrak{z} + u) \), which is closed, hence they are equal. This implies that

\[
G.u = \overline{G.\mathfrak{z}} \cap \mathfrak{N}.
\]

The proof is now complete, since we have described the Richardson orbit corresponding to \( p \) in terms of its Levi subalgebra \( l \) only.

For example, a nilpotent orbit closure in type A can have an arbitrary large number of different symplectic resolutions. Indeed, the type of a parabolic subalgebra \( p \) of \( \mathfrak{gl}_n \) is prescribed by the dimensions of the spaces in the flag it preserves. This can be any increasing sequence \( d_0 = 0, d_1, \ldots, d_k = n \). The Levi subalgebra of \( p \) is then isomorphic with the sum of the \( \mathfrak{gl}_i \), for \( m_i = d_i - d_{i-1} \), \( 1 \leq i \leq k \). We have \( d_i = m_1 + \ldots + m_i \), but if we permute the \( m_i \)'s arbitrarily we can get a different increasing sequence \( d'_0 = 0, d'_1, \ldots, d'_k = n \), hence another type of parabolic subalgebra defining the same Richardson orbit. Moreover the corresponding Springer type resolution of singularities of the orbit closure will be different, although they can be proved to be deformation equivalent.

A typical example is Mukai’s flop between two distinct resolutions of the closure of an orbit of type \( O_{2r+2r} \) in \( \mathfrak{sl}_n \). This is the orbit of matrices of rank \( r \) and square
zero. The image and the kernel provide us with maps to two different Grassmannians. Passing to the closure we get the following picture:

\[
\begin{array}{ccc}
\Omega_{X_+} & \xleftarrow{\text{flop}} & \Omega_{X_-} \\
X_+ = G(r,n) & \searrow & X_- = G(n-r,n) \\
\end{array}
\]

More explicitly, note that \(\Omega_{X_{\pm}}\) is the total space of \(\text{Hom}(Q,S)\) on the Grassmannian, where \(Q\) and \(S\) are the quotient and tautological bundle. This is the subspace in \(\text{Hom}(V,S)\) of morphisms vanishing on \(S\), hence the natural map to \(\mathfrak{sl}_n\) whose image is our orbit closure.

**Bala-Carter theory**

*Associated parabolic subalgebras.* Recall that if \((Y,H,X)\) is a \(\mathfrak{sl}_2\)-triple, the semisimple element \(H\) has integer eigenvalues. The decomposition of \(g\) into eigenspaces,

\[
g = \bigoplus_{i \in \mathbb{Z}} g_i,
\]

is a grading of \(g\), in the sense that \([g_i,g_j] \subset g_{i+j}\), as follows from the Jacobi identity. In particular \(g_0, g_{\geq 0}\) and \(g_{>0}\), with obvious notations, are graded subalgebras of \(g\). We have seen that we may suppose that \(H \in \mathfrak{h}\), in which case each \(g_i\) for \(i \neq 0\), is just a sum of root spaces. Moreover we may suppose that \(\alpha(H) \geq 0\) for any positive root \(\alpha\). This implies that \(g_{>0} \supset \mathfrak{b}\) is a parabolic subalgebra with radical \(g_{\geq 0}\) and Levi subalgebra \(g_0 = c_g(H)\), a reductive subalgebra containing \(\mathfrak{h}\).

Beware that this parabolic subalgebra does not characterize the orbit of \(X\): two weighted Dynkin diagrams with the same zeroes will define the same conjugacy class of parabolic subalgebras.

*Distinguished orbits.* The main idea of Bala-Carter theory is to use induction through proper Levi subalgebras. Indeed, if a nilpotent element is contained in a proper Levi subalgebra \(l\), then it is also contained in its semisimple part \([l,[l]]\), which has smaller rank than \(g\). Also recall that \(l\) has a non trivial center \(\mathfrak{z}\), consisting in semisimple elements, and that \(l\) and \(\mathfrak{z}\) are the centralizers in \(g\) one of each other.

A consequence is that if \(X \in l \neq g\), then \(c_g(X)\) contains a non zero semisimple element. This element must be contained in \(g_0\) and therefore \(\dim g_2 < \dim g_0\).

Conversely, suppose that \(\dim g_2 < \dim g_0\). Then the centralizer \(l\) of the full \(\mathfrak{sl}_2\)-triple is non zero. One can check that \(l\) is reductive, hence it contains some non zero semisimple element \(H\). Then the centralizer of \(H\) is a proper reductive subalgebra of \(g\) containing \(X\).
A nilpotent element \( X \in \mathfrak{g} \) is distinguished if it is not contained in any proper Levi subalgebra.

With the same notations as before, we have proved that \( X \) is distinguished if and only if \( \dim \mathfrak{g}_2 = \dim \mathfrak{g}_0 \). Independently of the grading, this can be rewritten as follows: we have the parabolic algebra \( \mathfrak{p} = \mathfrak{g}_{>0} \), with its nilpotent radical \( \mathfrak{u} = \{2} \mathfrak{g}_{>0} \) and its Levi part \( \mathfrak{l} = \mathfrak{g}_0 \). Then the condition can be rewritten as

\[
\dim \mathfrak{l} = \dim \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}].
\]

A parabolic subalgebra satisfying this condition will be called distinguished. Note that the dimension on the right hand side is in fact easy to compute: it is the number of indecomposable roots in \( \mathfrak{u} \), that is, roots contributing to \( \mathfrak{u} \) which cannot be written as the sum of two such roots.

In order to complete the induction, there remains to prove that each distinguished parabolic subalgebra \( \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \) in \( \mathfrak{g} \) comes from a distinguished nilpotent element. Let \( O \) be the unique nilpotent orbit meeting \( \mathfrak{u} \) along a dense subset, and \( Z \) be a point in this intersection. As before we know that \( [\mathfrak{p}, Z] = \mathfrak{u} \). Let \( \mathfrak{a} \) (resp. \( \mathfrak{a}_- \)) denote the sum of the root spaces in \( \mathfrak{u} \) corresponding to indecomposable roots (resp. to their opposite roots). Then \( \mathfrak{a} \) is a complement to \( [\mathfrak{u}, \mathfrak{u}] \) in \( \mathfrak{u} \), and \( [\mathfrak{a}, \mathfrak{a}_-] = \mathfrak{l} \). Recall that the hypothesis that \( \mathfrak{p} \) is distinguished means that \( \mathfrak{a} \) and \( \mathfrak{l} \) have the same dimension.

Write \( Z = U + U' \) with \( U \in \mathfrak{a} \) and \( U' \in [\mathfrak{u}, \mathfrak{u}] \). From \( [\mathfrak{p}, Z] = \mathfrak{u} \) we deduce that necessarily, \( [\mathfrak{l}, U] = \mathfrak{a} \). In particular \( [\mathfrak{g}, U] \supset \mathfrak{a} \), and by taking orthogonals with respect to the Killing form we get that \( c_{\mathfrak{g}}(U) \subset \mathfrak{a}^\perp \), hence \( c_{\mathfrak{g}}(U) \cap \mathfrak{a}_- \subset \mathfrak{a}_- \cap \mathfrak{a}^\perp = 0 \). We deduce that \( [\mathfrak{a}_-, U] \) has the same dimension as \( \mathfrak{a}_- \), hence \( \mathfrak{l} \), so that they must be equal.

Now we choose \( H \in \mathfrak{h} \) such that \( \alpha_\alpha(H) = 0 \) if the simple root \( \alpha_\alpha \) is a root of \( \mathfrak{l} \), and \( \alpha_\alpha(H) = 2 \) otherwise. Then \( \alpha_\alpha(H) = 2 \) for any root \( \alpha \) contributing to \( \mathfrak{a} \), hence \( \langle H, U \rangle = 2U \). Moreover, since \( [\mathfrak{a}_-, U] = \mathfrak{l} \) there exists \( V \in \mathfrak{a}_- \) such that \( [U, V] = H \). Finally \( U \) is a distinguished nilpotent element in \( \mathfrak{g} \) and \( \mathfrak{p} \) is the associated parabolic subalgebra. We have proved:

**Theorem 20.** There exists a bijection between nilpotent orbits in \( \mathfrak{g} \) and \( G \)-conjugacy classes of pairs \((\mathfrak{l}, \mathfrak{p})\), with \( \mathfrak{l} \) a Levi subalgebra in \( \mathfrak{g} \) and \( \mathfrak{p} \) a distinguished parabolic subalgebra of \([\mathfrak{l}, \mathfrak{l}]\).

Note that the previous proof also constructs for us a representative of the orbit \( O \) corresponding to \((\mathfrak{l}, \mathfrak{p})\). Indeed, we just need to decompose \( \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \) and then \( \mathfrak{u} = \mathfrak{a} \oplus [\mathfrak{u}, \mathfrak{u}] \), and we know that a generic element of \( \mathfrak{a} \) has to belong to \( O \).

**Example 14.** The Borel subalgebra \( \mathfrak{b} \) is always distinguished: it decomposes as \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \), and \( \mathfrak{n}/[\mathfrak{u}, \mathfrak{u}] \) is generated by the simple root spaces, so its dimension is the same as that of \( \mathfrak{h} \). The corresponding nilpotent orbit is the regular one.

**Example 15.** If \( \mathfrak{g} = sl_n \), a parabolic standard subalgebra has a Levi subalgebra consisting in block matrices of size \( l_1, \ldots, l_k \) with \( l_1 + \cdots + l_k = n \). Then \( \dim \mathfrak{l} = l_1^2 + \cdots + l_k^2 \).
\[ \cdots + l_k^2 - 1 \] while \( \dim a = l_1 l_2 + \cdots + l_{k-1} l_k \). One deduces that no parabolic subalgebra is distinguished apart from the Borel. More generally the Levi subalgebras of \( \mathfrak{g} \) do not contain other distinguished subalgebras than their Borel subalgebras. Therefore there is exactly one nilpotent orbit for each class of Levi subalgebra, hence for each partition of \( n \).

**Example 16.** We classify the nilpotent orbits in \( \mathfrak{g}_2 \). Let \( \alpha_1, \alpha_2 \) be the two simple roots, the first one being short. Up to conjugation there are four parabolic algebras \( \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{g}_2 \) and four Levi subalgebras \( \mathfrak{h}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{g}_2 \) with respective semisimple parts \( 0, \alpha_2, \alpha_1, \mathfrak{g}_2 \). The first three account for three orbits, those of \( 0, X_{\alpha_1}, X_{\alpha_2} \). There remains to classify distinguished parabolic subalgebras of \( \mathfrak{g}_2 \). As always \( b \) is distinguished but \( \mathfrak{g}_2 \) is not. The positive roots in \( \mathfrak{p}_1 \) (resp. \( \mathfrak{p}_2 \)) are \( \alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \) (resp. \( \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \)), among which the indecomposable ones are \( \alpha_1 \) and \( \alpha_1 + \alpha_2 \) (resp. \( \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2 \)). Since \( l_1 \) and \( l_2 \) have dimension four, we see that \( \mathfrak{p}_2 \), which gives four indecomposable roots, is distinguished, while \( \mathfrak{p}_1 \) is not.

**The subregular orbit**

Let us come back to the Springer resolution \( \pi : \Omega_B = G \times_B \mathfrak{n} \to \mathcal{X} \). We know it is an isomorphism over the regular orbit \( O_{\text{reg}} \). More precisely, we have seen that the regular orbit is the orbit of the sum of the simple root vectors

\[ X = X_{\alpha_1} + \cdots + X_{\alpha_r}, \]

to which we can add an arbitrary combination of the root vectors corresponding to non simple positive roots. Since the simple root vectors had been chosen arbitrarily (but non zero, of course), any linear combination

\[ X' = t_{\alpha_1} X_{\alpha_1} + \cdots + t_{\alpha_r} X_{\alpha_r} \]

with \( t_{\alpha_1} \cdots t_{\alpha_r} \neq 0 \) is also in the regular orbit. But if one of the coefficient is taken to be zero, then the centralizer of \( X' \) has dimension \( r + 2 \) and semisimple part isomorphic to \( \mathfrak{sl}_2 \). This shows that the pull-back by \( \pi \) of the complement of the regular orbit is \( G \times_B \mathfrak{t} \) where \( \mathfrak{t} \subset \mathfrak{n} \) is a union of hyperplanes indexed by the simple roots. Moreover its image is the closure of finitely many orbits of codimension two in \( \mathcal{X} \). Since two simple roots are conjugate under the Weyl group when they have the same length, two hyperplanes corresponding to two roots of the same length will yield the same orbit. It is not a priori clear that the same conclusion should hold for two simple roots \( \alpha \) and \( \alpha' \) of different lengths, but we already know that it is true when \( \mathfrak{g} \) has rank two: indeed we have seen that for \( \mathfrak{sp}_4 \) and \( \mathfrak{g}_2 \), there is only one nilpotent orbit of codimension two. But then the general case follows: indeed we can suppose that \( \alpha \) and \( \alpha' \) define adjacent vertices in the Dynkin diagram; then we can consider them inside the simple Lie algebra they generate, which has rank two. And the conclusion goes through. We have proved:
**Proposition 22.** The complement in \( \mathcal{N} \) of the regular orbit \( O_{\text{reg}} \) in the closure of a unique orbit of codimension two.

This orbit is called the subregular orbit and denoted \( O_{\text{subreg}} \). It is particularly important because the singular locus of \( \mathcal{N} \) is exactly the closure of \( O_{\text{subreg}} \). Moreover, taking a transverse slice to \( O_{\text{subreg}} \) at some point yields an affine surface with an isolated singularity. If \( g \) is simply laced, it has been proved by Grothendieck that this is an ADE singularity of the same type as \( g \) [49].

**The closure ordering**

There is a natural ordering on the set of nilpotent orbits, the closure ordering. We simply let \( O \geq O' \) if the Zariski closure \( \overline{O} \) contains \( O' \).

The closure ordering has been determined for all the simple Lie algebras. One of the most efficient tools is to use the desingularization \( G \times^P u \) of the orbit closure \( \overline{O} \) that we constructed below. It is clear that \( O \geq O' \) if and only if \( O' \) meets the linear space \( u \). We can for example deduce the following statement.

**Proposition 23 (Gerstenhaber).** Let \( \lambda \) be a partition of \( n \), and \( O_\lambda \subset \mathfrak{sl}_n \) the nilpotent orbit of matrices whose Jordan blocks have for sizes the parts \( \lambda_i \) of \( \lambda \). Then \( O_\lambda \geq O_\mu \) if and only if

\[
\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \quad \forall i.
\]

This condition defines a partial order on the set of partitions of a same integer, that we denote by \( \lambda \geq \mu \). This is called the dominance ordering.

**Proof.** Recall that \( x \) belongs to \( O_\lambda \) if and only if the kernel of \( x^k \) has dimension \( \lambda_i^* + \cdots + \lambda_k^* \) for each \( k \), where \( \lambda^* \) denotes the dual partition of \( \lambda \). Since the dimension of kernels can only increase when we go to the closure, the condition that \( O_\lambda \geq O_\mu \) implies that \( \lambda^* \leq \mu^* \) for the dominance ordering. So it implies that \( \lambda \geq \mu \) by the following lemma:

**Lemma 4.** The dominance ordering is reversed by duality: \( \lambda \geq \mu \) if and only if \( \lambda^* \leq \mu^* \).

**Example 17.** Recall that we have listed ten non-trivial nilpotent orbits in \( \mathfrak{sl}_6 \). The closure ordering is the following:
Proof of the Lemma. Write $\lambda \mapsto \mu$ if $\lambda > \mu$ and there is no partition $\nu$ such that $\lambda > \nu > \mu$. Then there exists a sequence of parts of $\lambda$ of the form $(p + 1, p^2, q)$, with $p > q$ and $a \geq 0$, such that the corresponding sequence of parts in $\mu$ is $(p^{a+1}, q + 1)$ – all the other parts being the same in $\lambda$ and $\mu$. This implies that $\lambda \mapsto \mu$ if and only if $\mu^* \mapsto \lambda^*$, hence the claim.

Conversely, suppose that $\lambda \mapsto \mu$. We need to prove that the closure of $O_\mu$ contains $O_\nu$. It is enough to check this for $\lambda = \mu$, and by the previous lemma it suffices to show that the closure of $O_{p+1,q}$ contains $O_{p,q+1}$. Consider $x$ with two Jordan blocks of size $p + 1$ and $q$, that is $x = \sum_{i \neq p+1} e_i^* \otimes e_{i-1}$. We can easily complete it into a $\mathfrak{sl}_2$-triple by treating separately the two blocks, hence reducing to the regular orbits in $\mathfrak{sl}_{p+1}$ and $\mathfrak{sl}_q$. In particular we get a semisimple element $h$ in diagonal form, to which we associate the linear space $u \subset \mathfrak{sl}_{p+q}$. Then the element

$$x' = \sum_{i=3}^{p+1} e_i^* \otimes e_{i-1} + e_{p+2}^* \otimes e_1 + \sum_{j=p+3}^{p+q+1} e_j^* \otimes e_{j-1}$$
belongs to $u$ and has Jordan type $(p, q + 1)$. This completes the proof. \hfill \Box

4. Prehomogeneous spaces, generalities

4.1. Prehomogeneous vector spaces

In this section we discuss prehomogeneous spaces, vector spaces on which a Lie group acts almost transitively. More than the fact that they are the natural affine generalizations of the homogeneous spaces, our motivation is that a large class of prehomogeneous spaces can be constructed inside the tangent spaces of the rational homogeneous varieties: these are called parabolic homogeneous spaces. Prehomogeneous spaces are also related to interesting birational transformations. They appear naturally in the geometry of Severi and Scorza varieties.

Let $G$ denote an affine algebraic group and $V$ a $G$-module.

**Definition 19.** The $G$-module $V$ is a prehomogeneous vector space if it contains an open $G$-orbit $O$. The isotropy subgroup $G_x$ of a point $x \in O$ is independent of $x$ up to conjugacy. It is called the generic isotropy group.

Note that $V$ remains prehomogeneous for any finite index subgroup of $G$. In particular we will suppose in the sequel that $G$ is connected.

**Example 18.** The group $\text{GL}_m(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ acts on the space $M_{m,n}(\mathbb{C})$ of matrices of size $m \times n$, and the set of matrices of maximal rank is an open orbit, so $M_{m,n}(\mathbb{C})$ is a prehomogeneous vector space. What is the generic isotropy group? More generally, take any $G$-module $V$, of dimension $n$. Then $V \otimes \mathbb{C}^m$ is a prehomogeneous vector space for $G \times \text{GL}_m(\mathbb{C})$ if $m \geq n$.

**Definition 20.** A non zero rational function $f$ on $V$ is a relative invariant if there exists a character $\chi \in X(G)$ such that $f(gx) = \chi(g)f(x)$ $\forall g \in G, x \in V$.

The hypersurface $(f = 0)$ in $V$ is then $G$-invariant, and since $G$ is connected each of its irreducible component is invariant. The converse also holds.

**Proposition 24.** Let $f_1, \ldots, f_m$ be equations of the irreducible components of codimension one of $V \setminus O$. They are algebraically independent relative invariants of $V$, called the fundamental invariants. Every relative invariant is, up to scalar, of the form $f_1^{p_1} \cdots f_m^{p_m}$ for some integers $p_1, \ldots, p_m \in \mathbb{Z}$.

**Proof.** Since the hypersurface $(f_i = 0)$ is $G$-invariant, for every $g \in G$, $f_i \circ g$ is again an equation of it, hence a multiple of $f_i$ by some scalar $\chi_i(g)$. Each $\chi_i$ is clearly a character of $G$. These characters are all different, because $\chi_i = \chi_j$ would imply that $f_i / f_j$ is an invariant rational function, in particular constant on $O$, hence on $V$. But by Dedekind’s
lemma, relative invariants with different characters are always linearly independent. Indeed, suppose we have a relation \( a_1 f_1 + \cdots + a_m f_m = 0 \), where the character of \( f_i \) is \( \chi_i \), and \( \chi_1, \ldots, \chi_m \) are pairwise distinct. Applying this relation to \( g x \), with \( g \in G \) and \( x \in V \), we get the relations

\[
a_1 \chi_1 (g) f_1 + \cdots + a_m \chi_m (g) f_m = 0, \quad \forall g \in G.
\]

Since pairwise distinct characters are linearly independent, this implies that \( a_1 = \cdots = a_m = 0 \). Moreover the same argument implies that any polynomial relation between \( f_1, \ldots, f_m \) must be trivial, that is, \( f_1, \ldots, f_m \) are algebraically independent.

Conversely, decompose any relative invariant into irreducible factors. Because \( G \) is connected, each of these factors is itself a relative invariant, and defines an irreducible component of \( V \setminus O \). So it must be one of the \( f_i \)'s.

**Proposition 25.** Suppose that the prehomogeneous vector space \( V \) is an irreducible \( G \)-module, the group \( G \) being reductive. Then \( V \) has at most one fundamental invariant.

**Proof.** Suppose we have two non proportional fundamental invariants \( f, g \), of degrees \( d, e \) respectively. Then \( f^e / g^d \) is an invariant rational function: indeed, since \( V \) is irreducible, the center of \( G \) acts by homotheties on \( V \) (Schur’s lemma), and the semisimple part has no non trivial character. But then \( f^e / g^d \) must be a constant, a contradiction. \( \square \)

**Example 19.** The general linear group \( GL_n \) acts on \( \mathbb{C}^n \) with only two orbits, the origin and its complement. In particular there is no relative invariant. This remains true for the action of \( Sp_n \) acts on \( \mathbb{C}^n \) (\( n \) even). On the contrary \( \mathbb{C}^n \) is not a prehomogeneous vector space for the action of \( SO_n \), but it is one for \( \mathbb{C}^* \times SO_n \), the invariant quadratic form \( q \) being a relative invariant.

**Example 20.** The action of \( GL_n \) on the space of symmetric or skew-symmetric two-forms is again prehomogeneous: there are finitely many orbits classified by the rank (which is always even in the skew-symmetric case). The determinant provides a relative invariant, non trivial except in the skew-symmetric case and in odd dimensions. In this case there is no relative invariant, since all the orbits have codimension bigger than one (except the open one of course). In even dimensions the determinant is not a fundamental invariant, being the square of the Pfaffian, which is the fundamental invariant.

**Example 21 (Pencils of skew-symmetric forms).** The strong difference between skew-symmetric forms in even and odd dimensions is of course also apparent when we consider the space \( V = \mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^n \), with the action of \( G = GL_2 \times SL_m \). A tensor in \( V \) which is not decomposable defines a pencil of skew-symmetric forms, and studying such pencils is essentially equivalent to study \( SL_2 \)-orbits in \( V \).

Since \( \dim V = n(n - 1) < \dim G = n^2 + 3 \) we could expect \( V \) to be a prehomogeneous vector space. This cannot be the case if \( n = 2m \) is even with \( m > 3 \). Indeed,
Consider an element of $V$ as a map from $\mathbb{C}^2$ to $\wedge^2 \mathbb{C}^n$. Taking the Pfaffian we get a polynomial function of degree $m$ on $\mathbb{C}^2$. When this polynomial is non-zero it defines an $m$-tuple of points on $\mathbb{P}^1$. Since $PSL_2$ does not act generically transitively on such $m$-tuples of points, the action cannot be prehomogeneous.

Of course the argument does not apply when $n$ is odd since the Pfaffian does not exist in this case. One can check that $V = \mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^2$ is in fact prehomogeneous under the action of $G = GL_2 \times SL_{2m+1}$. Indeed, a direct computation shows that the stabilizer of $v_1 \otimes \omega_1 + v_2 \otimes \omega_2$ has the correct dimension if

$$\omega_1 = e_1 \wedge e_2 + \cdots + e_{2m-1} \wedge e_{2m},$$

$$\omega_2 = e_2 \wedge e_3 + \cdots + e_{2m} \wedge e_{2m+1}.$$

Note that the $G$-orbit of $v_1 \otimes \omega_1 + v_2 \otimes \omega_2$ is open in $V$ if and only if the $PSL_n$-orbit of $\langle \omega_1, \omega_2 \rangle$ is open in the Grassmannian $G(2, V)$ parametrizing pencils of skew-symmetric forms (this is an important remark that we will use again when we will define castling transforms). Therefore the generic pencil is equivalent to $\langle \omega_1, \omega_2 \rangle$.

### 4.2. Regular invariants

Let $f$ be a relative invariant on the prehomogeneous vector space $V$. The rational map

$$\phi_f : V \dashrightarrow V^* \quad x \mapsto df(x)/f(x)$$

is $G$-equivariant (with respect to the dual action of $G$ on $V^*$). Moreover it is well-defined on the open orbit $O$. Its degree is $-1$, and $\phi_f$ should be considered as a kind of inverse mapping (see the examples below).

The differential of $\phi_f$ at a point $x \in O$ is a linear map from $V$ to $V^*$, whose determinant we denote by $H(f)(x)$. In a given basis, this is just the usual determinant of the matrix of second derivatives of $\log(f)$ – its Hessian.

**Definition 21.** A prehomogeneous vector space $V$ is **regular** if it admits a relative invariant $f$ whose Hessian $H(f)$ is not identically zero.

**Proposition 26.** The prehomogeneous vector space $V$ is regular if and only if there is a relative invariant $f$ such that the rational map $\phi_f : V \dashrightarrow V^*$ is dominant. In particular, $V^*$ is also prehomogeneous for the dual action of $G$.

**Example 22.** Consider the action on $V = \mathbb{C}^2$, of the stabilizer $G \subset GL_2(\mathbb{C})$ of some non-zero vector. Check that this is a prehomogeneous vector space whose dual space is not prehomogeneous.

In the cases we will be interested in in the sequel, the group $G$ will be reductive. Most irreducible prehomogeneous vector space of reductive groups are well-behaved, in the sense that they have a unique non-trivial relative invariant, and this relative invariant is regular. One can show that the existence of a **regular** fundamental invariant
We deduce a degree four relative invariant of $B$. But there is an open orbit, which is a prehomogeneous vector space by computing the isotropy algebra of an explicit point.

In practice this is an important criterion, since we often check that a vector space is a prehomogeneous vector space by computing the isotropy algebra of an explicit point. When we have done this computation, it is in general easy to decide whether this algebra is reductive or not. If it is reductive, we know that we have to go in search of a relative invariant. Then we face the problem that we don’t know a priori what should be the degree of such an invariant. We will discuss this question in the last section, in connection with projective duality.

What we can quickly decide is the number of fundamental invariants that we need to find. Indeed, let $v$ belong to the open orbit $O$ in $V$, and let $G_v$ denote its stabilizer (the generic stabilizer). If $f$ is a non trivial invariant with character $\chi$, the relation $f(gv) = \chi(g)f(v)$ implies that $\chi(G_v) = 1$. Conversely, if $\chi$ is a character of $G$ with this property, we can define a regular function on $O$ by letting $f(gv) = \chi(g)$ for any $g \in G$. This is a rational function on $V$, and its character $\chi$ is therefore the difference of the characters of two regular invariants.

Let us summarize this discussion.

**Proposition 27.** Let $V$ be a prehomogeneous vector space of a reductive group $G$, let $v$ belong to the open orbit. Then $V$ is regular if and only if the generic stabilizer $G_v$ is reductive. If this is the case, the number of fundamental invariants is the rank of the group

$$X(G,V) = \text{Ker}(X(G) \to X(G_v)).$$

**Example 23.** Consider $V = \wedge^3 \mathbb{C}^6$, acted on by $G = GL_6$. A straightforward computation shows that the point $v = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$ has a stabilizer $G_v$ whose connected component is isomorphic to $SL_3 \times SL_3$. In particular the $G$-orbit of $v$ is open.

Moreover, since $G_v$ is semi-simple and $X(G) = \mathbb{Z}$, the prehomogeneous vector space $V$ is regular and there is only one fundamental invariant, hence the complement of the open orbit must be an irreducible hypersurface.

**Example 24.** Let $G = SL_3 \times SO_3 \times Sp_{2n}$ act on $V = \mathbb{C}^3 \otimes \mathbb{C}^{2n}$. We claim that this is a prehomogeneous space for any $n \geq 2$. A relative invariant can be constructed as follows. Let $q$ be the quadratic form preserved by $SO_3$ and $\omega$ the symplectic form preserved by $Sp_{2n}$. We consider $V$ as the space of maps $\nu$ from $(\mathbb{C}^3)^*$ to $\mathbb{C}^{2n}$. Such a map will send $q$ to an element of $S^2 \mathbb{C}^{2n}$, hence $q^2$ to an element of $Q(\nu) \in S^2 (S^2 \mathbb{C}^{2n})$. But there is an $Sp_{2n}$-invariant linear form $\Omega$ on the latter space, defined by

$$\Omega(ab,cd) = \omega(a,c)\omega(b,d) + \omega(a,d)\omega(b,c).$$

We deduce a degree four relative invariant of $G$ by letting $f(v) = \Omega(Q(\nu))$. In coordinates, suppose given an orthonormal basis $(e_1,e_2,e_3)$ of $\mathbb{C}^3$ and let $v = e_1 \otimes v_1 + e_2 \otimes v_2 \otimes v_3$. Then $v$ is regular and there is only one fundamental invariant, hence the complement of the open orbit must be an irreducible hypersurface.
\[ v_2 + e_3 \otimes v_3. \]

Then
\[ \frac{1}{2} f(v) = \omega(v_1, v_2)^2 + \omega(v_2, v_3)^2 + \omega(v_3, v_1)^2. \]

This relative invariant is not regular (see [30], Example 2.30 page 70).

There is a nice interplay between a prehomogeneous vector space and its dual, that we will describe in some detail. The main motivation is a nice relation with Cremona transformations – birational transformations of projective spaces.

Over the complex numbers, a reductive group \( G \) is the complexification of a compact Lie group \( K \) (in fact, this is the original definition). For example \( G = GL_n(\mathbb{C}) \) is the complexification of the unitary group \( K = U_n \).

Now, our complex representation \( V \) of \( G \) can always be endowed with a \( K \)-invariant Hermitian form; otherwise said, we may suppose it is contained in the unitary group. But then the dual representation \( V^* \), restricted to \( K \), is equivalent to the complex conjugate representation \( \bar{V} \) since \( g = (g^{-1})' \) when \( g \in K \).

Suppose that \( V \) has a relative invariant \( f \) of degree \( d \geq 2 \). By letting \( f^*(x) = f(x) \), \( x \in V^* \), we obtain a relative invariant of \( K \) on \( V^* \cong \bar{V} \), of the same degree as \( f \). Since \( G \) is the complexification of \( K \), this will automatically be a relative invariant of \( G \). Moreover, \( f^* \) is clearly regular as soon as \( f \) is.

4.3. Cremona transformations

A very nice property of regular invariants is that it provides interesting examples of Cremona transformations.

**Theorem 21.** Suppose that \( G \) is reductive and that the prehomogeneous vector space \( V \) is regular, with a unique fundamental invariant \( f \). Then \( df : \mathbb{P}V \to \mathbb{P}V^* \) is a birational map, with inverse \( d f^* : \mathbb{P}V^* \to \mathbb{P}V \).

**Proof.** The main observation is that starting from the relative invariants \( f \) and \( f^* \), both of degree \( d \), we can construct a new relative invariant \( F = f^*(df) \). Since \( f^* \) is regular \( F \) cannot be zero, and since its degree is \( d(d-1) \) there exists a non zero constant \( c \) such that \( F = 2cf^{d-1} \). The result will be obtained by differentiating this identity.

For \( x \in O \) consider \( q_f(x) = df(x)/f(x) \in V^* \). Its differential \( dq_f(x) \) is a linear map from \( V \) to \( V^* \), and by the regularity hypothesis this map is invertible. For \( \delta \in V^* \), we claim that

\[ \langle x, dq_f(x)(\delta) \rangle = -2\langle \delta, q_f(x) \rangle. \]

This follows readily from the Euler identity, \( q_f \) being homogeneous of degree \(-1\).
Now the relation $F = 2cf^d - 1$ can be rewritten as $f^*(q_f(x)) = 2cf(x)^{-1}$. Differentiating, we obtain
\[
\langle df^*(q_f(x)), dq_f(x)(\delta) \rangle = -2cf(x)^{-2}\langle \delta, df(x) \rangle.
\]
Taking 3 into account and letting $\theta = dq_f(x)(\delta)$, this can be rewritten as
\[
\langle df^*(q_f(x)), \theta \rangle = cf(x)^{-1}\langle x, \theta \rangle.
\]
But since $dq_f(x)$ is invertible this is true for any $\theta$, and we deduce that $df^*(q_f(x)) = cf(x)^{-1}x$, or equivalently
\[
df^*(df(x)) = cf(x)^{d-2}x.
\]
This completes the proof.

**Example 25.** Consider $G = (\mathbb{C}^*)^n$ acting diagonally on $V = \mathbb{C}^n$. This is a prehomogeneous space, the open orbit being the set of vectors with only non zero coordinates. The relative invariant $f(x) = x_1 \cdots x_n$ is regular, and the associated Cremona transformation is
\[
\psi_f(x) = (x_1^{-1}, \ldots, x_n^{-1}).
\]

**Example 26.** Consider the action of $G = GL_n$ on $V = \text{Sym}^2 \mathbb{C}^n$, with its regular fundamental invariant given by the determinant. Since the differential of the determinant is the comatrix, which is proportional to the inverse, the associated Cremona transformation of $P(\text{Sym}^2 \mathbb{C}^n)$ is just the inverse map. The same conclusion holds for the action of $G = GL_n \times GL_n$ on $V = \mathbb{C}^n \otimes \mathbb{C}^n$ and the action of $G = GL_{2n}$ on $V = \wedge^2 \mathbb{C}^{2n}$.

**Example 27.** Consider $G = GL_2$ acting on $V = \text{Sym}^3(\mathbb{C}^2)^*$, the space of cubic polynomials. A direct computation shows that the stabilizer of the cubic $x^3 + y^3$ is finite, so that its orbit is open and $V$ is prehomogeneous. More is true: the stabilizer being reductive this prehomogeneous space must be regular. We can construct a relative invariant $f$ as follows: starting from a cubic $C$, its Hessian $He(C)$ is a quadratic polynomial, and we can let $f(C)$ equal its discriminant. Explicitly, for $C = a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3$ we get
\[
f(C) = 18a_1a_2a_3a_4 + a_2^2a_3^2 - 27a_1a_4^2 - 4a_1a_3^3 - 4a_2a_4.
\]
The theorem implies that the rational map
\[
(a_1, a_2, a_3, a_4) \mapsto (9a_2a_3a_4 - 27a_1a_4^2 - 2a_1a_3^3, 9a_1a_3a_4 + a_2a_4^2 - 6a_2a_4, 9a_1a_2a_4 + a_2^2a_3 - 6a_1a_3^2, 9a_1a_2a_3 - 27a_1a_4 - 4a_2^4)
\]
is an involutive Cremona transformation of $\mathbb{P}^3$.

Using the same ideas as above, one can prove the following statement:
THEOREM 22. Let $V$ be a prehomogeneous vector space of a reductive group $G$, with a relative polynomial invariant $f$, and let $f^*$ be the corresponding relative invariant of $V^*$. Then there exists a polynomial $b$ in one variable, of the same degree as $f$, such that

$$f^*(\partial) f^* = b(s)f^{s-1} \quad \forall s \in \mathbb{Z}.$$ 

The polynomial $b$ is called the $b$-function of the prehomogeneous vector space, and such polynomials have been studied extensively. One of the simplest examples is the case where $G = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acts on $V = \mathcal{M}_{n,\mathbb{Z}}(\mathbb{C})$. We know this is a prehomogeneous vector space with a regular relative polynomial invariant given by the determinant. The corresponding $b$-function is $b(s) = s(s+1)\cdots(s+n-1)$, and the identity

$$\det(\frac{\partial}{\partial x_j}) \det(x)^{s} = s(s+1)\cdots(s+n-1) \det(x)^{s-1}$$

was discovered by Cayley.

4.4. Castling transforms

In the sequel we only deal with reductive groups. An important idea in the classification of prehomogeneous vector spaces is to define equivalence classes. First, two prehomogeneous vector space $V$ and $V'$ on which the groups $G$ and $G'$ respectively act through the representations $\rho$ and $\rho'$, are strongly equivalent if there is an isomorphism of $V$ with $V'$ that identifies the actions of $\rho(G)$ and $\rho'(G')$. This means that we only care with the effective action of $\rho(G)$ of $V$, not really of $G$.

A more subtle idea is to use the following result.

PROPOSITION 28. Let $V$ be a $G$-module of dimension $n$, and $p,q$ be integers such that $p+q = n$. Then $V \otimes \mathbb{C}^p$ is a prehomogeneous vector space for $G \times GL_p(\mathbb{C})$, if and only if $V^* \otimes \mathbb{C}^q$ is one for $G \times GL_q(\mathbb{C})$.

Proof. Suppose that $V \otimes \mathbb{C}^p \simeq V \otimes \cdots \otimes V$ is a prehomogeneous vector space for $G \times GL_p(\mathbb{C})$. The open subset $\Omega$ of independent $p$-tuples $(v_1, \ldots, v_p)$ is $G \times GL_p(\mathbb{C})$-stable, hence contains the open orbit. Moreover, we have a map from $\Omega$ to the Grassmannian $G(p, V)$, sending $(v_1, \ldots, v_p)$ to the $p$-dimensional subspace $P$ of $V$ that they span. Note that the fiber of $P$ is the space of all its basis, on which $GL_p(\mathbb{C})$ acts transitively. Therefore, $G \times GL_p(\mathbb{C})$ has an open orbit in $V \otimes \mathbb{C}^p$, if and only if $G$ has an open orbit in the Grassmannian $G(p, V)$.

Now we use that fact that by duality, $G(p, V) \simeq G(q, V^*)$, in such a way that the action on the former is identified with the dual action on the latter. So $G$ has an open orbit on $G(p, V)$ if and only if it has an open orbit on $G(q, V^*)$, and by the same argument as above, this is equivalent to the fact that $V^* \otimes \mathbb{C}^q$ is a prehomogeneous vector space for $G \times GL_q(\mathbb{C})$.

REMARK 9. One can easily check that the generic isotropy groups are the same
in $V \otimes \mathbb{C}^p$ and $V^* \otimes \mathbb{C}^q$. In particular, if one of these spaces is regular, the other one is also regular.

**Remark 10.** If $V$ is an $n$-dimensional prehomogeneous vector space for $G$, it is also prehomogeneous for the action of $G \times GL_1(\mathbb{C}) = G \times \mathbb{C}^*$, where $\mathbb{C}^*$ acts by homotheties. We conclude that $V^* \otimes \mathbb{C}^{n-1}$ is a prehomogeneous vector space for $G \times GL_{n-1}(\mathbb{C})$. Applying this procedure again and again, we can construct infinite series of prehomogeneous vector spaces from a given one.

**Example 28.** Starting from the obvious fact that the action of $GL_1(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ on $\mathbb{C}^1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2$ is prehomogeneous. Let us denote $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ by $(a, b, c)$ for simplicity. By castling transforms we obtain the following sequence of prehomogeneous vector spaces:

$$(1, 2, 2) \quad (3, 2, 2) \quad (3, 2, 4) \quad (5, 2, 4) \quad (5, 2, 6) \quad (5, 18, 4) \quad (3, 10, 26) \quad (37, 10, 4)$$

**Definition 22.** Two irreducible prehomogeneous vector spaces $U$ and $U'$ are related by a castling transform if there is an $n$-dimensional irreducible $G$-module $V$, and integers $p, q$ with $p + q = n$, such that $U$ is strongly isomorphic to $V \otimes \mathbb{C}^p$ and $U'$ to $V^* \otimes \mathbb{C}^q$.

Two irreducible prehomogeneous vector spaces $U$ and $U'$ are equivalent if they can be related by a finite sequence of castling transforms.

**Proposition 29.** Every equivalence class of prehomogeneous vector spaces contains a unique element of minimal dimension, up to duality.

**Proof.** We can prove a stronger statement, which is that if two prehomogeneous vector spaces are related by a sequence of castling transforms, than the dimensions of the sequence of prehomogeneous vector spaces produced by these transforms must be monotonous. Suppose the contrary. Then we can find a prehomogeneous vector space $V$ for a group $G$ with two different castling transforms of smaller dimensions. These transforms need to correspond to two distinct factors in $G$, so we may suppose that
$G = H \times GL_a \times GL_b$ and $V = U \otimes \mathbb{C}^a \otimes \mathbb{C}^b$. Denote by $u$ the dimension of the $H$-module $U$. The two castling transforms are then $U \otimes \mathbb{C}^a \otimes \mathbb{C}^{a-b}$ and $U \otimes \mathbb{C}^{ub-a} \otimes \mathbb{C}^b$. If they are both of dimension smaller than $V$, then $ua < 2b$ and $ub < 2a$, therefore $u < 2$, a contradiction.

\begin{proof}

\end{proof}

4.5. The case of tensor products

We have seen that using castling transforms we can produce infinite sequences of tensor products $\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_m}$ for which the action of $GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_m}(\mathbb{C})$ is prehomogeneous. We would like to know which of these tensor products have finitely many orbits.

\begin{lemma}
\label{finite-orbits-lemma}
Let $V$ be a $G$-module, with $G$ reductive. If $V \otimes \mathbb{C}^a$ has finitely $G \times GL_a(\mathbb{C})$ orbits and $n \leq \dim V$, then $V \otimes \mathbb{C}^m$ has finitely $G \times GL_m(\mathbb{C})$ orbits for all $m \leq n$.
\end{lemma}

\begin{proof}
Again we consider $V \otimes \mathbb{C}^n$ as the space of $n$-tuples $(v_1, \ldots, v_n)$, and we have a stratification by the dimension $k$ of the span of these $n$-vectors. Each strata can be mapped to a Grassmannian $G(k, V)$, and using the same argument as in the proof of Proposition 28, there is a bijective correspondence between the $G \times GL_n(\mathbb{C})$-orbits in $V \otimes \mathbb{C}^n$ and the $G$-orbits in the disjoint union of the $G(k, V)$ for $0 \leq k \leq n$. The claim follows immediately.
\end{proof}

\begin{proposition}
\label{finite-orbits-prop}
Suppose $k_1 \geq \cdots \geq k_m > 1$ and $m \geq 3$.

If $GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_m}(\mathbb{C})$ has finitely many orbits in the tensor product $\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_m}$, then $m = 3$, $k_1 = 2$ and $k_2 \leq 3$.
\end{proposition}

\begin{proof}
By Lemma \ref{finite-orbits-lemma} it is enough to prove that $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ have infinitely many orbits. Since the center of each copy of $GL_2(\mathbb{C})$ acts by homotheties, the orbits of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ are the same as those of $\mathbb{C}^2 \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. But this group has dimension 13, so cannot have an open orbit in the 16-dimensional space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The second case can be treated in the same way. We leave the last case as an exercise: one has to prove that the generic isotropy group is five-dimensional.
\end{proof}

It turns out that there is no other obstruction.

\begin{theorem}
\label{finite-orbits-theorem}
The group $GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_m}(\mathbb{C})$ has finitely many orbits in the tensor product $\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_m}$, if and only if $m \leq 2$, or $m = 3$ and $(k_1, k_2, k_3) = (n, 2, 2)$ or $(n, 3, 2)$.
\end{theorem}

Parfenov has studied the orbits and their incidence relations. A remarkable fact is that the orbit structure stabilizes when $n$ is large enough, as indicated by the following table, which includes the degree of the fundamental relative invariant, when there is one.
Most of these prehomogeneous vector spaces are parabolic (see the next lecture). For example, the isotropic Grassmannian $G_Q(n - 2, 2n)$ is encoded in the weighted Dynkin diagram

The corresponding space with finitely many orbits is $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{n-2}$.

The cases of type $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^n$ come from the exceptional groups up to $n = 5$, which corresponds to the $E_8$-Grassmannian with weighted Dynkin diagram

**Remark 11.** Note that having finitely many orbits is a much stronger condition than being prehomogeneous vector space. In particular it is not preserved by castling transforms. Nevertheless, it follows from the classification theorems of Sato and Kimura that any prehomogeneous tensor product is castling equivalent to a prehomogeneous tensor product with finitely many orbits.

### 4.6. Relations with projective duality

There are several recipes that allow to compute the degree of the fundamental invariant of a simple prehomogeneous space $V$. Kimura gives a trace formula which boils down to a linear algebra computation (see [30], Proposition 2.19 page 34).

More geometrically, consider a $G$-orbit closure $Z$ in the projective space $\mathbb{P}V$. The projective dual variety $Z^*$ inside $\mathbb{P}V^*$ is $G$-stable, hence a $G$-orbit closure since $\mathbb{P}V^*$ has only finitely many orbits. Since projective duality is involutive, we deduce that:

<table>
<thead>
<tr>
<th>$(k_1, k_2, k_3)$</th>
<th># orbits</th>
<th>$\deg f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, 2, 2)$</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>$(3, 2, 2)$</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>$(4, 2, 2)$</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>$(n \geq 5, 2, 2)$</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$(3, 3, 2)$</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>$(4, 3, 2)$</td>
<td>24</td>
<td>12</td>
</tr>
<tr>
<td>$(5, 3, 2)$</td>
<td>26</td>
<td>0</td>
</tr>
<tr>
<td>$(6, 3, 2)$</td>
<td>27</td>
<td>6</td>
</tr>
<tr>
<td>$(n \geq 7, 3, 2)$</td>
<td>27</td>
<td>0</td>
</tr>
</tbody>
</table>
Proper Proposition 31. Projective duality defines a bijection between $G$-orbits in $\mathbb{P}V$ and $G$-orbits in $\mathbb{P}V^*$. Beware that this bijection is in general rather badly behaved. For example it is not compatible with the closure ordering on orbits.

In general, the projective dual of a variety $Z \subset \mathbb{P}V$ is expected to be a hypersurface, unless $Z$ has some special properties (for example $Z$ will be uniruled). Of course orbit closures are quite special and the proposition shows that their duals are almost never hypersurfaces. Nevertheless, one could hope that a minimal $G$-orbit (closure) $Z_{\text{min}}$ in $\mathbb{P}V$ has a codimension one dual variety. Its equation would then be a fundamental (semi)invariant of $V^*$.

For a smooth projective variety $Z \subset \mathbb{P}V$ of dimension $n$, there exist quite explicit formulas that allow to compute the degree of the dual variety, and decide whether it is a hypersurface or not. Denote by $h$ the hyperplane class on $Z$ and let
$$\delta_Z = \sum_{i \geq 0} (i+1) \int_Z c_{n-i}(\Omega_Z)h^i.$$ If $\delta_Z = 0$, then the dual variety $Z^*$ is not a hypersurface. If $\delta_Z \neq 0$, then $Z^*$ is a hypersurface of degree $\delta_Z$. In particular in the case where $Z$ is the minimal $G$-orbit in a simple prehomogeneous space, then $\delta_Z$ is the degree of the fundamental invariant.

Even if $\delta_Z = 0$, one can determine the dimension and the degree of the dual variety as follows. Consider the polynomial
$$P_Z(q) = \sum_{i \geq 0} q^{i+1} \int_Z c_{n-i}(\Omega_Z)h^i.$$ Proper Proposition 32 (Katz). Let $c$ be the minimal integer such that $P_Z^{(c)}(1) \neq 0$. Then $Z^*$ has codimension $c$ and degree $P_Z^{(c)}(1)/c!$.

Example 29. Consider $G = \text{GL}_m \times \text{GL}_m$ acting on $V = \mathfrak{gl}_m \simeq M_m(\mathbb{C})$ by multiplication on the left and on the right. The orbits are defined by the rank, so that the closure ordering is complete. Moreover projective duality exchanges the rank with the corank. In particular the minimal orbit is $\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$, and its dual is the determinant hypersurface.

The same remark applies to symmetric or skew-symmetric matrices. In the latter case, note that the prehomogeneous space $V = A_n(\mathbb{C}) \simeq \wedge^2 \mathbb{C}^n$ is regular only when $n = 2m$ is even. The minimal orbit in $\mathbb{P}V$ is the Grassmannian $G(2, 2m)$, whose dual variety is the Pfaffian hypersurface, of degree $m$.

For $n$ odd, $V$ has no semi-invariant and $\mathbb{P}V$ contains no invariant hypersurface. The complement of the open orbit is the dual variety of $G(2, 2m+1)$. This is a codimension three subvariety of $\mathbb{P}V^*$, of degree $2m+1$.

Example 30. Consider $G = \text{GL}_5$ or $\text{SL}_5$ acting on $V = \wedge^2 \mathbb{C}^5$. There are only three orbits, parametrizing tensors of rank 0, 2 or 4. In particular the Grassmannian $G(2, 5)$ must be self-dual.
The tensor product $\mathbb{C}^k \otimes V$ remains prehomogeneous under the action of $GL_k \times GL_5$ for $k \leq 4$. The minimal orbit $Z$ in $\mathbb{P}((\mathbb{C}^k \otimes V)$ is $\mathbb{P}^{k-1} \times G(2,5)$. Its codimension $c$ and degree $\delta$ are as follows:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$c$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>40</td>
</tr>
</tbody>
</table>

**Example 31.** We have shown how to produce infinite sequences of prehomogeneous spaces of type $\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_m}$ acted on by $GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_m}(\mathbb{C})$. We can even impose that these tensor products are regular with a unique fundamental invariant. How to define this invariant is not clear in general. As we suggested above, we can try to use the fact that the projectivization $\mathbb{P}((\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_m})$ contains $\mathbb{P}(\mathbb{C}^{k_1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{k_m})$, whose dual variety is expected to be a hypersurface. This is known to hold when the $m$-tuple $(k_1, \ldots, k_m)$ is balanced in the sense that $2k_i \leq k_1 + \cdots + k_m$ for each $1 \leq i \leq m$. When this is true the degree of the dual hypersurface, hence of the fundamental invariant, is computed in [24]. Unfortunately the condition that the $m$-tuple $(k_1, \ldots, k_m)$ is balanced is in general not preserved by castling transformations.

5. Prehomogeneous spaces of parabolic type

There is a close connection between parabolic subgroups of simple Lie groups and a large family of prehomogeneous vector spaces. These will be defined by gradings of semisimple Lie algebras.

5.1. Classification of $\mathbb{Z}$-gradings

**Definition 23.** A $\mathbb{Z}$-grading of a Lie algebra $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i,$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. In particular, $\mathfrak{g}_0$ is a subalgebra and each component $\mathfrak{g}_i$ is a $\mathfrak{g}_0$-module.

Our main example is the following. Take $\mathfrak{g}$ semisimple, $\mathfrak{t}$ a Cartan subalgebra and $\Delta$ a set of simple roots. Choose a subset $I \subset \Delta$, and let $H_I \in \mathfrak{t}$ be defined by the conditions that

$$\alpha(H_I) = 0 \quad \text{when} \quad \alpha \in I, \quad \alpha(H_I) = 1 \quad \text{when} \quad \alpha \notin I.$$

This element $H_I$ induces a $\mathbb{Z}$-grading of $\mathfrak{g}$ defined by letting

$$\mathfrak{g}_i = \{X \in \mathfrak{g}, \quad [H_I, X] = iX\}.$$
Note that a root space $g_i$, with $\beta = \sum_{\alpha \in A} n_\alpha \alpha$, is contained in $g_i$ for $i = \sum_{\alpha \notin \Delta} n_\alpha$. In particular, the subalgebra

$$g_0 = t \oplus \bigoplus_{\alpha \in \langle I \rangle} g_\alpha,$$

where $\langle I \rangle \subset \Delta$ is the set of roots that are linear combinations of the simple roots in $I$. In particular, the subalgebra $g_0 = t \oplus \bigoplus_{\alpha \in I} g_\alpha$ is contained in $g_i$ for $i = \sum_{\alpha \notin \Delta} n_\alpha$. The center of $g_0$ is $Z = \{ H \in t, (H) = 0 \forall \alpha \in I \}$, and the derived algebra $[g_0, g_0] = \bigoplus_{\alpha \in I} C H_\alpha \oplus \bigoplus_{\alpha \in \langle I \rangle} g_\alpha$ is semisimple. Its Dynkin diagram can be deduced from that of $g$ just by erasing the vertices corresponding to the simple roots not belonging to $I$.

More generally, we can associate a $\mathbb{Z}$-grading to any element $H \in t$ such that $\alpha_i(H) \in \mathbb{Z}$ for any simple root $\alpha_i$.

**Proposition 33.** Up to conjugation, any $\mathbb{Z}$-grading of the semisimple Lie algebra $g$ is defined by $H \in t$, and one can suppose that $\alpha_i(H) \in \mathbb{Z}^+$.  

**Remark 12.** Let us return to the previous situation. Then $p_I = \bigoplus_{i \geq 0} g_i$ is the Lie algebra of the parabolic subgroup $P_I$ of $G$. Moreover, we get a filtration of $g$ by $p_I$-modules if we let $g_{i,j} = \bigoplus_{i,j \geq 0} g_i$. The homogeneous bundle defined by the adjoint action of $P_I$ on $g_{i,j}$ is isomorphic with the cotangent bundle of $G/P_I$.

### 5.2. Parabolic prehomogeneous spaces

Let us suppose for simplicity that $I$ is the complement of a single simple root, which means that $P_I$ is a maximal parabolic subgroup of $G$. In this case $G/P_I$ is sometimes called a generalized Grassmannian.

**Proposition 34.** In this case, each component $g_i$ of the grading, for $i \neq 0$, is an irreducible $g_0$-module.

**Proof.** The main observation is that $g_i$ is a sum of root spaces in $g$ with the same coefficient on the simple root which is not in $I$. This means that, considered as a module over the semisimple part of $g_0$, each $g_i$ has only weights of multiplicity one, which are all congruent with respect to its root lattice.

This is enough to ensure the irreducibility. Indeed, suppose that we have two highest weights $\mu$ and $\nu$, congruent with respect to the root lattice. Looking at the signs of the coefficients of $\mu - \nu$ expressed in terms of simple roots, we see that there exist two disjoint sets $J, K$ of simple roots, and two positive linear combinations $\mu_J, \nu_K$ of simple roots in $J, K$, respectively, such that $\mu - \mu_J = \nu - \nu_K$. Call this weight $\lambda$. If $i \notin J$,

$$\langle \lambda, \alpha_i^\vee \rangle = \langle \mu - \mu_J, \alpha_i^\vee \rangle \geq 0$$

since $\langle \mu, \alpha_i^\vee \rangle \geq 0$ and $\langle \alpha_i, \alpha_i^\vee \rangle \leq 0$ for any $j \neq i$. Similarly $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ when $i \notin K$. But then this holds for any $i$, hence $\lambda$ is dominant. Being smaller that $\mu$ it must be a
weight of $V$, and also of $V_6$ for the same reason. But then it would have multiplicity at least two in $g_i$, a contradiction.

The case of $g_1$ is particularly simple, since we know in this case that $\alpha_i$ is the only lowest weight, if $i$ does not belong to $I$. This allows to determine pictorially the representation $g_1$ of the semisimple part of $g_0$.

**Remark 13.** A particularly nice case is when $g_2 = 0$. Then $g$ has a three-step grading

$$g = g_{-1} \oplus g_0 \oplus g_1,$$

hence a particularly simple structure. The generalized Grassmannian $G/P_I$ is then called a *cominuscule* homogeneous space. Equivalently, the isotropy representation of $P_I$ on the tangent space $g/p_I$ is irreducible.

This happens exactly when the highest root has coefficient one on some of the simple roots. Therefore one can easily list all the cominuscule homogeneous spaces by looking at the highest roots of the simple Lie algebras. For example, from the Dynkin diagrams of types $D_6$ and $E_6$ we get the following gradings:

\[
\begin{align*}
&\circ \circ \circ \circ \bullet \quad \text{so}_{12} = \Lambda^2(C^6)^* \oplus \mathfrak{gl}_6 \oplus \Lambda^2(C^6) \\
&\circ \circ \circ \circ \circ \quad \text{e}_6 = \Delta^- \oplus (C \oplus \text{so}_{10}) \oplus \Delta^+ \quad \text{for } D_6
\end{align*}
\]

Let $G_0$ denote the connected subgroup of $G$ with Lie algebra $g_0$. The adjoint action of $G$ on $g$, when restricted to $G_0$, stabilizes the subspaces $g_i$.

**Theorem 24 (Vinberg).** The action of $G_0$ on $g_1$ has a finite number of orbits. In particular, it is prehomogeneous.

**Definition 24.** A prehomogeneous vector space obtained by this procedure will be called a parabolic prehomogeneous vector space.

**Proof.** The idea is to deduce this statement from the finiteness of nilpotent orbits in $g$. Indeed, $g_1$ is contained in the nilpotent cone of $g$, so it is enough to prove that any nilpotent $G$-orbit $O$ in $g$ intersects $g_1$ along the union of finitely many $G_0$-orbits. We will prove a more precise statement: each irreducible component of $O \cap g_1$ is a $G_0$-orbit.

This follows from an infinitesimal computation. Let $x$ be a $G$-orbit of $x$. The tangent space at $x$ to this (schematic) intersection is $T_x O \cap g_1 = g_0 \cap g_1$. The tangent space to the $G_0$-orbit of $x$ is $g_0 \cdot x$. We just need to check that they are equal. Obviously $g_0 \cdot x \cap g_1 \supset g_0 \cdot x$. 

Conversely, let $X \in g$, and decompose it as $\sum X_k$ according to the grading. For $Y = [X,x]$ to belong to $g_1$, we need that $[X_k,x] = 0$ for $k \neq 0$. But then $Y = [X_0,x]$ belongs to $g_0x$.

5.3. Classification of orbits

The previous statement does not really provide us with a convenient tool for classifying the $G_0$-orbits in $g_1$. Indeed, the intersection of a nilpotent $G$-orbit with $g_1$ can be very non transverse; it will often be empty, and when it is not, the number of its connected component seems difficult to control a priori. That is why other classification schemes have been developed, more in the spirit of Bala-Carter theory. The following definition is due to Vinberg.

**Definition 25.** A graded reductive subalgebra $s$ of $g$ is:

- regular if it is normalized by a Cartan subalgebra of $g_0$; then $s$ is the direct sum of a subspace of this Cartan subalgebra with some of the root spaces – in particular regular subalgebras can be classified combinatorially;
- complete if it is not contained in a bigger graded reductive subalgebra $s'$ of $g$ of the same rank;
- locally flat if $s_0$ and $s_1$ have the same dimension; an important example is obtained by taking any semisimple Lie algebra $s$, and letting $s_0$ be a Cartan subalgebra and $s_1$ be the sum of the simple root spaces.

**Theorem 25.** There is a bijection between the $G_0$-orbits in $g_1$ and the $G_0$-conjugacy classes of graded semi-simple subalgebras $s$ of $g$ that are regular, complete and locally flat.

Let us explain how this bijection is defined.

Starting from an element $e$ in $g_1$, we start by completing it into a $\mathfrak{sl}_2$-triple $(e,h,f)$, with $h \in g_0$ and $f \in g_{-1}$. Consider in $G_0$ the normalizer $N_0(e)$ of the line generated by $e$. For any element $u$ of its Lie algebra, there is a scalar $\psi(u)$ such that

$$[u,e] = \psi(u)e.$$ 

In particular $\psi(h) = 2$. Let $H$ be a maximal torus in $N_0(e)$, whose Lie algebra $\mathfrak{h}$ contains $h$. Define, for each integer $k$,

$$g_k(h) = \{ x \in g_k, \quad [u,x] = k\psi(u)x \quad \forall x \in \mathfrak{h} \}. $$

The direct sum $g(h)$ of these spaces is a graded subalgebra of $g$. By construction $g_0(h) = g_0(h) \supset \mathfrak{h}$ and $e \in g_1(h)$. One can show that $g(h)$ is always reductive. Let $s$ be its semi-simple part, called the support of $e$. Vinberg proves that $s$ is regular, complete and locally flat. Moreover the $S_0$-orbit of $e$ is dense in $s_1$ (this is where the flatness condition comes from).
The latter property shows how to define the reverse bijection. Starting from a graded semi-simple subalgebra $s$ of $g$, which is regular, complete and locally flat, we simply choose for $e$ a generic element in $s_1$ – more precisely, an element of the open $S_0$-orbit.

So we have replaced the problem of classifying $G_0$-orbits in $g_1$ by the problem of classifying $G_0$-conjugacy classes of graded semi-simple regular, complete and locally flat subalgebras $s$ of $g$. As we noticed, the regularity property implies that this problem can be treated as a problem about root systems, hence purely combinatorial.

Concretely, one can start with $s$, an abstract graded locally flat semi-simple algebra, with a Cartan subalgebra and a root space decomposition. Then we try to embed it in $g$ as a regular subalgebra. Such an embedding is defined by the images of the simple root vectors of $s$. Up to conjugation it is determined by associating to each simple root $\alpha_i$ a root $\beta_i$ in the root system of $g$. These roots need not be simple, but they have to share the same pattern as the simple roots of $s$ – this will guarantee that there is a compatible embedding of $s$ inside $g$ as a Lie subalgebra.

One important ingredient that we miss is the classification of regular, complete and locally flat graded semi-simple algebras. This was obtained by Vinberg. The most important class is that of principal gradings, which are obtained as follows. Consider any semi-simple Lie algebra $s$, choose a Cartan subalgebra $t$ and a set of simple roots. Then define a grading of $s$ by letting $s_0 = t$ and $s_1$ be the sum of the simple root spaces. Obviously this grading is regular and locally flat. It is also complete.

Some non principal gradings can be obtained by giving degree zero to some of the simple roots, and degree one to the remaining ones. It is easy to see that the local flatness condition implies that the vertex of the Dynkin diagram corresponding to the degree zero root is a triple vertex, which is quite restrictive (we will call these non principal gradings elementary). This indicates that the non principal gradings are relatively scarce and can be classified.

**Example 32.** Start with $g = \mathfrak{e}_7$ with the grading defined by the node at the end of the shortest arm of the Dynkin diagram. This is a five-step grading, namely

$$\mathfrak{e}_7 = \mathbb{C}^7 \oplus \wedge^3(\mathbb{C}^7)^* \oplus \mathfrak{gl}_7 \oplus \wedge^3(\mathbb{C}^7) \oplus (\mathbb{C}^7)^*.$$

The $GL_7$-orbits in $\wedge^3(\mathbb{C}^7)$ have been first determined by Schouten in 1931. There are exactly ten orbits including zero. As was already known to E. Cartan, the generic stabilizer is (up to a finite group) a form of the exceptional group $G_2$. Moreover a normal form (different from the one below) of the generic three-form encodes the multiplication table of the Cayley octonion algebra.

The correspondence between orbits and graded subalgebras allows us to exhibit a representative $\omega$ of each orbit. This goes as follows, where we denote by $O^d$ the unique orbit of dimension $d$. (Note that this contains the classification of $GL_6$-orbits inside $\wedge^3(\mathbb{C}^6)$: there are exactly four orbits of dimensions $0, 10, 15, 19, 20$. We will come back to this example later on.)
EXAMPLE 33. Let us consider the grading of \( e_6 \) defined by the central vertex of the Dynkin diagram of type \( E_6 \). Then we need to determine what can be the restriction \( Q \) the Killing form of \( e_6 \) to the weights of \( g_1 = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \), which we represent as triples \((ijk)\) with \( 1 \leq i \leq 2 \) and \( 1 \leq j, k \leq 3 \). We do not need to make an explicit computation, which could be tedious: we just need to observe that the result must be invariant by the Weyl group of \( g_0 \), which is the product of the permutation groups of the indices \( i, j \) and \( k \) respectively. Moreover, since \( e_6 \) is simply-laced, we know that we can normalize \( Q \) in such a way that \( Q(ijk, ijk) = 2 \) and \( Q(ijk, lmn) \in \{-1, 0, 1\} \) if \( ijk \) and \( lmn \) are distinct. The only possibility is that \( Q(ijk, lmn) = \delta_{ij} + \delta_{jm} + \delta_{kn} - 1 \).

Then we must first analyze the embeddings of the semisimple Lie algebras with their principal gradings. Concretely we take a Dynkin diagram and try to associate to each node a basis vector of \( g_1 \), in such a way that the scalar products match. We get the following lists of possibilities, up to isomorphism:

\[
\begin{align*}
O & \quad g & \quad \omega \\
O^{35} & 3A_1 \times A_2 & e_{123} + e_{145} + e_{167} + e_{246} + e_{357} \\
O^{34} & 2A_1 \times A_2 & e_{123} + e_{145} + e_{246} + e_{357} \\
O^{31} & A_1 \times A_2 & e_{123} + e_{246} + e_{357} \\
O^{28} & 4A_1 & e_{123} + e_{145} + e_{167} + e_{246} \\
O^{26} & A_2 & e_{123} + e_{456} \\
O^{25} & 3A_1 & e_{123} + e_{145} + e_{246} \\
O^{24} & 3A_1 & e_{123} + e_{145} + e_{167} \\
O^{20} & 2A_1 & e_{123} + e_{145} \\
O^{13} & A_1 & e_{123} \\
\end{align*}
\]
Then there remains to deal with the non principal gradings. One can check that this gives only one additional case, coming from an elementary non principal grading of type $D_4$, that we denote $D_4(\alpha_2)$.

$$D_4(\alpha_2) = (111)(122)(133) - \alpha_1$$

where $\alpha_1$ is the positive root of the $\mathfrak{sl}_2$ factor.

Finally we get 18 orbits in $g_1 = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ (including the origin) and we can provide explicit representatives.

It is rather remarkable that the classification of orbits in these two spaces of tensors, and a few others, are in fact controled by the exceptional Lie algebras!

5.4. The closure ordering

The Zariski closure on a $G_0$-orbit $O$ in $g_1$ is a union of $G_0$-orbits. We define the closure ordering on the set of orbits by letting

$$O \geq O' \quad \text{when} \quad \overline{O} \supset \overline{O}'.$$

It is an interesting problem to determine the closure ordering. Unfortunately it is not known how to interprete it in terms of supports.

**Example 34.** In the space of matrices $M_{m,n}$ with its usual action of $GL_m \times GL_n$, the orbits $O_r$ are classified by the rank $r$. Since the rank is semi-continuous, $O_r \geq O_s$ if and only if $r \geq s$.

In simple cases it is possible to define semi-continuous invariants: typically when we deal with tensors we can consider the ranks of auxiliary maps that can be constructed. This is sometimes sufficient to determine completely the closure ordering.

**Example 35.** Let us come back to the case $\wedge^3(\mathbb{C}^7)$, whose $GL_7$-orbits we have just described. One can define two auxiliary invariants, the rank and the 2-rank, that we are going to describe.

Each element $\omega \in \wedge^3(\mathbb{C}^7)$ defines a map

$$a_\omega : \wedge^2(\mathbb{C}^7) \to \wedge^5(\mathbb{C}^7) \simeq \wedge^2(\mathbb{C}^7)^*,$$

and the rank of $\omega$ is the rank of this map. The definition of the 2-rank is more subtle. Observe that there exists a $GL_7$-equivariant map $S^3(\wedge^3(\mathbb{C}^7)) \to S^2 \mathbb{C}^7$ defined up to constant. Indeed, let us choose a generator $\Omega$ of $\det(\mathbb{C}^7)$. For $u \in (\mathbb{C}^7)^*$ denote by $\omega_u \in \wedge^3(\mathbb{C}^7)$ the contraction of $\omega$ by $u$. We define a quadratic form $q_\omega$ on $(\mathbb{C}^7)^*$ by the formula

$$\omega \wedge \omega_u \wedge \omega_u = q_\omega(u) \, \Omega.$$

The 2-rank is the rank of this quadratic form. Note that $q_\omega$ is non-degenerate if and only if $\omega$ belongs to the open orbit, and this yields the classical embedding $G_2 \subset SO_7$. 
Since the rank and the 2-rank cannot increase by specialization, they impose strong restrictions of the possibilities for an orbit to belong to the closure of an open orbit. Note also that the determinant of $a_o$ defines a semi-invariant of degree 21. However, since the rank drops by three on the codimension one orbit, this determinant must be a cube, and indeed the fundamental invariant has degree seven. It can also be computed as the degree of the dual hypersurface to the Grassmannian $G(3, 7)$.

With the help of these two invariants, and a little extra work, one shows that the closure ordering on orbits is represented by the following diagram. Here we denoted the orbits by $O^d(r, \rho)$ rather than simply $O^d$, with $r$ the 2-rank and $\rho$ the rank.

\[
\begin{array}{c}
O^{35}(21, 7) \\
\downarrow \\
O^{34}(18, 4) \\
\downarrow \\
O^{31}(16, 2) \\
\end{array}
\]

\[
\begin{array}{c}
O^{35}(21, 7) \\
\downarrow \\
O^{34}(18, 4) \\
\downarrow \\
O^{31}(16, 2) \\
\end{array}
\]

\[
\begin{array}{c}
O^{28}(16, 1) \\
\downarrow \\
O^{28}(16, 1) \\
\downarrow \\
O^{24}(15, 1) \\
\end{array}
\]

\[
\begin{array}{c}
O^{25}(12, 0) \\
\downarrow \\
O^{25}(12, 0) \\
\downarrow \\
O^{20}(10, 0) \\
\end{array}
\]

\[
\begin{array}{c}
O^{20}(10, 0) \\
\downarrow \\
O^{13}(6, 0) \\
\end{array}
\]

\[
\begin{array}{c}
O^{13}(6, 0) \\
\downarrow \\
0 \\
\end{array}
\]

In more complicated cases, invariants like the rank or the two-rank of the previous example are not necessary available, or they do not allow to determine the closure ordering completely.

### 5.5. Desingularizations of orbit closures

Let $O$ be the $G_0$-orbit of some element $e \in g_1$, and let us complete it into a $\mathfrak{sl}_2$-triple $(e, h, f)$ with $h \in g_0$ and $f \in g_{-1}$. The semi-simple element $h$ acts on $g_0$ and $g_1$ with integer eigenvalues. The direct sum of its non-negative eigenspaces in $g_0$ is a parabolic
subalgebra \( p \); the direct sum in \( g_1 \) of its eigenspaces associated to eigenvalues at least equal to two is a \( p \)-submodule \( u \) of \( g_1 \). Let \( P \) denote the parabolic subgroup of \( G_0 \) with Lie algebra \( p \).

**Proposition 35.** The natural map \( \pi : G_0 \times^P u \to g_1 \) defines a \( G_0 \)-equivariant resolution of singularities of \( \overline{O} \).

This can be used to determine the closure ordering.

**Corollary 7.** Let \( O' \) be another \( G_0 \)-orbit in \( g_1 \). Then \( O \geq O' \) if and only if \( O' \) meets \( u \).

**Example 36.** Let us come back to the case of \( \wedge^3 C^6 \) and its five orbit closures. We have seen that the unique \( GL_6 \)-orbit \( O \) of codimension five is represented by the point \( \omega = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5 \). Let \( U_1 \) and \( U_5 \) be the tautological vector bundles of rank one and five over the flag variety \( F_{1,5} \), and let \( E \) denote the vector bundle \( U_1 \wedge (\wedge^2 U_5) \) over \( F_{1,5} \). The total space of \( E \) projects to \( \wedge^3 C^6 \) and gives the canonical desingularization of \( O \). But there are two other natural desingularizations, obtained as follows. Denote by \( V_1 \) the tautological line bundle on \( \mathbb{P}(C^6) \), and by \( V_5 \) the tautological hyperplane bundle on the dual \( \mathbb{P}(C^6)^* \). Let \( F \) be the vector bundle \( V_1 \wedge (\wedge^2 C^6) \) over \( \mathbb{P}(C^6) \), and let \( G \) be the vector bundle \( \wedge^3 V_5 \) over \( \mathbb{P}(C^6)^* \). Then both the total spaces of \( F \) and \( G \) are desingularizations of \( O \), and there is a diagram

![Diagram](image)

and the birational transformation between \( \text{Tot}(F) \) and \( \text{Tot}(G) \) is a flop. In particular the canonical desingularization of \( O \) is not minimal. One can check that the two other desingularizations are crepant, while the canonical one is not.

It is a rather general phenomenon that a \( G_0 \)-orbit closure in \( g_1 \) has many different equivariant resolutions of singularities by total spaces of vector bundles (and even more alterations). Understanding which are the best behaved is an open problem. For a systematic treatment of the exceptional cases, except \( E_8 \), see [33, 34].

### 5.6. Classification of irreducible prehomogeneous vector spaces

The classification of irreducible prehomogeneous vector spaces of reductive complex algebraic groups has been obtained by Sato and Kimura [46]. Recall that we used
castling transforms to define an equivalence relation on prehomogeneous spaces, and that each equivalence class contains a unique reduced prehomogeneous space, that is, of minimal dimension.

**Theorem 26.** Every irreducible prehomogeneous vector space $V$ of a reductive complex algebraic group $G$ is castling equivalent to either:

1. a trivial prehomogeneous vector space, that is, some $U \otimes \mathbb{C}^n$ acted on by $H \times \text{SL}_n$, where $U$ is any irreducible representation of the reductive group $H$, of dimension smaller than $n$;

2. a parabolic prehomogeneous vector space;

3. the restriction of a parabolic prehomogeneous vector space to some subgroup;

4. $GL_{2m+1} \times SL_2$ or $SL_{2m+1} \times SL_2$ acting on $\Lambda^2 \mathbb{C}^{2m+1} \otimes \mathbb{C}^2$.

This follows only a posteriori from the explicit classification. Let us be more precise.

We can list all the parabolic prehomogeneous vector spaces by listing the Dynkin diagrams and their vertices. Each case provides us with a reductive group $G_0$ with center $\mathbb{C}^*$, with a prehomogeneous action on $V = g_1$. Most of them are regular with the following exceptions (recall that non regularity is equivalent to the condition that the complement of the open orbit has codimension at least two):

<table>
<thead>
<tr>
<th>$G_0$</th>
<th>$V$</th>
<th>$G$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GL_{2m+1}$</td>
<td>$\Lambda^2 \mathbb{C}^{2m+1}$</td>
<td>$D_{2m+1}$</td>
<td>$\alpha_{2m+1}$</td>
</tr>
<tr>
<td>$GL_{2m+1} \times SL_2$</td>
<td>$\Lambda^2 \mathbb{C}^{2m+1} \otimes \mathbb{C}^2$</td>
<td>$E_{2m+2}$</td>
<td>$\alpha_2$</td>
</tr>
<tr>
<td>$GL_{2m+1} \times SP_{2m}$</td>
<td>$\mathbb{C}^{2m+1} \otimes \mathbb{C}^{2n}$</td>
<td>$C_{n+2m+1}$</td>
<td>$\alpha_{2m+2}$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times Spin_{10}$</td>
<td>$\Delta_1$</td>
<td>$E_6$</td>
<td>$\alpha_1$</td>
</tr>
</tbody>
</table>

Here in the column denoted by $G$ we give the type of the simple Lie group whose Lie algebra admits a grading yielding the prehomogeneous vector space $(G_0, V)$. In the column $P$ we give the simple root that defines the grading. Those prehomogeneous vector spaces have no non trivial relative invariant. Note that the case coming from $E_{2m+2}$ is parabolic only for $m \leq 3$, but yields prehomogeneous vector spaces for arbitrary $m$!

The fact that there is no relative invariant can be shown to imply that the action remains prehomogeneous when restricted to a subgroup which is a complement to the center of $G_0$ — in each case a copy of $\mathbb{C}^*$.

There remains only seven “accidental” cases obtained by restriction from $G_0$ to
a subgroup $H$. The list is the following:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$V$</th>
<th>$G$</th>
<th>$P$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GL_2 \times SO_8$</td>
<td>$\mathbb{C}^2 \otimes \mathbb{C}^8$</td>
<td>$D_6$</td>
<td>$\alpha_2$</td>
<td>$GL_2 \times Spin_7$</td>
</tr>
<tr>
<td>$GL_3 \times SO_8$</td>
<td>$\mathbb{C}^3 \otimes \mathbb{C}^8$</td>
<td>$D_7$</td>
<td>$\alpha_3$</td>
<td>$GL_3 \times Spin_7$</td>
</tr>
<tr>
<td>$C^* \times SO_{16}$</td>
<td>$\mathbb{C}^{16}$</td>
<td>$D_9$</td>
<td>$\alpha_1$</td>
<td>$C^* \times Spin_9$</td>
</tr>
<tr>
<td>$C^\ast \times Spin_{12}$</td>
<td>$\mathbb{C}^{32}$</td>
<td>$E_7$</td>
<td>$\alpha_1$</td>
<td>$C^\ast \times Spin_{11}$</td>
</tr>
<tr>
<td>$C^* \times SO_7$</td>
<td>$\mathbb{C}^7$</td>
<td>$B_4$</td>
<td>$\alpha_4$</td>
<td>$C^\ast \times G_2$</td>
</tr>
<tr>
<td>$GL_2 \times SO_7$</td>
<td>$\mathbb{C}^2 \otimes \mathbb{C}^7$</td>
<td>$B_5$</td>
<td>$\alpha_2$</td>
<td>$GL_2 \times G_2$</td>
</tr>
</tbody>
</table>

These cases have the same flavor: they come from special low-dimensional embeddings of simple Lie groups in slightly bigger groups, namely $Spin_7 \subset SO_8$ and $Spin_9 \subset SO_{16}$ (these embeddings are defined by the spin representations of $Spin_7$ and $Spin_9$, the natural embedding $Spin_{11} \subset Spin_{12}$, and the embedding $G_2 \subset SO_7$, which will be the starting point of the last chapter.

**About the proof.** A substantial part of the proof of the classification theorem by Sato and Kimura consists in listing all the irreducible $G$-modules $V$ such that $\dim G \geq \dim V$, which is an obvious condition for $V$ to be prehomogeneous. The Weyl dimension formula is used to express the dimension of an irreducible module in a convenient way, in terms of its highest weight. Of course we can eliminate the trivial prehomogeneous vector spaces and the non reduced ones. Then it is not difficult to show that $G$ cannot have more than three simple factors. After some work Sato and Kimura obtain an explicit finite list. Then they consider each case one by one and manage to decide, essentially by explicit computations, which is prehomogeneous and which is not.

### 5.7. $\mathbb{Z}_m$-graded Lie algebras

There is a nice variant of the theory that is, in some way, closer to the case of nilpotent orbits we started with. We have explained how to classify $\mathbb{Z}$-gradings of semisimple Lie algebras. It is natural to ask about gradings over more general groups, or even monoids. We will focus on $\mathbb{Z}_m$-gradings.

The first observation is that defining a $\mathbb{Z}_m$-grading on a Lie algebra $\mathfrak{g}$ is equivalent to giving a group homomorphism $f$ from $\mathbb{Z}_m$ to $Aut(\mathfrak{g})$: the graded parts of $\mathfrak{g}$ being the eigenspaces of $\theta = f(1)$. If $\mathfrak{g}$ is semisimple, the algebraic group $Aut(\mathfrak{g})$ is a finite extension of the group of inner automorphisms, which is the adjoint group $G$ (the quotient is the group of outer automorphisms, it is isomorphic with the automorphism group of the Dynkin diagram). The $\mathbb{Z}_m$-grading of $\mathfrak{g}$ is called *inner* if $\theta = Ad(h)$ is an inner automorphism. Since $\theta^m = 1$, the element $h \in G$ is semisimple and its eigenvalues are $m$-th roots of unity. One can show that up to conjugation, such an $h$ can be prescribed by the following recipe.

Let us suppose that $\mathfrak{g}$ is simple. Choose as usual a Cartan subalgebra and a set of simple roots $\Delta$. Let $\psi$ denote the highest root.

**Definition 26.** The affine Dynkin diagram of $\mathfrak{g}$ is obtained by adding to the Dynkin diagram a vertex (representing $-\psi$), and connecting it to the vertex represent-
ing the simple root $\alpha_i$ by $\langle \psi, \alpha_i \rangle$ edges (oriented as usual from the long to the short root in the non simply laced case).

Let $n_i$ be the coefficient of $\psi$ on the simple root $\alpha_i$, and let $n_0 = 1$. Then up to conjugation, the inner automorphism $\theta$ is uniquely defined by non-negative integers $\theta_0$ and $\theta_i, i \in \Delta$, such that

$$n_0 \theta_0 + \sum_{i \in \Delta} n_i \theta_i = m.$$  

For example, from the affine Dynkin diagrams of types $E_7$ and $E_8$ we get the following gradings, over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ respectively:

$$E_7 = \mathfrak{sl}_8 \oplus \wedge^4 (\mathbb{C}^8)$$

$$E_8 = \wedge^3 (\mathbb{C}^9)^* \oplus \mathfrak{sl}_9 \oplus \wedge^3 (\mathbb{C}^9)$$

An important difference with the case of $\mathbb{Z}$-gradings is that a subspace $g_i, i \neq 0$, need not be contained in the nilpotent cone. Nevertheless, the same argument as in the $\mathbb{Z}$-graded case shows that there are only finitely many $G_0$-orbits of nilpotent elements in $g_i$. Moreover they can again be classified by their support, and their Zariski closures admit equivariant desingularizations by total spaces of vector bundles.

For example nilpotent orbits in $\wedge^4 (\mathbb{C}^8)$ or $\wedge^3 (\mathbb{C}^9)$ can be classified (see [52, 53]).

6. The magic square and its geometry

6.1. The exceptional group $G_2$ and the octonions

Stabilizers of three-forms

The classical groups $O(V), Sp(V), SL(V)$ are all defined to be groups preserving some generic tensor. In addition to these, there is just one more class of simple group that can be defined as the group preserving a generic tensor of some type on a vector space. The reason there are so few is that the tensor spaces $S_3 V$ almost always have dimension greater than that of $SL(V)$, and then the subgroup of $SL(V)$ preserving a generic element will be zero dimensional.

Remark 14. The group preserving a generic tensor is not always simple, for example the odd symplectic group $Sp_{2n+1}$ preserving a generic $\omega \in \Lambda^2 \mathbb{C}^{2n+1}$ is not even reductive. However it does have interesting properties.
Other than $S^2 V$ and $\wedge^2 V$, the only examples of Schur powers $S_2 V$ of smaller dimension than $SL(V)$ are $A^3 V$ (and their duals) for $n = 6, 7$ and $8$. We expect the generic stabilizers to be groups respectively of dimensions $16, 14$ and $8$. Note that $2^3 V$, for $n = 6, 7, 8$, is a parabolic prehomogeneous vector space coming from the shortest arm of the Dynkin diagram of $E_n$.

$n = 6$. We have already obtained a normal form for the generic element $\omega$ in $\wedge^3 \mathbb{C}^6$: in a well-chosen basis we can write it as

$$\omega = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6.$$  

The subgroup of $GL_6$ fixing $\omega$ must preserve the pair of vector spaces $\langle e_1, e_2, e_3 \rangle$, $\langle e_4, e_5, e_6 \rangle$. We thus get for generic stabilizer an extension of $SL_3 \times SL_3$ by $Z_2$.

$n = 8$. We begin with a general construction. Let $g$ be a complex Lie algebra with Killing form $K$. We define the Cartan 3-form $\phi \in A^3 g^*$ by the formula

$$\phi(X,Y,Z) = K(X, [Y,Z]), \quad X,Y,Z \in g.$$  

This is a $g$-invariant 3-form. Moreover, if $g$ is semisimple the infinitesimal stabilizer of the Cartan 3-form is exactly $g$.

Suppose that $g = sl_3$, which has dimension eight. Up to a finite group the stabilizer of the Cartan 3-form $\phi$ is $SL_3$, in particular the dimension of its $GL_8$-orbit is $\dim GL_8 - \dim SL_3 = 64 - 8 = 56$. So $\phi$ must belong to the open orbit in $\wedge^3 (sl_3)^*$ and the generic stabilizer is a finite extension of $SL_3$.

These two cases are a bit disappointing, since the generic stabilizers are well-known groups. The case $n = 7$ turns out to be more interesting.

The exceptional group $G_2$

In the case $n = 7$, first note that a form $\phi \in A^3 V^*$ and a volume form $\Omega \in A^7 V^*$ determine a quadratic form as follows. For $v \in V$ denote by $\phi_v \in \wedge^2 V^*$ the contraction of $\phi$ by $v$. Then we define $q_\phi \in S^2 V^*$ by the identity

$$\phi_v \wedge \phi_w \wedge \phi = q_\phi(v,w)\Omega, \quad v,w \in V.$$  

For a generic $\phi$, the symmetric bilinear form $q_\phi$ will be non degenerate. In particular the stabilizer of $\phi$ is a subgroup of the orthogonal group $O(q_\phi) \cong O_7$.

A little more calculation (see [25], Theorem 6.80) shows that one obtains a simple Lie group of dimension $14$. In the Cartan-Killing classification there is only one such group, $G_2$.

The octonions

The group $G_2$ is commonly defined in terms of octonions. The Cayley algebra of octonions is an eight-dimensional non-commutative and non-associative algebra of the real
numbers, that we denote by $\mathcal{O}$. It admits a basis $e_0, e_1, \ldots, e_7$ in which the multiplication is particularly simple: $e_0 = 1$ is the unit, and for $i \neq j \geq 1$,
\[ e_i^2 = -1, \quad e_i e_j = \pm e_k \]
where $k$ is determined by the diagram above\(^1\).

This must be understood as follows: in the diagram any two vectors $e_i$ and $e_j$ are joined by a unique line (including the central circle), and there is a unique third vector on the line, which is $e_k$; the sign is $+1$ is the arrow goes from $e_i$ to $e_j$, and $-1$ otherwise; in particular $e_i e_j = -e_j e_i$.

Note that each of the four-dimensional subspaces generated by $(e_0, e_i, e_j, e_k)$ is a copy of the more familiar Hamilton algebra $\mathbb{H}$ of quaternions, which is non commutative but is associative.

One of the most remarkable properties of the octonion algebra is that it turns out to be a normed algebra, in the sense that it admits a norm such that the norm of a product is the product of the norms. The norm $|x|$ of $x = x_0 e_0 + \cdots + x_8 e_8$ is defined by $|x|^2 = x_0^2 + \cdots + x_8^2 = q(x)$. We also let $x \mapsto x$ be the orthogonal symmetry with respect to the unit vector. Then $x + \overline{x} = 2x_0 e_0$ and $x_0$ is called the real part of $x$. Octonions with zero real part are called imaginary, and we denote by $\text{Im}(\mathcal{O})$ the space of imaginary octonions, spanned by $e_1, \ldots, e_7$.

**Proposition 36.** The Cayley algebra has the following properties.

1. $|xy| = |x| \times |y| \quad \forall x, y \in \mathcal{O}$.

---

\(^1\)This diagram is usually called the Fano plane: it represents the projective plane over $\mathbb{F}_2$, with its seven points and seven lines.
2. $\mathcal{O}$ is alternative, in the sense that the subalgebra generated by any two elements is associative.

3. As a consequence, the associator $\omega(x,y,z) = (xy)z - x(yz)$ is skew-symmetric.

It is a classical theorem that there exists only four normed algebras over the field of real numbers, forming the series $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathcal{O}$.

Proposition 37. The automorphism group of $\mathcal{O}$ is a compact connected Lie group of type $G_2$.

Note that every automorphism of $\mathbb{H}$ is inner, so that $\text{Aut}(\mathbb{H}) \simeq SO_3$. Also $\text{Aut}(\mathbb{C}) \simeq \mathbb{Z}_2$ (conjugation is the only non trivial automorphism) and $\text{Aut}(\mathbb{R}) = 1$.

Any automorphism of $\mathcal{O}$ fixes the unit element and preserves the norm. Therefore it preserves $\text{Im}(\mathcal{O})$, and it is naturally a subgroup of $SO_7$. Moreover it preserves the three-form $\phi(x,y,z) = \text{Re}((xy)z - x(yz))$.

Note that $\phi$ expressed in the canonical basis of $\text{Im}(\mathcal{O})$ has exactly seven terms, one for each line in the Cayley plane.

One can check that the stabilizer of this form is exactly $\text{Aut}(\mathcal{O})$. In particular $\phi$ is a generic three form in seven variables.

Triality

Given any algebra $\mathcal{A}$, its automorphism group $\text{Aut}(\mathcal{A}) := \{g \in \text{GL}(\mathcal{A}) \mid g(xy) = g(x)g(y) \forall x,y \in \mathcal{A}\}$ is an affine algebraic group. The associated Lie algebra is the derivation algebra $\text{Der}(\mathcal{A}) := \{X \in \text{gl}(\mathcal{A}) \mid X(xy) = (Xx)y + x(Xy) \forall x,y \in \mathcal{A}\}$.

We may enlarge $\text{Aut}(\mathcal{A})$ to define the triality group of $\mathcal{A}$,

$$T(\mathcal{A}) := \{(g_1,g_2,g_3) \in \text{GL}(\mathcal{A})^{\times 3} \mid g_1(xy) = g_2(x)g_3(y) \forall x,y \in \mathcal{A}\}.$$

When $\mathcal{A}$ is equipped with a quadratic form, we require that $(g_1,g_2,g_3) \in \text{SO}(\mathcal{A})^{\times 3}$. One also has the corresponding triality algebra

$$t(\mathcal{A}) := \{(X_1,X_2,X_3) \in \text{so}(\mathcal{A})^{\times 3} \mid X_1(xy) = (X_2(x)y + x(X_3(y)) \forall x,y \in \mathcal{A}\}.$$

This definition, due to B. Allison [2], generalizes the most important case of $\mathcal{A} = \mathcal{O}$, which is due to Elie Cartan.

There are three natural actions of $T(\mathcal{A})$ (or $t(\mathcal{A})$) on $\mathcal{A}$ corresponding to the three projections to $\text{GL}(\mathcal{A})$, and we denote these representations by $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$. 
Proposition 38 (Cartan's triality principle). The triality group of the Cayley algebra is $T(\mathbb{O}) \simeq Spin_8$, each projection $T(\mathbb{O}) \to SO(\mathbb{O}_i)$, for $i = 1, 2, 3$, being a double covering.

Moreover the representations $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ are non equivalent: they are the three fundamental representations of $Spin_8$ other than the adjoint representation, and they are permuted by the outer automorphisms of $T(\mathbb{O})$. This is encoded in the triple symmetry of the Dynkin diagram of type $D_4$:

\[\begin{array}{ccc}
& O_2 & \\
O_1 & \longrightarrow & O_3 \\
& ad & \\
\end{array}\]

6.2. The magic square

Once we have constructed $G_2$ as the automorphism groups of the octonions, it is natural to ask whether we can construct the other exceptional groups in a similar fashion. In the 1950's, Freudenthal and Tits found a way to attach a Lie algebra to a pair of normed algebras. When one of the algebras is $\mathbb{O}$, one obtains the exceptional Lie algebras $f_4, e_6, e_7$ and $e_8$. The first construction that will be explained below is different, and more elementary than the original construction of Tits and Freudenthal, but it will yield in the end exactly the same Lie algebras (at least over the complex numbers; the same constructions over the real numbers can yield different real forms of the same complex Lie algebras).

The magic square from triality

Let $A$ and $B$ be two normed algebras. We will also allow the possibility that $B = 0$. Consider the vector space

$g = g(A, B) = t(A) \times t(B) \oplus (A_1 \otimes B_1) \oplus (A_2 \otimes B_2) \oplus (A_3 \otimes B_3)$.

We define a Lie algebra structure, that is, a Lie bracket on $g(A, B)$, by the following conditions:

- $t(A) \times t(B)$ is a Lie subalgebra, and the bracket with an element of $A_i \otimes B_i$ is given by the natural action of $t(A_i)$ on $A_i$ and $t(B_i)$ on $B_i$;

- the bracket of two elements in $A_i \otimes B_i$ is given by the natural map

$\wedge^2 (A_i \otimes B_i) = \wedge^2 A_i \otimes S^2 B_i \oplus S^2 A_i \otimes \wedge^2 B_i \to \wedge^2 A_i \oplus \wedge^2 B_i \to t(A_i) \times t(B_i)$,

where the first arrow follows from the quadratic forms given on $A_i$ and $B_i$, and the second arrow is dual to the map $t(A_i) \to \wedge^2 A_i \subset End(A_i)$ (and similarly
for \(B\) prescribing the action of \(t(\mathbb{A})\) on \(\mathbb{A}_i\) (which, by definition, preserves the quadratic form on \(\mathbb{A}_i\)). Here duality is taken with respect to a \(t(\mathbb{A})\)-invariant quadratic form on \(t(\mathbb{A})\), and the quadratic form on \(\wedge^2 \mathbb{A}_i\) induced by that on \(\mathbb{A}_i\);

• finally, the bracket of an element of \(\mathbb{A}_i \otimes \mathbb{B}_i\) with one of \(\mathbb{A}_j \otimes \mathbb{B}_j\), for \(i \neq j\), is given by the following rules, with obvious notations:

\[
\begin{align*}
[u_1 \otimes v_1, u_2 \otimes v_2] &= u_1 u_2 \otimes v_1 v_2 \in \mathbb{A}_3 \otimes \mathbb{B}_3, \\
[u_2 \otimes v_2, u_3 \otimes v_3] &= u_3 u_2 \otimes v_3 v_2 \in \mathbb{A}_1 \otimes \mathbb{B}_1, \\
[u_3 \otimes v_3, u_1 \otimes v_1] &= \bar{u}_1 u_3 \otimes \bar{v}_1 v_3 \in \mathbb{A}_2 \otimes \mathbb{B}_2.
\end{align*}
\]

**Theorem 27.** This defines a structure of semi-simple Lie algebra on \(g(\mathbb{A},\mathbb{B})\).

This is proved in [37], see also [5]. Note that the construction works over the real or the complex numbers as well. Let us work over the complex numbers to simplify the discussion. Letting \(\mathbb{A},\mathbb{B}\) equal \(\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\) we obtain the following square of Lie algebras, call the *Tits-Freudenthal magic square*:

\[
\begin{array}{cccc}
\mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{O} \\
\mathbb{R} & \mathfrak{so}_3 & \mathfrak{sl}_3 & \mathfrak{sp}_6 & \mathfrak{f}_4 \\
\mathbb{C} & \mathfrak{sl}_3 & \mathfrak{sl}_3 \times \mathfrak{sl}_3 & \mathfrak{sl}_6 & \mathfrak{e}_6 \\
\mathbb{H} & \mathfrak{sp}_6 & \mathfrak{sl}_6 & \mathfrak{so}_{12} & \mathfrak{e}_7 \\
\mathbb{O} & \mathfrak{f}_4 & \mathfrak{e}_6 & \mathfrak{e}_7 & \mathfrak{e}_8
\end{array}
\]

There exists a rather strange formula for the dimensions of \(g(\mathbb{A},\mathbb{B})\) in terms of \(a = \dim \mathbb{A}\) and \(b = \dim \mathbb{B}\), namely

\[
\dim g(\mathbb{A},\mathbb{B}) = 3 \frac{(ab + 4a + 4b - 4)(ab + 2a + 2b)}{(a+4)(b+4)}.
\]

Formulas of that type were first obtained by Vogel and Deligne using ideas from knot theory. See [37] for more on this.

**The magic square from Jordan algebras**

The original version of the magic square was not obviously symmetric. Given a normed algebra \(\mathbb{A}\), Freudenthal and Tits considered the space \(f_3(\mathbb{A})\) of \(3 \times 3\) Hermitian matrices with coefficients in \(\mathbb{A}\):

\[
f_3(\mathbb{A}) = \left\{ \begin{pmatrix} r & x & y \\ x & s & z \\ y & z & t \end{pmatrix} : r, s, t \in \mathbb{R}, x, y, z \in \mathbb{A} \right\}.
\]

There is a natural algebra structure on \(f_3(\mathbb{A})\): although the product of two Hermitian matrices in not in general Hermitian, the symmetrized product

\[
X.Y = \frac{1}{2}(XY + YX), \quad X, Y \in f_3(\mathbb{A}),
\]
will be. This product is of course commutative, and although it is not associative it has some nice properties.

**Proposition 39.** The algebra \( J_3(A) \) is a Jordan algebra: for any \( X \in J_3(A) \), the operators of multiplication by \( X \) and \( X^2 \) commute.

Any Jordan algebra \( J \) is power associative, which means that the subalgebra generated by any element \( X \in J \) is associative. In particular the powers \( X^m \) are well-defined. One defines the rank of \( J \) as the dimension of this subalgebra for \( X \) generic.

One can check that \( J_3(A) \) has rank three. Therefore there exists a polynomial identity

\[
X^3 - f_1(X)X^2 + f_2(X)X - f_3(X)I = 0, \quad \forall X \in J_3(A),
\]

where \( f_i \) is a polynomial of degree \( i \). When \( A = \mathbb{R} \) or \( \mathbb{C} \) this is of course the usual formula. In particular \( f_3(X) \) is the determinant, and we will use the notation \( \text{Det}(X) = f_3(X) \) for any \( X \in J_3(A) \) and any \( A \).

The Jordan algebra \( J_3(\mathbb{O}) \) is called the exceptional Jordan algebra. Its first connection with the exceptional Lie groups is the following result of Chevalley.

**Theorem 28.** The automorphism group of the exceptional Jordan algebra \( J_3(\mathbb{O}) \) is a Lie group of type \( F_4 \).

This implies that the derivation algebra \( \text{Der}(J_3(\mathbb{O})) \) is simple of type \( f_4 \). The Tits-Freudenthal construction expands this identity in order to obtain the other exceptional Lie algebras. For a pair \( A, B \) of normed algebras we let

\[
g^{TF}(A,B) = \text{Der}(A) \times \text{Der}(J_3(B)) \oplus \text{Im}(A) \otimes J_3(B)_0,
\]

where \( J_3(B)_0 \subset J_3(B) \) is the hyperplane of traceless matrices. The resulting square of Lie algebras is again the Tits-Freudenthal magic square. In fact we can enlarge it a little bit by allowing \( B \) to be zero, in which case we take for \( J_3(B) \) the space of diagonal matrices – or we could even take \( J_3(B) = 0 \), which we denote formally \( B = \emptyset \). With this notation we have

\[
g^{TF}(A,\emptyset) = \text{Der}(A), \quad g^{TF}(A,0) = \mathfrak{t}(A).
\]

We get (over the complex numbers) the following **magic rectangle**:

<table>
<thead>
<tr>
<th></th>
<th>( \Delta )</th>
<th>0</th>
<th>( \mathbb{R} )</th>
<th>( \mathbb{C} )</th>
<th>( \mathbb{H} )</th>
<th>( \mathbb{O} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{R} )</td>
<td>0</td>
<td>0</td>
<td>( \mathfrak{so}_3 )</td>
<td>( \mathfrak{sl}_3 )</td>
<td>( \mathfrak{sp}_6 )</td>
<td>( \mathfrak{f}_4 )</td>
</tr>
<tr>
<td>( \mathbb{C} )</td>
<td>0</td>
<td>( \mathbb{C}^3 )</td>
<td>( \mathfrak{sl}_3 )</td>
<td>( \mathfrak{sl}_3 \times \mathfrak{sl}_3 )</td>
<td>( \mathfrak{sl}_6 )</td>
<td>( \mathfrak{e}_6 )</td>
</tr>
<tr>
<td>( \mathbb{H} )</td>
<td>( \mathfrak{so}_5 )</td>
<td>( \mathfrak{so}_3 )</td>
<td>( \mathfrak{sp}_6 )</td>
<td>( \mathfrak{sl}_6 )</td>
<td>( \mathfrak{so}_{12} )</td>
<td>( \mathfrak{e}_7 )</td>
</tr>
<tr>
<td>( \mathbb{O} )</td>
<td>( \mathfrak{g}_2 )</td>
<td>( \mathfrak{so}_8 )</td>
<td>( \mathfrak{f}_4 )</td>
<td>( \mathfrak{e}_6 )</td>
<td>( \mathfrak{e}_7 )</td>
<td>( \mathfrak{e}_8 )</td>
</tr>
</tbody>
</table>
6.3. The geometry of the magic rectangle

In a long series of papers, Freudenthal showed that to each ordered pair of normed algebras \( \mathbb{A}, \mathbb{B} \), one can associate a special geometry, with a group of natural transformations whose Lie algebra is precisely \( g^{TF}(\mathbb{A}, \mathbb{B}) \). The type of this geometry depends only on \( \mathbb{A} \), for example for \( \mathbb{A} = \mathbb{C} \) one obtains a plane projective geometry. This geometry is defined in terms of certain objects (points and lines for a plane projective geometry) which are parametrized by homogeneous varieties, and may have certain incidence relations (typically, a line can contain a point, or not). These homogeneous varieties have especially interesting properties.

We will take a different perspective, more in the spirit of our study of parabolic prehomogeneous vector spaces. Indeed we have noticed the relations between parabolic prehomogeneous vector spaces and tangent spaces of homogeneous spaces. Geometrically, the most interesting tangent lines are those that remain contained in the variety. So we introduce the following definition [38].

**Definition 27.** Let \( X = G/P \subset PV \) be a projective homogeneous space, considered in its minimal equivariant embedding. The reduction of \( X \) is defined as the variety \( Y \subset PTX \) of tangent directions to lines through \( x \) contained in \( X \), where \( x \) is a given point of \( X \).

Of course, up to projective equivalence \( Y \) does not depend on \( x \), by homogeneity. By reduction we will construct series of interesting varieties. Our starting point will be the exceptional adjoint varieties.

**Adjoint varieties**

Let \( \mathfrak{g} \) be a simple Lie algebra.

**Definition 28.** The adjoint variety of \( \mathfrak{g} \) is the projectivization of the minimal nilpotent orbit,

\[
X_{ad}(\mathfrak{g}) = \mathbb{P}O_{min} \subset \mathbb{P}\mathfrak{g}.
\]

In other words, \( X_{ad}(\mathfrak{g}) \) is the only closed orbit inside \( \mathbb{P}\mathfrak{g} \), for the action of the adjoint group. The adjoint varieties of the classical Lie algebras are the following.

- \( SL_n \) \( \mathbb{F}_{1,n-1} = \mathbb{P}(T^{\mathbb{P}n-1}) \),
- \( SO_n \) \( OG(2,n) \),
- \( Sp_{2n} \) \( v_2(\mathbb{P}^{2n-1}) \).

Being the projectivization of conic symplectic varieties, the adjoint varieties have an induced structure, called a contact structure.

**Definition 29.** A contact structure on a variety \( X \) is a hyperplane distribution with the following property. If the distribution is given by a hyperplane bundle \( H \) of
the tangent bundle, and if $L$ denotes the quotient line bundle, then the skew-symmetric bilinear map
\[ \omega : \wedge^2 H \to L \]
induced by the Lie bracket is non degenerate.

Each nilpotent orbit $O \subset \mathfrak{g}$ is a symplectic cone, so its projectivization $\mathbb{P}O \subset \mathbb{P}\mathfrak{g}$ is a contact variety (see [6] for more details). The adjoint variety is the only one which is closed.

Once we have fixed a Cartan subalgebra and a set of positive roots, we get a point of $X_{ad}(\mathfrak{g}) = \mathbb{P}O_{\text{min}}$ by taking the root space of the highest root $\psi$.

**Lemma 6.** For any root $\alpha$ one has $\alpha(H_\psi) \leq 2$, with equality if and only if $\alpha = \psi$.

Another way to understand this statement is the following: if we consider the five-step grading
\[ \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \]
defined by $H_\psi$, then $\mathfrak{g}_2 = \mathfrak{g}_0$ is only one-dimensional. The parabolic algebra stabilizing the line $\mathfrak{g}_0$ is $\mathfrak{p} = \mathfrak{g}_{\geq 0}$. The tangent bundle of $X_{ad}(\mathfrak{g}) = G/P$ is the homogeneous bundle associated to the $P$-module $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. Note that $\mathfrak{g}_{-1}$ is a $P$-submodule (but not $\mathfrak{g}_{-2}$). The exact sequence
\[ 0 \to \mathfrak{g}_{-1} \to \mathfrak{g}/\mathfrak{p} \to \mathfrak{g}_{-2} \to 0 \]
of $P$-modules induces an exact sequence of vector bundles on $X_{ad}(\mathfrak{g})$
\[ 0 \to H \to TX_{ad}(\mathfrak{g}) \to L \to 0, \]
in particular a hyperplane distribution in $X$. Moreover the skew-symmetric bilinear map $\omega : \wedge^2 \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ can be identified at our special point with the restriction of the Lie bracket $\wedge^2 \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$. In particular it must be non-degenerate, since $\mathfrak{g}_{-1}$ is irreducible.

Projective contact manifolds are expected to be rather uncommon. The projectivization $\mathbb{P}(\Omega_Y)$ of the cotangent bundle of any smooth projective variety has a contact structure inherited from the natural symplectic structure on $\Omega_Y$. But if we impose some restrictions, for example that the Picard number is one, only few examples are known.

An important conjecture is the following.

**Conjecture (Le Brun–Salamon [39]).** Any Fano contact variety is the adjoint variety of a simple Lie algebra.

So there would be three series of classical Fano contact varieties: the projective spaces of odd dimensions, the projectivized cotangent bundles the projective spaces, and the Grassmannians of lines contained in a smooth quadric. Plus five exceptional examples coming from the exceptional Lie algebras – their dimensions being $5, 15, 21, 33, 57$. 
Legendrian varieties

Now we apply the reduction procedure we have introduced above. We choose a point in the adjoint variety, and then we look at lines passing through that point and contained in \(X_{ad}(\mathfrak{g}) = \mathbb{P}O_{\text{min}}\). It is a general fact that the tangent directions to the lines must be contained inside the contact distribution, so we get a variety \(Y \subset \mathbb{P}_{\mathfrak{g}_{-1}}\). Recall that \(\mathfrak{g}_{-1}\) is equipped with a \((\mathfrak{g}_{-2}\)-valued) symplectic form induced by the Lie bracket (in particular it must be even dimensional). A subspace of \(\mathfrak{g}_{-1}\) is called Lagrangian if it is isotropic and of maximal dimension for this property (that is, half the dimension of \(\mathfrak{g}_{-1}\)).

**Proposition 40.** The variety \(Y \subset \mathbb{P}_{\mathfrak{g}_{-1}}\) is Legendrian: each of its affine tangent spaces is a Lagrangian subspace of \(\mathfrak{g}_{-1}\).

Of course we know that \(\mathbb{P}_{\mathfrak{g}_{-1}}\) has only finitely many \(G_0\)-orbits. Moreover \(Y\) is the only closed one. The orbit structure is in fact quite simple. Apart from \(Y\) and \(\mathbb{P}_{\mathfrak{g}_{-1}}\) itself there are only two orbit closures, namely the dual hypersurface \(Y^*\) of \(Y\), and its singular locus \(\sigma_+(Y)\), with the simplest possible closure ordering:

\[Y \subset \sigma_+(Y) \subset Y^* \subset \mathbb{P}_{\mathfrak{g}_{-1}}.\]

An equation of the dual hypersurface of \(Y\), which has degree four, can be defined in terms of the five-step grading of \(\mathfrak{g}\): we choose a generator of \(\mathfrak{g}_{2}\), namely \(X_{q}\), and we let for \(X \in \mathfrak{g}_{-1}\),

\[P(X) = K(X_{q}, ad(X)^4X_{q}).\]

This is obviously a \(G_0\)-invariant polynomial, hence the equation of an invariant hypersurface, which must be \(Y^*\).

The open orbit in \(\mathbb{P}_{\mathfrak{g}_{-1}}\) is the complement of this quartic hypersurface, and the associated birational map is a cubo-cubic Cremona transformation of \(\mathbb{P}_{\mathfrak{g}_{1}}\).

Starting from the exceptional Lie algebras, we get five exceptional Legendrian varieties that we can relate to the Jordan algebras \(J_3(A)\). Observe that \(\mathfrak{g}_{-1}\) has dimension \(6a + 8\), which is twice the dimension of \(J_3(A)\) plus two. In fact one can make an identification

\[\mathfrak{g}_{-1} \simeq \mathbb{C} \oplus J_3(A) \oplus J_3(A) \oplus \mathbb{C}\]

in such a way that the following statement does hold:

**Proposition 41.** The Legendrian varieties associated to the exceptional Lie algebras \(\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\) are the twisted cubic “curves” over the Jordan algebras \(J_3(A)\), for \(A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\).

By twisted cubic “curves” over the Jordan algebras \(J_3(A)\) we mean the following. Recall that we have defined a “determinant” on \(J_3(A)\), whose existence we related to the Cayley-Sylvester identity \(X^3 - f_1(X)X^2 + f_2(X)X - f_3(X)I = 0\), for \(X \in J_3(A)\). We have \(f_1(X) = \text{Trace}(X)\) and \(f_3(X) = \text{Det}(X)\) is our determinant. As usual we can
rewrite the previous identity as $X \text{Com}(X) = \text{Com}(X)X = f_3(X)I$ where the “comatrix” $\text{Com}(X) = X^2 - f_1(X)X + f_2(X)I$ is to be thought of as giving the “inverse” matrix when $X$ is “invertible”. Then the map

$$X \in \mathfrak{h}_3(A) \mapsto [1, X, \text{Com}(X), \text{Det}(X)] \in \mathbb{P}^{g-1}$$

parametrizes a dense open subset of the Legendrian variety $Y = Y(\mathfrak{h}_3)$. This description extends to $\mathfrak{so}_8$ if we let $\mathfrak{h}_3 = 0$, and also to $\mathfrak{g}_2$ for $\mathfrak{h}_3 = \mathfrak{g}_2$; in the latter case the associated Legendrian variety is the usual twisted cubic in $\mathbb{P}^3$.

A more explicit description of the varieties $Y(\mathfrak{h}_3)$ is as follows:

- $Y(\Delta) = v_3(\mathbb{P}^1)$,
- $Y(0) = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,
- $Y(\mathbb{R}) = LG(3, 6)$,
- $Y(\mathbb{C}) = G(3, 6)$,
- $Y(\mathbb{H}) = OG(6, 12)$,
- $Y(\mathbb{O}) = E_7/\mathbb{P}_7$.

Using (for example) the description above one can show that:

**PROPOSITION 42.** The Legendrian variety $Y$ has the following properties:

1. It is non degenerate, in the sense that its secant variety is the whole projective space $\mathbb{P}^{g-1}$.
2. The tangent variety of $Y$ coincides with its dual hypersurface.
3. Any point outside the tangent variety belongs to a unique secant line to $Y$; in particular $Y$ is a variety with one apparent double point.
4. Any point on the smooth locus of the tangent variety belongs to a unique tangent to $Y$.

One can wonder if these properties can be used to approach the LeBrun–Salamon conjecture. Starting from a Fano contact manifold $X$ with its contact distribution $H$, one can look at rational curves of minimal degree through a general point $x$ of $X$. Their tangent directions define a subvariety $Y_x \subset \mathbb{P}H_x \subset \mathbb{P}T_x X$. Kebekus proved that $Y_x$ is a smooth Legendrian variety. It would be nice to prove that $Y_x$ is one of the homogeneous Legendrian varieties, and then to deduce that $X$ itself must be homogeneous. A step in this direction is the following statement.

**PROPOSITION 43 (Buczyński [8]).** Let $Y \subset \mathbb{P}V$ be a Legendrian variety whose ideal is generated by quadrics. Then $Y$ must be homogeneous.

**Severi varieties**

Starting from the homogeneous Legendrian varieties we can apply once more our reduction procedure. In the exceptional cases, let us consider our description of these varieties as twisted cubics over the Jordan algebras $\mathfrak{h}_3(\mathfrak{h}_3)$. It is then clear that the tangent space at the point $[1, 0, 0, 0]$ is precisely $\mathfrak{h}_3(\mathfrak{h}_3)$. So by reduction we get a subvariety $Z(\mathfrak{h}_3) \subset \mathbb{P}\mathfrak{h}_3(\mathfrak{h}_3)$, with an action of a Lie group $H(\mathfrak{h}_3)$. 
**Proposition 44.** The Lie group $H(\mathbb{A})$ is the subgroup of $GL(J_3(\mathbb{A}))$ preserving the determinant. Its orbit closures in $\mathbb{P}J_3(\mathbb{A})$ are

$$Z(\mathbb{A}) \subset \text{Sec}(Z(\mathbb{A})) \subset \mathbb{P}J_3(\mathbb{A}),$$

to be considered as the sets of matrices of "rank" at most one, two and three respectively.

In particular $Z(\mathbb{A})$ is smooth and is acted on transitively by $H(\mathbb{A})$. Its equations are the "$2 \times 2$ minors", the derivatives of the determinant. In particular one can show that a dense open subset of $Z(\mathbb{A})$ can be parametrized as the set of matrices of the form

$$\begin{pmatrix}
1 & u & v \\
|u|^2 & \pi & |v|^2 \\
|v|^2 & \pi u & |v|^2
\end{pmatrix}, \quad u, v \in \mathbb{A}.$$ 

This implies that $Z(\mathbb{A})$ has dimension $n = 2a$, but $\text{Sec}(Z(\mathbb{A}))$ has dimension $3a + 1 = \frac{3}{2}n + 1$, while we would expect that $\text{Sec}(Z(\mathbb{A})) = \mathbb{P}J_3(\mathbb{A})$. It turns out that Severi cases are exactly at the boundary of Zak’s famous theorem on linear normality.

**Theorem 29 (Zak [56]).** Let $Z \subset \mathbb{P}^N$ be a smooth, linearly non degenerate variety, with $N < \frac{3}{2}n + 2$. Then $\text{Sec}(Z) = \mathbb{P}^N$.

Moreover, if $N < \frac{3}{2}n + 2$ and $\text{Sec}(Z) \neq \mathbb{P}^N$, then $Z$ is one of the $Z(\mathbb{A})$.

Explicitly, we get the four Severi varieties

$$Z(\mathbb{R}) = v_2(\mathbb{P}^2), \quad Z(\mathbb{C}) = \mathbb{P}^2 \times \mathbb{P}^2, \quad Z(\mathbb{H}) = G(2, 6), \quad Z(\mathbb{O}) = E_6/P_1.$$ 

Of course the secant variety $\text{Sec}(Z(\mathbb{A}))$ is the cubic hypersurface defined by the determinant. The derivatives of this polynomial define a quadro-quadric Cremona transformation of $\mathbb{P}J_3(\mathbb{A}) = \mathbb{P}^{3a+2}$, whose base locus is exactly $Z(\mathbb{A})$. In particular this base locus is smooth and connected. This is again a very exceptional property.

**Theorem 30 (Ein & Shepherd-Barron [16]).** Consider a quadro-quadric Cremona transformation, and suppose that the base locus is smooth and connected. Then it must be defined by the quadrics containing one of the Severi varieties.

**Conclusion**

We have sketched a kind of geometric version of the Tits-Freudenthal magic square by associating to any pair of normed algebras a variety $X(\mathbb{A}, \mathbb{B})$ whose automorphism group has Lie algebra $\mathfrak{g}(\mathbb{A}, \mathbb{B})$. 
The projective varieties in this table have extremely special and particularly interesting properties. Moreover:

1. Their geometric properties are essentially the same in a given line.
2. One can pass from one line to another by the reduction procedure, which is the geometric version of the algebraic reduction that passes from a Z-graded Lie algebra to the degree one component of this grading.

In fact one can associate to each box of the magic square not only one but several projective varieties with astonishing relations between them. These were explored in detail in the works of H. Freudenthal (see [18] and references therein).

References


AMS Subject Classification: 11S90, 14L30, 14M17, 17B08, 17B25

Laurent MANIVEL,
Institut Fourier, Université Joseph Fourier
100 rue des Maths, 38402 Saint Martin d’Hères, FRANCE
e-mail: Laurent.Manivel@ujf-grenoble.fr

Lavoro pervenuto in redazione il 18.06.2013.
Abstract. We introduce invariant rings for forms (homogeneous polynomials) and for \( d \) points on the projective space, from the point of view of representation theory. We discuss several examples, addressing some computational issues. We introduce the graphical algebra for the invariants of \( d \) points on the line. This is an expanded version of the notes for the School on Invariant Theory and Projective Geometry, Trento, September 17-22, 2012.

Le teorie vanno e vengono ma le formule restano.\textsuperscript{5} G.C. Rota

Contents

1. Introduction and first examples .................................................. 120
   1.1. What is invariant theory, and the more modest aim of these lectures .... 120
   1.2. The Veronese variety and its equations .................................. 121
   1.3. The split variety and its equations ...................................... 125
2. Facts from representation theory .............................................. 126
   2.1. Basics about representations ............................................ 126
   2.2. Young diagrams and symmetrizers ..................................... 127
   2.3. Representations of finite groups and of \( \Sigma_d \) ......................... 128
   2.4. Representations of \( GL(n+1) \) and \( SL(n+1) \), Schur functors .......... 130
   2.5. The Lie algebra \( sl(n+1) \) and the weight structure of its representations .... 133
   2.6. Schur-Weyl duality ...................................................... 138
3. Invariants of forms and representation theory ............................ 140
   3.1. Invariance for the torus ............................................... 140
   3.2. Counting monomials of given weight ................................... 141
   3.3. Lie algebra action on forms .......................................... 142
   3.4. Cayley-Sylvester formula for the number of invariants of binary forms ... 144
   3.5. Counting partitions and symmetric functions .......................... 146
   3.6. Generating formula for the number of invariants of ternary forms ....... 148
   3.7. The Reynolds operator and how to compute it. Hilbert finiteness theorem .. 150
   3.8. Tableau functions. Comparison among different applications of Young diagrams 152
   3.9. The symbolic representation of invariants .............................. 154
   3.10. The two Fundamental Theorems for invariants of forms ................ 156
4. Hilbert series of invariant rings. Some more examples of invariants .... 157
   4.1. Hilbert series ......................................................... 157
   4.2. Covariant ring of binary cubics ...................................... 159
   4.3. Apolarity and transvectants .......................................... 161
   4.4. Invariant ring of binary quartics .................................... 162

\textsuperscript{5}The theories may come and go but the formulas remain
4.5. \textit{SL}(2) as symplectic group. Symplectic construction of invariants for binary quartics .......................................................... 165
4.6. The cubic invariant for plane quartics .................................................. 166
4.7. The Aronhold invariant as a pfaffian ................................................. 167
4.8. Clebsch and Lüroth quartics. Theta characteristics ................................. 169

5. Invariants of points. Cremona equations for the cubic surface and
invariants of six points ......................................................................... 172
5.1. The two Fundamental Theorems for invariants of points ......................... 172
5.2. The graphical algebra for the invariants of d points on the line. Kempe’s Lemma 175
5.3. Molien formula and elementary examples ............................................. 178
5.4. Digression about the symmetric group \( \Sigma_d \) and its representations .......... 180
5.5. The invariant ring of six points on the line ............................................ 182
5.6. The invariant ring of six points on the plane. Cremona hexahedral equation for the cubic surface ................................................ 187

References ................................................................................................ 191

1. Introduction and first examples

1.1. What is invariant theory, and the more modest aim of these lectures

Invariant theory is a classical and superb chapter of mathematics. It can be pursued from many points of view, and there are several excellent introductions to the subject (\[3,13,15,28,30,31,39,41,46,47,53\] and many others).

Given a group \( G \) acting on a variety \( X \), we want to describe the invariant subring \( A(X)^G \) inside the coordinate ring \( A(X) \). This framework is very general, in the spirit of Klein’s Erlangen Program.

Most of the classical work on the topic was done on invariants of forms and invariants of sets of points. In the case of invariant of forms, \( V \) is a complex vector space, \( G = \text{SL}(V) \) and \( X \) is the natural embedding of \( PV \) in \( PS^d V \), which is called the \( d \)-Veronese variety (\( S^d V \) is the \( d \)-th symmetric power of \( V \)). In the case of invariants of points, there are two interesting situations. When the points are \textit{ordered}, we have \( G = \text{SL}(V) \) acting on \( d \) copies of \( PV \), that is on the Segre variety \( PV \times \ldots \times PV \). When the points are \textit{unordered}, we quotient the Segre variety by the symmetric group \( \Sigma_d \). Note that when \( \dim V = 2, d \) unordered points are described in equivalent way by a homogeneous polynomial of degree \( d \) in two variables, and we reduce again to the case of forms, here \( X \) is the rational normal curve. Note that, in the dual description, points correspond to hyperplanes and we get the well known “arrangement of hyperplanes”.

In XIX century invariants were constructed by means of the two fundamental theorems, that we review in §3.10 for invariants of forms and in §5.1 for invariants of points. The First Fundamental Theorem (1FT for short) claims that all the invariants can be constructed by a clever combinatorial procedure called “symbolic representation” (see §3.9). Our approach is close to [31,47], where the 1FT is obtained as a consequence of Schur-Weyl duality. The classical literature is rich of interesting examples and computations. As main textbooks from the classical period we recommend [21]. Also several parts from [17,50] are developed in the setting of invariant theory. Hilbert
lectures [24] deserve a special mention. They are the translation of handwritten notes of a course held by Hilbert in 1897 at Göttingen. The reading of such master work is particularly congenial to understand the modern development of invariant theory. The “symbolic representation” of XIX century was a hidden way to introduce Schur functors and representation theory, not yet having a formal setting for them.

In the case of forms, the invariants of degree $m$ for $d$-forms are the $SL(V)$-invariant subspace of $S^m(S^dV)$. The decomposition of this space as a sum of irreducible representations is a difficult problem called plethysm. Although there are algorithms computing this decomposition, for any $m$, $d$, a simple description is missing.

The case $\dim V = 2$ is quite special. The elements of $S^dV$ are called binary forms of degree $d$ (ternary forms correspond to $\dim V = 3$ and so on) and most of classical results regarded this case. The common zero locus of all the invariant functions is called the “nullcone”, and it coincides with the nilpotent cone of Manivel’s lectures [37] when $X$ is the Lie algebra of $G$ and the action is the adjoint one.

In these lectures we try to give some tools to apply and use projective invariants, possibly in related fields, involving algebraic geometry or commutative algebra. This aim should guide the volunteered reader into useful and beautiful mathematics.

There are several approaches to invariant theory. Some examples look “natural” and “easy” just from one point of view, while they look more sophisticated from other points of view. So it is important to have a plurality of descriptions for the invariants, and to look for several examples. During lectures, understanding is more important than efficiency, so we may prove the same result more than once, from different points of view.

Computers opened a new era in invariant theory. Anyway, there are basic cases where the needed computations are out of reach, even with the help of a computer, and even more if the computer is used in a naive way. We will try to sketch some computational tricks that we found useful in mathematical practice. Our basic computational sources are [13, 53].

I am indebted to M. Bolognesi, C. Ciliberto, A. Conca, I. Dolgachev, D. Faenzi, J.M. Landsberg, L. Manivel, L. Oeding, C. Ritzenthaler, E. Sernesi, B. Sturmfels and J. Weyman for several discussions. The §4.5 is due to an unpublished idea of my diploma advisor, F. Gherardelli. These notes are an expanded version of the notes prepared for the School on Invariant Theory and Projective Geometry held in Trento, in September 2012, organized by CIRM. I wish to thank the CIRM and V. Baldoni, G. Casnati, C. Fontanari, F. Galluzzi, R. Notari, F. Vaccarino for the wonderful organization. I wish to thank all the participants for their interest, it is amazing that a question left open during the course, about Lüroth quartics, has been immediately attacked and essentially solved [2].

1.2. The Veronese variety and its equations

Let $V$ be a (complex) vector space of dimension $n + 1$. We denote by $S^dV$ the $d$-th symmetric power of $V$. The $d$-Veronese variety embedded in $\mathbb{P}S^dV$ is the image of the
map

\[ \mathbb{P}^V \rightarrow \mathbb{P} S^d V \]

\[ v \mapsto v^d \]

We denote it by \( v_d(P^V) \), it consists of all homogeneous polynomials of degree \( d \) in \( n + 1 \) variables which are the \( d \)-th power of a linear form. Historically, this construction gave the main motivation to study algebraic geometry in higher dimensional spaces. Its importance is due to the fact that hypersurfaces of degree \( d \) in \( P^V \) are cut out by hyperplanes in the Veronese embedding.

These elementary remarks are summarized in the following.

**THEOREM 1.**

- (i) A linear function on \( S^d V \) is uniquely determined by its restriction to the Veronese variety.
- (ii) Linear functions over \( S^d V \) correspond to homogeneous polynomials of degree \( d \) over \( V \).

**Proof.** Let \( H \) be a linear function which vanishes on the \( d \)-th Veronese variety. It induces a homogeneous polynomial of degree \( d \) which vanishes for all the values of the variables. Hence the polynomial is zero, proving (i). (ii) is proved in the same way, since both spaces have the same dimension. \( \square \)

In equivalent way, Theorem 1 says that \( \mathbb{P} S^d V \) is spanned by elements lying on the Veronese variety. To make effective the previous Theorem, compare a general polynomial

\[ f = \sum \frac{d!}{i_0! \ldots i_n!} a_{i_0 \ldots i_n} x_0^{i_0} \ldots x_n^{i_n} \]

with the \( d \)-th power

\[ (b_0 x_0 + \ldots + b_n x_n)^d = \sum \frac{d!}{i_0! \ldots i_n!} b_{i_0}^{i_0} \ldots b_{i_n}^{i_n} x_0^{i_0} \ldots x_n^{i_n} \]

getting the correspondence

\[ b_{i_0}^{i_0} \ldots b_{i_n}^{i_n} \mapsto a_{i_0 \ldots i_n} \]

(1)

It will be the basic tool for the symbolic representation of an invariant that we will see in §3.9.

The conormal space (which is the annihilator of the tangent space) at a point \( x \in v_d(P^V) \) can be identified with the space of hyperplanes in \( \mathbb{P} S^d V \) containing the tangent space at \( x \). These hyperplanes correspond to the hypersurfaces of degree \( d \) which are singular at \( x \), which give the vector space \( H^0(I_x(x^d)) \). This can be summarized with

**PROPOSITION 1** (Lasker Lemma). The conormal space of \( v_d(P^V) \) at \([x]\) is isomorphic to \( H^0(I_x(x^d)) \).
Theorem 2. The Veronese variety is defined as scheme by the quadrics which are the $2 \times 2$-minors of the contraction

$$V^\vee \overset{C_f}{\rightarrow} S^{d-1}V$$

for $f \in S^d V$.

Proof. We have to prove that $f$ is a power of a linear form if and only if $\text{rk } C_f = 1$. The elements in $V^\vee$ can be seen as differential operators of first order. If $f = l^d$ then $\frac{\partial f}{\partial x_i} = d \frac{\partial l^d}{\partial x_i} = 0$ so that $\text{Im } C_f$ is spanned by $l^{d-1}$. It follows $\text{rk } C_f = 1$. Conversely, assume that $\text{rk } C_f = 1$ and let $l \in V$ be a generator of the one dimensional annihilator of $\ker C_f$. We may assume $l = x_0$. Then $\frac{\partial f}{\partial x_i} = 0$ for $i > 0$ implies that $f$ is a multiple of $x_0^d$. This proves the result set-theoretically. The proof can be concluded by a infinitesimal computation. If $f = x^d$, then it is easy to check that $\ker C_x = H^0(I_x(1))$, and $(\text{Im } C_f)^\perp = H^0(I_x(d - 1))$.

The conormal space of the determinantal locus at $[x]$ is given by the image of the natural map

$$H^0(I_x(1)) \otimes H^0(I_x(d - 1)) \rightarrow H^0(I_x(d))$$

It is easy to check that this map is surjective, and from Proposition 1 the result follows.

The quadratic equations which define the Veronese variety can be considered as a $\text{SL}(V)$-module inside $S^2(S^d V)$.

We have the decomposition

$$S^2(S^d V) = \bigoplus_{i=0}^{\lfloor \frac{d}{2} \rfloor} S^{2d-2i,2i}$$

so that the quadratic part of the ideal $I$ of the Veronese variety is

$$I_2 = \bigoplus_{i=1}^{\lfloor \frac{d}{2} \rfloor} S^{2d-2i,2i},$$

$I_2$ corresponds indeed to the $2 \times 2$ minors of $C_f$ in (2). A stronger and nontrivial result is true.

Theorem 3. The ideal $I$ of the Veronese variety is generated by its quadratic part $I_2$.

This Theorem is a special case of result, due to Kostant, holding for any rational homogeneous variety $G/P$. For a proof in the setting of representation theory see [32] Theorem 16.2.2.6, or [47] chap. 10 §6.6. For a somewhat different approach, generalized to flag varieties, see [55] Prop. 3.1.8.

The composition of two symmetric powers like $S^k(S^d V)$ is quite hard to be computed. The formula solving this problem in the case $\dim V = 2$ is due to Cayley and Sylvester and it is one of the most beautiful achievements of XIX century invariant theory. We will review it in §3.4.
For \( d = 3 \) we have
\[
S^2(S^3V)) = S^4V \oplus S^{4,2}V
\]
and the equations consist of the irreducible module \( S^{4,2}V \).
This is given by the \( 2 \times 2 \)-minors of the matrix
\[
\begin{vmatrix}
x_0 & x_1 & x_2 \\
x_1 & x_2 & x_3 \\
\end{vmatrix}
\]
In classical notation, there are “dual” variables \( y_i \) and the single Hessian covariant
\[
(3) \quad H = \begin{vmatrix}
x_0 & x_1 & x_2 \\
x_1 & x_2 & x_3 \\
y_0 & y_0y_1 & y_1^2
\end{vmatrix} = 0
\]
For \( d = 4 \) we have
\[
S^2(S^4V)) = S^8V \oplus S^{6,2}V \oplus S^{4,4}V
\]
and the equations consist of the sum of the two last modules \( S^{6,2}V \oplus S^{4,4}V \). The module structure is not evident at a first glance from the minors and it is commonly not considered when these equations are encountered. There is a single invariant quadric, which is called the equianharmonic quadric and it is classically denoted by \( I \). It corresponds to 4-ples which are apolar to themselves (we will describe apolarity in §4.3).
We have, given a binary quartic \( f \)
\[
f = \sum_{i=0}^{4} \binom{4}{i} f_i x^{4-i} y^i
\]
the expression
\[
(4) \quad I = f_0 f_4 - 4 f_1 f_3 + 3 f_2^2
\]
This expression gives the most convenient way to check if a given \( f \) is anharmonic. Note that from this expression the invariance is not at all evident.
We will prove in Theorem 25 that the ring of invariants for a binary quartic is generated by \( I \) and another invariant \( J \), which has a simpler geometric construction. It is the equation of the 2nd secant variety \( \sigma_2(v_4(\mathbb{P}^1)) \), that is, it is the equation of the locus spanned by the secant lines at the rational normal quartic curve \( v_4(\mathbb{P}^1) \).
We get
Projective Invariants

(5) \[ J = \det \begin{bmatrix} f_0 & f_1 & f_2 \\ f_1 & f_2 & f_3 \\ f_2 & f_3 & f_4 \end{bmatrix} \]

Indeed the above matrix has rank 1 on \( v_4(\mathbb{P}^1) \), hence it has rank \( \leq 2 \) on any secant line.

The weight of a monomial \( f_0^{n_0} \cdots f_4^{n_4} \) is by definition \( \sum_{i=0}^4 i n_i \). A (homogeneous) polynomial is called isobaric if all its monomials have the same weight. The monomials \( f_0 f_4, f_1 f_3, f_2^2 \) are all the monomials of degree 4 and weight 2, and we will check (see the Proposition 7) that they are the only ones that can appear in (4).

But why the coefficients in the expression (4) have to be proportional at \((1, -4, 3)\)? During these lectures, we will answer three times to this question, respectively in §4.3, §3.3, in §3.7.

These answers follow different approaches that are useful ways to look at invariants.

Remark 1. The dual variety to the Veronese variety is the discriminant hypersurface of degree \((n + 1)(d - 1)\) in \( \mathbb{P}S^d \mathbb{P} V \), parametrizing all singular hypersurfaces of degree \( d \) in \( \mathbb{P}V \). More generally, the dual variety to the \( k \)-secant variety (see [32] 5.1) to the \( d \)-Veronese variety (which is denoted by \( \vdash_k(\mathbb{P}^n) \)) consists of all hypersurfaces of degree \( d \) in \( \mathbb{P}V \) with at least \( k \) double points. This follows by Terracini Lemma ([32] 5.3).

For example the dual to \( \vdash_k(\mathbb{P}^n) \) (symmetric matrices of rank \( \leq k \)) is given by \( \sigma_{n+1-k}(\vdash_k(\mathbb{P}^n)) \) (symmetric matrices of rank \( \leq n+1-k \)).

The dual to \( \vdash_2(\mathbb{P}^2) \) is given by plane cubic curves with two double points, that is by reducible cubics. The dual to \( \vdash_3(\mathbb{P}^2) \) is given by cubics with three double points, that is by triangles. This is called a split variety, and we will consider it in next subsection.

1.3. The split variety and its equations

The split variety in \( \mathbb{P}S^d \mathbb{P} V \) consists of all polynomials which split as a product of linear factors. This subsection is inspired by §8.6 in [32], where the split variety is called Chow variety. The first nontrivial example is the variety of “triangles” in \( \mathbb{P}S^3 \mathbb{C}^3 \), which is a 6-fold of degree 15.

We consider the natural map

\[ S^k S^d \mathbb{P} V \rightarrow S^d S^k \mathbb{P} V \]

constructed by dividing \( V^{dk} \) represented by a \( d \times k \) rectangle, first by rows (in the source) and then first by columns (in the target).

The above map takes \((x_1^d \cdots x_k^d)\) to \((x_1 \cdots x_k)^d\), so it is nonzero on the coordinate ring of the split variety.
Theorem 4 (Brion [4]). The kernel of the above map gives the degree \( k \) part of the ideal of the split variety of \( d \)-forms on \( \mathbb{P}V \).

Note that in case \( \text{dim}\, V = 2 \) we have that all \( d \)-forms split and indeed the previous map is an isomorphism (Hermite reciprocity).

Example 1. \( S^2 S^3 \mathbb{C}^3 = S^6 \mathbb{C}^3 \oplus S^{4,2} \mathbb{C}^3 \),
\[ S^3 S^2 \mathbb{C}^3 = S^6 \mathbb{C}^3 \oplus S^{4,2} \mathbb{C}^3 \oplus S^{2,2} \mathbb{C}^3. \]

Indeed conics which split in two lines have a single invariant in degree 3, which is the determinant of the symmetric matrix defining them, while cubics which split in triangles have no equations in degree two.

Even the case \( k = d \) is interesting, the natural map

\[ S^d S^d \mathbb{C}^3 \rightarrow S^d S^d \mathbb{C}^3 \]

obtained by reshuffling between rows and columns, turns out to be a isomorphism for \( d \leq 4 \), but is is degenerate for \( d = 5 \). When \( d = 5 \) get \( \binom{5}{2} = 126 \) and \( \binom{126+4}{5} = 286, 243, 776 \), so the question corresponds to the rank computation for a square matrix of this size and it is already a computational challenge. It has been performed by Müller and Neunhofer in [40]. It should be interesting to understand theoretically this phenomenon.

So there are equations of degree 5 for the split variety of “pentahedra” in \( \mathbb{P}^d \).

It is interesting the split variety of triangles in the plane, which has quartic equations. These equations correspond to the proportionality between a cubic form \( f \in S^3 \mathbb{C}^3 \) and its Hessian \( H(f) \).

Remark 2. The next interesting variety for invariant theory is certainly the Grassmannian. We give for granted its description and the fact that its ideal is generated by the Plücker quadrics. For a proof, like in the case of Veronese variety, we may refer again to [32] Theorem 16.2.2.6. Let just remind the shape of the Plücker quadrics. The coordinates in the Plücker embedding of the Grassmannian of \( k+1 \) linear subspaces of \( V \) are indexed by sequences \([i_0 \ldots i_k]\) where \( 0 \leq i_0 < i_1 < \ldots < i_k \leq n \). Fix a subset of \( k+2 \) elements \( i_0, \ldots i_{k+1} \) and a set of \( k \) elements \( j_0 \ldots j_{k-1} \). Then the Plücker relations are

\[
\sum_{r=0}^{k+1} (-1)^r [i_0 \ldots \hat{i_r} \ldots i_{k+1}] [i_r j_0 \ldots j_{k-1}] = 0
\]

which hold for any subsets of respectively \( k+2 \) and \( k \) elements.
2. Facts from representation theory

2.1. Basics about representations

In this section we recall basic facts about representation theory, that can be found for example in [19]. From a logical point of view, the facts in this section are the foundations of the following sections. Anyway, the reader may find useful reading the section 3 for a better understanding of the use of representations, and going back when needed.

We will need to study representations of two basic groups, namely the finite symmetric group \( \Sigma_d \) of permutations on \( d \) elements, and the group \( SL(n+1) \) of \( (n+1) \times (n+1) \) matrices having \( \det = 1 \). Both are reductive groups.

A representation of a group \( G \) is a group morphism \( \rho: G \rightarrow GL(W) \), where \( W \) is a complex vector space. We say that \( \rho \) is \( \rho(G) \)-equivariant, namely it satisfies \( g \cdot w = \rho(g)(w) \) for any \( g \in G, w \in W \). This notation underlines that \( G \) acts over \( W \). This action satisfies the following properties, which follow immediately from the definitions

\[
g \cdot (w_1 + w_2) = g \cdot w_1 + g \cdot w_2, \quad \forall g \in G, w_1, w_2 \in W,
\]

\[
g \cdot \lambda w = \lambda g \cdot w, \quad \forall g \in G, w \in W, \lambda \in \mathbb{C},
\]

\[
(g_1 g_2) \cdot w = g_1 \cdot (g_2 \cdot w).
\]

These properties resume the fact that a representation is a \emph{linear} action.

Given two \( G \)-modules \( V, W \), then \( V \oplus W \) and \( V \otimes W \) are \( G \)-modules in a natural way, namely

\[
g \cdot (v + w) := (g \cdot v) + (g \cdot w),
\]

\[
g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w).
\]

The \( m \)-th symmetric power \( S^m W \) is a \( G \)-module, satisfying \( g \cdot (v^m) := (g \cdot v)^m \).

A morphism between two \( G \)-modules \( V, W \) is a linear map \( f: V \rightarrow W \) which is \( G \)-equivariant, namely it satisfies \( f(g \cdot v) = g \cdot f(v) \) \( \forall g \in G, v \in V \).

Every \( G \)-module \( V \) corresponding to \( \rho: G \rightarrow GL(V) \) has a character \( \chi_V: G \rightarrow \mathbb{C} \) defined as \( \chi_V(g) = \text{trace } \rho(g) \).

It satisfies the property \( \chi_V(h^{-1} gh) = \chi_V(g) \) hence the characters are defined on conjugacy classes in \( G \).

Moreover \( \chi_{V \oplus W} = \chi_V + \chi_W : \chi_{V \otimes W} = \chi_V \chi_W \).

Characters are the main tool to work with representations and to identify them.

Every \( G \)-module has a \( G \)-invariant submodule

\[
V^G = \{ v \in V | g \cdot v = v \quad \forall g \in G \}.
\]

In other words \( G \) acts trivially over \( V^G \). The character of the trivial representation of dimension \( r \) is constant, equal to \( r \) on every conjugacy classes.

For any \( G \)-module \( V, G \) acts on the graded ring \( \mathbb{C}[V] = \oplus_{m=0}^{\infty} S^m V \).

Since the sum and the product of two invariant elements are again invariant, we have the invariant subring \( \mathbb{C}[V]^G = \oplus_{m=0}^{\infty} (S^m V)^G \).
2.2. Young diagrams and symmetrizers

A Young diagram denoted by \( \lambda = (\lambda_1, \ldots, \lambda_k) \), where \( \lambda_1 \geq \lambda_2 \geq \ldots \) consists of a collection of boxes ordered in consecutive rows, where the \( i \)-th row has exactly \( \lambda_i \) boxes.

The number of boxes in \( \lambda \) is denoted by \( |\lambda| \).

All Young diagrams with \( d \) boxes correspond to the partitions of \( d \), namely \( \lambda = (\lambda_1, \ldots, \lambda_d) \) corresponds to \( |\lambda| = \lambda_1 + \ldots + \lambda_k \).

The following are the Young diagram corresponding to \((2, 1, 1)\) and \((4, 4)\):

Any filling of \( \lambda \) with numbers is called a tableau.

Just to fix a convention, for a given Young diagram, number consecutively the boxes like

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
\end{array}
\]

Here we used all numbers from 1 to \( d \) to fill \( d \) boxes. More generally, a tableau can allow repetitions of numbers, as we will see in the sequel.

Let \( \Sigma_d \) be the symmetric group of permutations over \( d \) elements. Due to the filling, we can consider the elements of \( \Sigma_d \) as permuting the boxes. Let \( R_\lambda \subseteq \Sigma_d \) be the subgroup of permutations preserving each row.

Let \( C_\lambda \subseteq \Sigma_d \) be the subgroup of permutations preserving each column.

**Definition 1.** The element

\[
c_\lambda = \sum_{\tau \in R_\lambda} \sum_{\sigma \in C_\lambda} \epsilon(\sigma) \sigma \tau \in \Sigma_d
\]

is called the Young symmetrizer corresponding to \( \lambda \).

Note immediately that it depends on \( \lambda \) but also on the filling of \( \lambda \), see Remark 3.

2.3. Representations of finite groups and of \( \Sigma_d \)

Let \( G \) be a finite group.

**Proposition 2.** There are exactly \( n(G) \) irreducible representations of \( G \), where \( n(G) \) is the number of conjugacy classes of \( G \).
Proof. [19] Prop. 2.30. \(\square\)

If \(V_i, i = 1, \ldots, n(G)\) are the irreducible representations of \(G\) and \(g_j, j = 1, \ldots, n(G)\) are representatives in the conjugacy classes of \(G\) then the square matrix \(\chi_V(g_j)\) is called the character table of \(G\).

Example 2. The group \(G = \Sigma_2\) has two elements 1, \(-1\), each one is a conjugacy class. Besides the trivial representation \(V_2\), we have another representation \(V_{1,1}\) defined on its basis element \(w\) by \(1 \cdot w = w, \ (-1) \cdot w = -w\).

The character table is

\[
\begin{array}{c|cc}
V_2 & 1 & -1 \\
V_{1,1} & 1 & -1 \\
\end{array}
\]

Note that \(V_2^G = V_2, V_{1,1}^G = 0\).

We will see in section 5.4 the character table of \(\Sigma_6\).

The proof of the following proposition is straightforward

Proposition 3. There is a \(G\)-surjective morphism \(R : V \rightarrow V^G\) which is defined as \(R(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v\) which satisfies \(R(v) = v \forall v \in V^G\).

The main theorem on finite groups is the following

Theorem 5. Let \(G\) be a finite group.

- (i) Given any \(G\)-module \(V\), there are uniquely determined nonnegative \(a_i\) for \(i = 1, \ldots, n(G)\) such that \(V = \bigoplus V_i^{a_i}\).

- (ii) The \(a_i\) are the unique solution to the square linear system

\[\chi_V(g_j) = \sum_i a_i \chi_V(g_j), \quad j = 1, \ldots, n(G).\]

In particular, from the character \(\chi_V\), it can be computed the dimension of the invariant part \(V^G\).

Proof. (i) is [19], Prop. 1.8. (ii) follows from [19] Coroll. 2.14, Prop. 2.30. \(\square\)

In the case of the symmetric group \(G = \Sigma_d\) its irreducible representations are in bijective correspondence with the conjugacy classes of \(\Sigma_d\), which correspond to the cycle structures of permutations and so they are described by partitions \(\lambda\) of \(d\).

Recall that for any finite group \(G\), the group algebra \(\mathbb{C}G\) is defined in the following way. The underlying vector space of dimension \(|G|\) has a basis \(e_g\) corresponding to the elements \(g \in G\) and the algebra structure is defined by the rule \(e_g \cdot e_h = e_{gh}\).

\(\Sigma_d\) acts on the vector space \(\mathbb{C}[\Sigma_d]\) by \(\sigma \cdot e_a = e_{\sigma a}\), which extends by linearity to \(\sigma : \mathbb{C}[\Sigma_d] \rightarrow \mathbb{C}[\Sigma_d]\).
DEFINITION 2. The Young symmetrizer $c_\lambda$ defined in (6) acts by right multiplication on $C\Sigma_d$, its image is a $\Sigma_d$-module that we denote by $V_\lambda$.

We remark that the notations we have chosen in Example 2 are coherent with this definition.

Let $T$ be a tableau corresponding to a filling of the Young diagram $\lambda$ with $d$ boxes with the numbers $\{1,\ldots,d\}$. A tableau is called standard if the filling is strictly increasing on rows and columns. A tableau corresponds to a permutation $\sigma_T \in \Sigma_d \subset C\Sigma_d$. By definition $V_\lambda$ is spanned by $c_\lambda T$ for any tableau $T$.

REMARK 3. $\Sigma_d$ acts by conjugation over $C\Sigma_d$. Any conjugate $\sigma^{-1}c_\lambda \sigma$ acts by right multiplication on $C\Sigma_d$, its image is isomorphic to $V_\lambda$, although they may be embedded in different ways. These different copies can be obtained, in equivalent way, starting by a different tableau in Def. 1, see [47] chap. 9, remark 2.2.5.

THEOREM 6 ([47] chap. 9, §2.4 and §9.2).

• (i) $V_\lambda$ is an irreducible representation of $\Sigma_d$.
• (ii) All irreducible representations of $\Sigma_d$ are isomorphic to $V_\lambda$ for some Young diagram $\lambda$.
• (iii) If $\lambda$ and $\mu$ are different partitions, then $c_\lambda c_\mu = 0$ and $c_\lambda^2$ is a scalar multiple of $c_\lambda$.
• (iv) A basis of $V_\lambda$ is given by $c_\lambda T$ where $\sigma_T \in \Sigma_d \subset C\Sigma_d$ corresponds to standard tableau $T$. In particular $\dim V_\lambda$ is equal to the number of standard tableaux on $\lambda$.

THEOREM 7 ([19] Prop. 3.29). We have the isomorphism of algebras

\[ C\Sigma_d = \bigoplus_{\{\lambda,|\lambda|=d\}} \text{End}(V_\lambda). \]  

According to the isomorphism (7), any $c_\mu$ is a diagonalizable endomorphism of rank one (and nonzero trace) in $\text{End}(V_\mu)$ and it is zero in $\text{End}(V_\lambda)$ for $\lambda \neq \mu$.

Note the two extreme cases, when $\lambda = d$ we get the trivial one dimensional representation of $\Sigma_d$, while if $\lambda = (1,\ldots,1) = 1^d$ (for short) we get the one dimensional representation given by sign, that is the action on a generator $e$ is given by $\sigma \cdot e = \varepsilon(\sigma)e$.

2.4. Representations of $GL(n+1)$ and $SL(n+1)$, Schur functors

THEOREM 8. Let $f: GL(n+1) \to \mathbb{C}$ be a polynomial function invariant by conjugation, that is $f(G^{-1}AG) = f(A)$ for every $G,A \in GL(n+1)$.

Then $f$ is a polynomial symmetric function of the eigenvalues of $A$. 
Proof. Let $D(d_1, \ldots, d_{n+1})$ be the diagonal matrix having $d_i$ on the diagonal. If $\tau \in \Sigma_{n+1}$ and $M_{\tau}$ is the corresponding permutation matrix, then

$$M_{\tau}^{-1}D(d_1, \ldots, d_{n+1})M_{\tau} = D(d_{\tau(1)}, \ldots, d_{\tau(n+1)}).$$

It follows that $f(D(d_1, \ldots, d_{n+1}))$ is a polynomial symmetric function of $d_1, \ldots, d_{n+1}$. By the Main Theorem on symmetric polynomials (see [53] Theor. 1.1.1) there is a polynomial $g \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ such that

$$f(D(d_1, \ldots, d_{n+1})) = g(\sigma_1(d_1, \ldots, d_{n+1}), \ldots, \sigma_{n+1}(d_1, \ldots, d_{n+1})),
$$

where $\sigma_i$ is the $i$-th elementary symmetric polynomials. For any matrix $A$, denote $\det(A - I) = \sum_{i=0}^{n+1}(-1)^i c_{n+1-i}(A)$. Note that $\sigma_i(d_1, \ldots, d_{n+1}) = c_i(D(d_1, \ldots, d_{n+1}))$, and that $c_i(A)$ is the $i$-th elementary symmetric function of the eigenvalues of $A$. We have proved that $f(A) = g(c_1(A), \ldots, c_{n+1}(A))$ for every diagonal matrix $A$. Both sides are invariant by conjugation, then the equality is satisfied by any diagonalizable matrix. Since diagonalizable matrix form a dense subset, it follows that $f(A) = g(c_1(A), \ldots, c_{n+1}(A))$ for any matrix $A$, \hfill \Box

The same argument works for polynomial functions $f : SL(n+1) \to \mathbb{C}$ which are invariant by conjugation. See [54] for an extension to invariants of several matrices.

COROLLARY 1. Characters of $GL(n+1)$ (and of $SL(n+1)$) are symmetric functions of the eigenvalues.

Let $V = \mathbb{C}^{n+1}$, the group $\Sigma_{n+1}$ acts on $\otimes^d V$ by permuting the factors. We define the Schur projection $c_{\lambda} : \otimes^d V \to \otimes^d V$ from the Young symmetrizer $c_{\lambda}$ defined in (6).

We fill the Young diagram with numbers from 1 to $n+1$, allowing repetitions. After a basis of $V$ has been fixed, any such tableau $T$ gives a basis vector $v_T \in \otimes^d V$.

A tableau is called semistandard if it has nondecreasing rows and strictly increasing columns.

THEOREM 9.

- The image of $c_{\lambda}$ is a irreducible $GL(n+1)$-module, which is nonzero if and only if the number of rows is $\leq n+1$, denoted by $S^d V$.

- All irreducible $GL(n+1)$-modules are isomorphic to $S^d V$ for some Young diagram $\lambda$, with number of rows $\leq n+1$.

- Any $GL(n+1)$-module is a sum of irreducible ones.

- The images $c_{\lambda}(v_T)$ where $T$ is a semistandard tableau give a basis of $S^d V$.

Proof. [19] Prop. 15.47. \hfill \Box
A particular case, very useful in the sequel, is when $\lambda$ consists of $g$ columns of length $n+1$. This happens if and only if $S^gV$ is one dimensional.

The theory of $SL(n+1)$-representations is very similar. The basic fact is that if $S^1, S^2, \ldots, S^{n+1} V$ and $S^1, S^2, \ldots, S^{n+1} V$ are isomorphic as $SL(n+1)$-modules, indeed

$$S^1, S^2, \ldots, S^{n+1} V \cong \wedge^{n+1} V \otimes S^1, S^2, \ldots, S^{n+1} V.$$

In other words, all columns of length $n+1$ correspond to the one dimensional determinantal representation, which is trivial as $SL(n+1)$-module. Removing these columns, we get another Young diagram with the number of rows $\leq n$.

**Theorem 10.**

- All irreducible $SL(n+1)$-modules are isomorphic to $S^gV$ for some Young diagram $\lambda$, with number of rows $\leq n$.
- Any $SL(n+1)$-module is a sum of irreducible ones.

**Proof.** [19]Prop. 15.15, Theor. 14.18 , [47] chap. 9 §8.1

**Remark 4.** The construction $W \mapsto S^gW$ is indeed functorial, in the sense that a linear map $W_1 \to W_2$ induces a linear map $S^gW_1 \to S^gW_2$ with functorial properties, see [47] chap. 9 §7.1. See also [55] chap. 2.

**Definition 3.** The character of $S^gV$ is a symmetric polynomial $s_\lambda$ which is called a Schur polynomial.

There is a way to write $s_\lambda$ as the quotient of two Vandermonde determinants ([36] Prop. 1.2.1), and more efficient ways to write down explicitly $s_\lambda$ which put in evidence the role of tableau ([36] Theor. 1.4.1).

If $\lambda = d$, then $s_\lambda(t_1, \ldots, t_n)$ is the sum of all possible monomials of degree $d$ in $t_1, \ldots, t_n$. If $\lambda = 1, \ldots, 1$, then $s_\lambda(t_1, \ldots, t_n)$ is the $i$-th elementary symmetric polynomial in $t_1, \ldots, t_n$.

$SL(n+1)$ contains the torus $(C^*)^n$ of diagonal matrices

$$T = \{ D \left( t_1, \ldots, t_n, \frac{1}{(t_1 \cdots t_n)} \right) | t_i \neq 0 \}.$$

Given $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ we have the one dimensional (algebraic) representation $\rho: T \to C^*$ defined by

$$\rho \left( D \left( t_1, \ldots, t_n, \frac{1}{(t_1 \cdots t_n)} \right) \right) := t_1^{a_1} \cdots t_n^{a_n}.$$

We denote it by $V_{a_1, \ldots, a_n}$.
**Proposition 4.**

- Every irreducible (algebraic) representation of $T$ is isomorphic to $V_{a_1, \ldots, a_n}$ for some $(a_1, \ldots, a_n) \in \mathbb{Z}^n$.
- Every (algebraic) $T$-module is isomorphic to the direct sum of irreducible representations.

**Proof.** [47] chap. 7, §3.3

**Theorem 11 (Cauchy Identity).**

\[
S^\lambda(V \otimes W) = \bigoplus \lambda S^\lambda V \otimes S^\lambda W
\]

where the sum is extended to all partitions $\lambda$ of $p$.

**Proof.** By using characters, the proof reduces to a nontrivial identity on symmetric functions, see [47] (6.3.2) or [36] 1.4.2.

2.5. The Lie algebra $\mathfrak{sl}(n+1)$ and the weight structure of its representations

We denote by $\mathfrak{sl}(n+1)$ the Lie algebra of $SL(n+1)$. It corresponds to the traceless matrices of size $(n+1)$, where the bracket is $[A, B] = AB - BA \forall A, B \in \mathfrak{sl}(n+1)$. The tangent space at the identity of $SL(n+1)$ is naturally isomorphic to $\mathfrak{sl}(n+1)$.

A representation of the Lie algebra $\mathfrak{sl}(n+1)$ is a Lie algebra morphism $\mathfrak{sl}(n+1) \to \mathfrak{sl}(W)$. The derivative (computed at the identity) of a group representation $SL(n+1) \to SL(W)$ is a representation of the Lie algebra $\mathfrak{sl}(n+1)$.

Since $SL(n+1)$ is simply connected, there is a natural bijective correspondence between $SL(n+1)$-modules and $\mathfrak{sl}(n+1)$-modules, in the sense that every Lie algebra morphism $\mathfrak{sl}(n+1) \to \mathfrak{sl}(W)$ is the derivative of a unique group morphism $SL(n+1) \to SL(W)$. In particular all $\mathfrak{sl}(n+1)$-modules are direct sum of irreducible ones.

This definition behaves in a different way when applied on direct sums and tensor products. Let $\mathfrak{g}$ be a Lie algebra. If $V$ and $W$ are two $\mathfrak{g}$-modules then $V \oplus W$ and $V \otimes W$ are $\mathfrak{g}$-modules in a natural way, namely

\[
g \cdot (v + w) := (g \cdot v) + (g \cdot w),
g \cdot (v \otimes w) := (g \cdot v) \otimes (w) + (v) \otimes (g \cdot w).
\]

The $m$-th symmetric power $S^m W$ is a $\mathfrak{g}$-module, satisfying

\[
g \cdot (v^m) := m(g \cdot v)v^{m-1}.
\]

A morphism between two $\mathfrak{g}$-modules $V, W$ is a linear map $f : V \to W$ which is $\mathfrak{g}$-equivariant, namely it satisfies $f(g \cdot v) = g \cdot f(v) \forall g \in \mathfrak{g}, v \in V$.

Every $\mathfrak{g}$-module has a $\mathfrak{g}$-invariant submodule

\[
V^\mathfrak{g} = \{ v \in V | g \cdot v = 0 \ \forall g \in \mathfrak{g} \}. 
\]
The adjoint representation of $SL(n + 1)$ is the group morphism

$$SL(n + 1) \to GL(\mathfrak{sl}(n + 1)),$$

$$g \mapsto (h \mapsto g^{-1}hg).$$

Its derivative is the Lie algebra morphism

$$\mathfrak{sl}(n + 1) \to \text{End}(\mathfrak{sl}(n + 1))$$

$$g \mapsto (h \mapsto [h, g]).$$

The diagonal matrices $H \subset \mathfrak{sl}(n + 1) = \mathfrak{g}$ make a Lie subalgebra which can be identified with $\text{Lie}(T)$. It is abelian, that is $[H, H] = 0$, and it is called a Cartan subalgebra. Write generators as

$$H = \{D(t_1, \ldots, t_{n+1}) \mid \sum_i t_i = 0\}.$$

A basis of $H$ is given by $\{D(1, 0, \ldots, 0, -1), D(0, 1, 0, \ldots, 0, -1), \ldots\}$. In dual coordinates, we have a basis $h_i = (0, \ldots, 0, 1, 0, \ldots, 0, -1) \in H^\ast$ for $i = 1, \ldots, n$. We can consider $h_i$ as Lie algebra morphisms $h_i : H \to \mathbb{C}$.

Representations of $\mathfrak{sl}(n + 1)$, when restricted to $H$, satisfy the analogous to Proposition 4.

**Proposition 5.** Let $W$ be a Lie algebra representation of $\mathfrak{sl}(n + 1)$. When restricting the representation to $H$, it splits in the direct sum of irreducible representations, each one isomorphic to an integral combination $\sum_{i=1}^n a_i h_i$ with $a_i \in \mathbb{Z}$. These representations are the derivative of the representations $V_{a_1, \ldots, a_n}$ of Prop. 4.

The strictly upper triangular matrices

$$N^+ := \{g \in \mathfrak{sl}(n + 1) \mid g_{ij} = 0 \text{ for } i < j\}$$

make a nilpotent subalgebra, in the sense that the descending chain

$$\mathfrak{g}^+ \supset [\mathfrak{g}^+, \mathfrak{g}^+] \supset [[\mathfrak{g}^+, \mathfrak{g}^+], \mathfrak{g}^+] \supset \cdots$$

terminates to zero. It holds $[N^+, H] \subset H$, which means that $N^+$ is an invariant subspace for the adjoint representation restricted to $H$, indeed it splits as the sum of one dimensional representations, which are spanned by the elementary matrices $e_{ij}$ which are zero unless one upper triangular entry which is 1.

More precisely, there are certain $\alpha = \sum_{i=1}^n a_i h_i \in H^\ast$ as in Prop. 5 and $n_\alpha \in \mathbb{N}$ such that

$$[h, n_\alpha] = \alpha(h)n_\alpha \quad \forall h \in H. \quad (9)$$

Such $\alpha$ are called (positive) roots, and $N^+$ has a basis of eigenvectors $n_\alpha$.

In the same way we can define the subalgebra of strictly lower triangular matrices $N^-$, which has a similar basis of eigenvectors. The corresponding roots are called negative.
Theorem 12 (Weight Decomposition). Let $W$ be a $\mathfrak{sl}(n+1)$-module. $W$ is also a $H$-module, and it splits as the sum of $H$-representations $W(\lambda_i)$ where $\lambda_i \in H^*$ is called a weight and satisfies the following property:

$$W(\lambda_i) = \{ w \in W | h \cdot w = \lambda_i(h)w, \forall h \in H \}.$$ 

The elements in the weight space $W(\lambda_i)$ are called $H$-eigenvectors with weight $\lambda_i$.

Theorem 13. Let $W$ be an irreducible $\mathfrak{sl}(n+1)$-module. Then there is a unique (up to scalar multiples) vector $w \in W$ satisfying $N^- \cdot w = 0$. $w$ is called a maximal vector and it is an $H$-eigenvector.

The representation $W$ is spanned by repeated applications of elements $g \in N^+$ to $w$. More precisely, $e_{i_1j_1} \cdots e_{i_sq}w$ span $W$ for convenient $e_{i_kj_k} \in N^+$.

Proposition 6. Let $n_\alpha \in N^+$ be an eigenvector with eigenvalue the root $\alpha$ as in (9). Let $\lambda \in H^*$ be a weight of a representation $W$. Then

$$n_\alpha(W(\lambda)) \subseteq W(\alpha+\lambda).$$

Proof. Let $w \in W_\lambda$. For any $h \in H$ we have

$$h(n_\alpha(w)) = [h, n_\alpha](w) + n_\alpha(h(w)) = \alpha(h)n_\alpha(w) + n_\alpha(h(w)) = (\alpha(h) + \lambda(h))(n_\alpha(w)).$$

This proves that $n_\alpha(w) \in W(\alpha+\lambda)$ as we wanted. \hfill \Box

It is instructive to draw pictures of eigenspaces decomposition, denoting any weight space as a vertex, identifying the action of elements $n_\alpha$ as in Proposition 6 with arrows from $W(\lambda)$ to $W(\alpha+\lambda)$.

We begin with $\mathfrak{sl}(2)$, which has dimension three, spanned by $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

satisfying

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$ 

Each irreducible $\mathfrak{sl}(2)$-module is isomorphic to $S^m\mathbb{C}^2$ for an integer $m \in \mathbb{Z}_{\geq 0}$. If $v$ is the maximal vector, the space $S^m\mathbb{C}^2$ is isomorphic to $\oplus_{i=0}^m \langle x^i \cdot v \rangle$ for $i = 0, \ldots, m$ as in the following picture

$$x^m \cdot v \leftarrow x^{m-1} \cdot v \leftarrow \cdots \leftarrow x \cdot v \leftarrow v.$$ 

If we want to emphasize the dimensions of the weight spaces, we just write

$$1 \leftarrow 1 \leftarrow 1 \leftarrow \cdots \leftarrow 1 \leftarrow 1.$$
The exterior skeleton of $S^p\Sigma^2$ has the following pattern (the maximal vector is marked)

Note that $x^{m+1} = 0$. We have $h \cdot v = mv$, and in general $h \cdot (x^i \cdot v) = (2m - 2i)(x^i \cdot v)$. The natural construction in the setting of invariant theory is the following. Each $H$-eigenspace is generated by the monomial $s^{m-i}$. The monomial $s^m$ corresponds to the maximal vector. According to (8) we compute that $(x^i \cdot s^m)$ is a scalar multiple of the monomial $s^{m-i}$.

Each irreducible $sl(3)$-module is isomorphic to $S^{a,b}\Sigma^3$ for a pair of integers $a, b \in \mathbb{Z}_{\geq 0}$ with $a \geq b$.

We have the following pictures, which describe the general pattern that “weights increase by one on hexagons and remain constant on triangles”, see [19] chapter 6. The maximal vectors are marked in bold.

In these pictures

(10) the arrow $\bullet \rightarrow \bullet$ corresponds to the action of $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(11) the arrow $\bullet \downarrow \bullet$ corresponds to the action of $A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

and the arrow $\bullet \leftarrow \bullet$ corresponds to the action of $[A_1, A_2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
\[(S^3 C^3)^\vee = S^{3,3} C^3\]
The exterior skeleton of $S^p q C^3$ has the following pattern (the maximal vector is marked differently)

\[ S^p q C^3 \]

2.6. Schur-Weyl duality

**Theorem 14 (Schur-Weyl duality).** There is a $\Sigma_d \times SL(V)$-decomposition

\[ V^\otimes k = \bigoplus \lambda V_\lambda \otimes S^\lambda V \]

where the sum is extended on all Young diagrams with $k$ boxes, $V_\lambda$ has been defined in Def. 2 and $S^\lambda V$ has been defined in Theorem 9.

**Proof.** [47] chap. 9 (3.1.4).

Now fill the Young tableau with numbers from 1 to $n + 1$ in such a way that they are nondecreasing on rows and strictly increasing on columns. For example we have the following
Projective Invariants

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\
2 & 2 & 2 & 2 & 3 & 3 & 3 & 3
\end{array}
\]

Each filling describes a vector in \( V^{\otimes d} \). The image of these vectors through \( c_{\lambda} \) give a basis of \( S^d V \). In other words, the elements \( c_{\lambda}(v_T) \) where \( T \) is any tableau span \( S^d V \).

Composing with a permutation of all the boxes, we may find different isomorphic copies of the same representation \( V_{\lambda} \) inside \( V^{\otimes d} \).

This construction is quite important and make visible that \( \text{Im} c_{\lambda} \) defines just one copy of the representation \( S^d V \) inside \( V^{\otimes d} \), which is not intrinsic but it depends on the convention we did in the definition of \( c_{\lambda} \). For example by swapping in the definition 6 of \( c_{\lambda} \) the order of \( R_{\lambda} \) and \( C_{\lambda} \) we get another copy of \( S^d V \), in general skew with respect to the previous one.

But the main reason is that the order we chose in the \( d \) copies of \( V \) is arbitrary, so acting with \( \Sigma_d \) we can find other embeddings of \( S^d V \), all together spanning \( V_{\lambda} \otimes S^d V \) which is canonically embedded in \( V^{\otimes d} \), and it is \( \Sigma_d \times SL(V) \)-equivariant.

For example in in \( \mathbb{C}^{2^\otimes d} \) there is, corresponding to \( \lambda = (2, 2) \) the invariant subspace \( \mathbb{C}^2_{2,2} \otimes S^{2,2} \mathbb{C}^2 \).

Note that \( \text{dim} \mathbb{C}^2_{2,2} = 2 \), while \( \text{dim} S^{2,2} \mathbb{C}^2 = 1 \).

This 2-dimensional space of invariants is spanned by the three functions that take

\[
x_1 \otimes x_2 \otimes x_3 \otimes x_4 \in \left( \mathbb{C}^{2V} \right)^{\otimes 4}
\]

respectively in \((x_1 \wedge x_2)(x_3 \wedge x_4), (x_1 \wedge x_3)(x_2 \wedge x_4), (x_1 \wedge x_4)(x_2 \wedge x_3)\).

Note that we have the well known relation

\[
(x_1 \wedge x_2)(x_3 \wedge x_4) - (x_1 \wedge x_3)(x_2 \wedge x_4) + (x_1 \wedge x_4)(x_2 \wedge x_3) = 0.
\]

The image of the Schur symmetrizer \( c_{2,2} \) is a scalar multiple of \( f = (x_1 \wedge x_4)(x_2 \wedge x_3) + (x_1 \wedge x_3)(x_2 \wedge x_4) \). Note that applying the permutation \((12) \in \Sigma_4 \), we get \((12) \cdot f = f \), while applying the permutation \((13) \in \Sigma_4 \) we get an independent \( SL(2) \)-invariant element \((13) \cdot f \), which together with \( f \) spans the \( \Sigma_4 \times SL(2) \)-invariant subspace \( \mathbb{C}^2_{2,2} \otimes S^{2,2} \mathbb{C}^2 \).

This example generalizes to the following

**Theorem 15.** If \( \sigma \in \Sigma_d \), the elements \( \sigma c_{\lambda}(v_T) \) where \( T \) is any tableau, span a \( SL(n+1) \)-module isomorphic to \( S^d V \), lying in the subspace \( V_{\lambda} \otimes S^d V \).

The whole subspace \( V_{\lambda} \otimes S^d V \) is spanned by these copies of \( S^d V \), for any \( \sigma \in \Sigma_d \).

On the other hand, if \( T \) is a fixed tableau such that \( c_{\lambda}(v_T) \neq 0 \), the elements \( \sigma c_{\lambda}(v_T) \) for \( \sigma \in \Sigma_d \) span a \( \Sigma_d \)-module isomorphic to \( V_{\lambda} \) lying in the subspace \( V_{\lambda} \otimes S^d V \).
3. Invariants of forms and representation theory

Again we denote \( V = \mathbb{C}^{n+1} \). The subject of this section is the action of \( SL(V) \) over \( \otimes m S^d(V) \). The invariant subspace \( S^m (S^d V) ^{SL(V)} \) is, by definition, the space of invariants of degree \( m \) for forms of degree \( d \). We have from §2.5 the decomposition \( \mathfrak{sl}(V) = H \oplus N^+ \oplus N^- \) and we study separately the actions of \( H \) and \( N^+ \).

3.1. Invariance for the torus

Denote the coefficients of a form \( f \in S^d V \) of degree \( d \) in \( n+1 \) variables in the following way

\[
 f = \sum_{i_0 + \ldots + i_n = d} \frac{d!}{i_0! \ldots i_n!} f_{i_0, \ldots, i_n} x_0^{i_0} \ldots x_n^{i_n}.
\]

The space \( S^m S^d V \) is spanned by monomials \( f_{i_0, \ldots, i_n} \ldots f_{i_0, \ldots, i_n} \).

The weight of the monomial \( f_{i_0, \ldots, i_n} \ldots f_{i_0, \ldots, i_n} \) is the vector

\[
 \left( \sum_{j=1}^m i_{0,j}, \sum_{j=1}^m i_{1,j}, \ldots, \sum_{j=1}^m i_{n,j} \right) \in \mathbb{Z}_{\geq 0}^{n+1}.
\]

As we will see in the proof of Proposition 7, the subspaces of monomial of a given weight, are exactly the \( H \)-eigenspaces for the action of \( \mathfrak{sl}(n+1) \) on \( S^m S^d \mathbb{C}^{n+1} \) seen in Theorem 12.

A monomial is called isobaric if its weight has all equal entries ("democratic"). Consider the double sum

\[
 \sum_{k=0}^m \sum_{j=1}^m i_{k,j} = \sum_{j=1}^m 1 \ldots 1 = md.
\]

So the weight of an isobaric monomial in \( S^m S^d V \) is \( \left( \frac{md}{n+1}, \ldots, \frac{md}{n+1} \right) \), in particular isobaric monomials exist if and only if \( n+1 \) divides \( md \).

An polynomial \( I \in S^m S^d V \) has degree \( m \) in the coefficients \( f_{i_0, \ldots, i_n} \).

**Proposition 7.** \( I \in S^m S^d V \) is invariant for the action of the torus of diagonal matrices \( (\mathbb{C}^\ast)^n \subset SL(V) \) if and only if it is sum of isobaric monomials.

Note that it is enough to check if a monomial of given degree is isobaric for \( n \) places in the \( (n+1) \)-dimensional weight vector. In particular for binary forms it is enough to check the condition just for one place.

**Proof.** Consider the diagonal matrix with entries \( \left( \frac{1}{t_1 \ldots t_n} t_1, \ldots, t_n \right) \) which acts on \( f_{i_0, \ldots, i_n} \) by multiplying for \( (t_1 \ldots t_n)^{-1} t_1^{i_0} \ldots t_n^{i_n} \).

Acting on the monomial

\[
 f_{i_0,1} \ldots f_{i_0, m} \]


we multiply it for \((t_1 \ldots t_n)^{-\sum_i i_0/j_i \sum_j i_{i,j} \ldots i_{n,j}}\) and this is equal to 1 if and only if \(\sum_j i_{k,j}\) does not depend on \(k\).

**COROLLARY 2.** A necessary condition for the existence of a nonzero \(I \in S^m S^d V\) which is invariant for \(\text{SL}(V)\) is that \(n+1\) divides \(md\).

**Proof.** If \(I\) is \(\text{SL}(V)\)-invariant then it is also invariant for the torus of diagonal matrices.

An equivalent way to express the fact that the polynomial \(I\) has to be isobaric is through the action of the Lie algebra \(H\) of diagonal matrices. This translates to the fact that \(I\) satisfies the system of differential equations (see [53] Theor. 4.5.2)

\[(12) \quad \sum_{i_0 \ldots i_n} i_j f_{i_0 \ldots i_n} \frac{\partial I}{\partial f_{i_0 \ldots i_n}} = \frac{md}{n+1} I \quad \forall j = 0, \ldots, n.\]

There is a second set of differential equations for the action of the triangular part \(N^+ \subset \text{sl}(n+1)\), see Propositions 9, 12.

**EXAMPLE 3.** In \(S^3(S^4 \mathbb{C}^3)\) there are 23 isobaric monomials among 680 monomials. In \(S^6(S^3 \mathbb{C}^3)\) there are 103 isobaric monomials among 5005. In Prop. 8 we will give a generating function that allows to compute these numbers.

### 3.2. Counting monomials of given weight

We compute now the number of monomials with a given weight.

Let \(H_{g,d,p_0,\ldots,p_n}\) be the space of monomials in \(S^g(S^d \mathbb{C}^{n+1})\) of weight \((p_0, \ldots, p_n)\). Since \(p_0 + \ldots + p_n = dg\), it is enough to record the last \(n\) entries of weight vector.

We have

**PROPOSITION 8.**

\[\sum_{g=0}^{\infty} \sum_{p_0+\ldots+p_n=dg} H_{g,d,p_0,\ldots,p_n} x_1^{p_1} \ldots x_n^{p_n} y^g = \prod_{i_1+\ldots+i_n \leq d} \frac{1}{1-x_1^{i_1} \ldots x_n^{i_n} y}.\]

**Proof.** The variable \(a_{i_0,\ldots,i_n}\) has degree 1 and weight \((i_0, \ldots, i_n)\). The left hand side is the Hilbert series for the multigraded ring \(K[a_{i_0,\ldots,i_n}]\). The graded ring in just the variable \(a_{i_0,\ldots,i_n}\) has Hilbert series \(\frac{1}{1-x_1^{i_1} \ldots x_n^{i_n} y} = 1 + x_1^{i_1} \ldots x_n^{i_n} y + x_1^{2i_1} \ldots x_n^{2i_n} y^2 + \ldots\). Taking into account all the variables, we have the product of the corresponding Hilbert series.

Recall that for invariants of weight \((p_0, \ldots, p_n)\) we have \(\sum_i p_i = dg\).

In case of binary forms, let \(H_{g,p,d}\) be the space of homogeneous polynomials of degree \(g\) and weight \((p, dg-p)\) in \(a_0, \ldots, a_d\).
THEOREM 16 (Cayley-Sylvester).

\[ \sum_{p} \dim H_{g,p,d} x^p = \frac{(1-x^d+1) \ldots (1-x^{d+g})}{(1-x)^{d+1}}. \]

Proof. Write (after Proposition 8)

\[ \phi_d(x,y) := \sum_{p \in \mathbb{N}} \dim H_{g,p,d} x^p y^q \cdot \prod_{i=0}^{d} \frac{1}{1-x^i} = \sum_{g=0}^{\infty} C_{g,d}(x)y^g, \]

so that the expression to be computed is \( C_{g,d}(x) \).

We get

\[ (1-y)\phi(x,y) = \prod_{i=1}^{d} \frac{1}{1-x^i} = (1-x^{d+1}y) \prod_{i=1}^{d-1} \frac{1}{1-x^i} y = \]

\[ (1-x^{d+1}y) \prod_{i=0}^{d} \frac{1}{1-x^{d+1}i} = (1-x^{d+1}y)\phi(x,xy). \]

Hence

\[ (1-y) \sum_{j=0}^{\infty} C_{j,d}(x)y^j = (1-x^{d+1}y) \sum_{j=0}^{\infty} C_{j,d}(x)x^j y^j. \]

Comparing the coefficients of \( y^j \) we get

\[ C_{j,d} - C_{j-1,d} = C_{j,d}x^j - C_{j-1,d}x^{d+j}, \]

hence

\[ C_{j,d}(x) = \frac{1-x^{d+j}}{1-x^d} C_{j-1,d}(x). \]

Since \( C_{0,d}(x) = 1 \), by induction on \( j \) we get the thesis.

Let’s state the proposition 8 in the case of \( SL(3) \), for future reference. We denote by \( H_{g,p_0,p_2,p_2} \) be the space of monomials in \( S^g(S^dC^3) \) of weight \( (p_1,p_2,p_2) \), and we get

\[ \sum_{p_0+\ldots+p_n=d,g} H_{g,d,p_0,p_2,p_2} x_1^{p_1} x_2^{p_2} y^g = \prod_{i_1=0}^{d} \prod_{i_2=0}^{d-i_1} \frac{1}{1-x_1^{i_1} x_2^{i_2} y}. \]

3.3. Lie algebra action on forms

We give a \( SL(2) \) example, which illustrates the general situation. Recall that the generator \( x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in sl(2) \) integrates in the Lie group to the one parameter subgroup

\[ e^{\alpha t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \]
The main application of the previous proposition is that it allows to compute explicitly invariants. Remind the equianharmonic quadric \( I = f_0 f_4 - 4f_1 f_3 + 3 f_2^2 \in S^2 S^4 C^2 \) and our question posed in §1.2, namely why the coefficients \( (1, -4, 3) \) ?

Now the coefficients can be computed by Prop. 9.

Call \( \alpha, \beta, \gamma \) unknown coefficients and apply \( D(\alpha f_0 + \beta f_1 + \gamma f_2) = f_0 f_3 (4\alpha + \beta) + f_1 f_3 (3\beta + 4\gamma) = 0 \). We get that \( (\alpha, \beta, \gamma) \) is proportional to \( (1, -4, 3) \).

In the same way we can prove the following dual version.

**Proposition 10.**

(i) \( I \in S^n S^4 C^2 \) is invariant with respect to the subgroup \( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \) if and only if it is invariant with respect to the subalgebra \( N^- \) if and only if \( \Delta = 0 \) where

\[
\Delta = \sum_{i=0}^{d-1} (d-i) a_{i+1} \frac{\partial}{\partial a_i} = da_1 \frac{\partial}{\partial a_0} + (d-1) a_2 \frac{\partial}{\partial a_1} + \ldots
\]

(ii) \( I \in S^n S^4 C^2 \) is \( SL(2) \)-invariant if and only if it is isobaric and \( \Delta = 0 \).

The following Proposition is a computation contained in [24], it is interesting because gave a motivation to study Lie algebras.

**Proposition 11.**

(i) \( DA - AD = \sum_{i=0}^{d} (d-2i) a_i \frac{\partial}{\partial a_i} \)

(ii) \( (DA - AD)(a_0^{\nu_0} \ldots a_d^{\nu_d}) = \sum_{i=0}^{d} (d-2i) \nu_i (a_0^{\nu_0} \ldots a_d^{\nu_d}) = (dg - 2p)(a_0^{\nu_0} \ldots a_d^{\nu_d}) \).

Hilbert proved from this proposition that an isobaric polynomial \( F \in S^g (S^4 C^2) \) (all its monomials have weight \( p \) where \( dg - 2p = 0 \)) satisfying \( DF = 0 \) must satisfy also \( \Delta F = 0 \), which is nowadays clear from the structure of \( sl(2) \)-modules (see §2.5 and also §3.7). Indeed their weight are segments centered around \( (\frac{dg}{2}, \frac{dg}{2}) \).
Note that we have in the case of the quadrics generating the twisted cubic (see (22))

\[ a_1a_3 - a_2^2 \frac{\partial}{\partial \Delta} a_0a_3 - a_1a_2 \frac{\partial}{\partial \Delta} a_0a_2 - a_1^2. \]

For ternary forms we introduce the differential operators

\[ D_1 = \sum_{i_0 + \ldots + i_2 = d} i_1f_{i_0+1,i_1-1,i_2} \frac{\partial}{\partial f_{i_0-i_2}} = 0, \]

\[ D_2 = \sum_{i_0 + \ldots + i_2 = d} i_2f_{i_0,i_1+1,i_2-1} \frac{\partial}{\partial f_{i_0-i_2}} = 0. \]

Note that \( D_1 \) adds \((1, -1, 0)\) to the weight, while \( D_2 \) adds \((0, 1, -1)\) to the weight.

They correspond to \( A_1 \) and \( A_2 \) in \( \S 2.5 \) and give the action on the directions depicted in (10) and (11). The following result is the natural extension to \( sl(3) \) of Prop. 9, the proof is the same.

**Proposition 12.**

(i) \( I \in S^mS^d \mathbb{C}^3 \) is invariant with respect to the subalgebra \( N^+ \) if and only if \( D_1I = D_2I = 0 \).

(ii) \( I \in S^mS^d \mathbb{C}^3 \) is \( SL(3) \)-invariant if and only if it is isobaric and \( D_1I = D_2I = 0 \).

### 3.4. Cayley-Sylvester formula for the number of invariants of binary forms

We recall now Cayley-Sylvester computation of the dimension of invariants and covariants for binary forms.

We have already seen the Hessian \( f_{xx}f_{yy} - f_{xy}^2 \) of a binary form \( f \in S^d \mathbb{C}^2 \), which can be considered as a module \( S^{2d-4} \mathbb{C}^2 \subset S^2(S^d \mathbb{C}^2) \). Indeed it is a polynomial of degree \( 2d - 4 \) in \( x, y \) with coefficients of degree 2 in \( f \). This is an example of a **covariant** of \( f \). In general a covariant of degree \( g \) of \( f \) is any module \( S^g \mathbb{C}^2 \subset S^g(S^d \mathbb{C}^2) \), so it corresponds to a \((SL(2))-invariant\) polynomial of degree \( e \) in \( x, y \) with coefficients of degree \( g \) in \( f \).

One of the most advanced achievement of classical period was the following computation of the number of covariants, which is a nontrivial example of \( SL(2) \)-plethysm.

We recall from \( \S 3.1 \) that \( H_{g,p,d} \) is the space of monomials in \( S^gS^d \mathbb{C}^2 \) of weight \((p, dg - p)\).

Let \( D \) be the differential operator defined in Proposition 9.

Let \( I_{g,p,d} \) be the kernel of the map \( H_{g,p,d} \rightarrow H_{g,p-1,d} \) of degree \( g \) and weight \( p \) in \( \alpha_0, \ldots, \alpha_d \).
THEOREM 17 (Cayley-Sylvester). (i) Let $p \leq \frac{dg}{2}$. Then $\dim I_{g,p,d}$ is the degree $p$ coefficient in
\[
\frac{(1 - x^{d+1}) \cdots (1 - x^{d+g})}{(1 - x^2) \cdots (1 - x^g)}.
\]

(ii) Let $2p = dg$. Then $\dim I_{g,\frac{dg}{2},d}$ is the dimension of the space of invariants so it is the coefficient of degree $\frac{dg}{2}$ in
\[
\frac{(1 - x^{d+1}) \cdots (1 - x^{d+g})}{(1 - x^2) \cdots (1 - x^g)}.
\]

(iii) More generally we have the $\text{SL}(2)$-decomposition
\[
S^g(S^d \mathbb{C}^2) = \bigoplus S^e \mathbb{C}^2 \otimes I_{g,\frac{dg-e}{2},d}
\]
where $\dim I_{g,\frac{dg-e}{2},d}$ is equal to the coefficient of degree $\frac{dg-e}{2}$ in
\[
\frac{(1 - x^{d+1}) \cdots (1 - x^{d+g})}{(1 - x^2) \cdots (1 - x^g)}.
\]

Proof. (i) Any $\mathfrak{sl}(2)$ representations splits as a sum of irreducible representations, each one centered around the weight $\left(\frac{dg}{2}, \frac{dg}{2}\right)$ (this is meaningful even if $\frac{dg}{2}$ is not an integer). Hence for $p \leq \frac{dg}{2}$ the differential $D$ is surjective and the result follows from Theorem 16.

(ii) follows from (i) and 9.

(iii) By the description in (i), the number of irreducible representations isomorphic to $S^e$ can be computed looking at the space of monomials of weight $\left(\frac{dg-e}{2}, \frac{dg+e}{2}\right)$ which are killed by $D$. \hfill $\square$

REMARK 5. For $p \leq \frac{dg}{2}$, $\dim I_{g,p,d}$ may be computed by Proposition 8 also as the coefficient of $x^p y^g$ in
\[
(1 - x)^d \prod_{i=0}^{d} \frac{1}{1 - x^i y}.
\]

The following proof is borrowed from [24]. We include it to show that concrete applications of Cayley-Sylvester formula can be painful. We will give an alternative proof in Corollary 7.

COROLLARY 3. Let $d = 3$. Let $I_{\frac{g}{2},\frac{dg}{2},d}$ be the dimension of the space of invariants of degree $g$ of the binary cubic. Then
\[
\sum_{g=0}^{\infty} I_{\frac{g}{2},\frac{dg}{2},3} x^g = \frac{1}{1 - x^3}.
\]

The ring of invariants is freely generated by the discriminant $D$, that is
\[
\bigoplus_m S^m(S^3(\mathbb{C}^2))^{|\text{SL}(2)|} = \mathbb{C}[D].
\]
Proof. We have from Theorem 17 that

\[
I_{g, \frac{3g}{2}, \frac{3g}{2}} = \frac{(1 - x^{g+1}) \cdots (1 - x^{g+3})}{(1 - x^2) \cdots (1 - x^6)} = \frac{(1 - x^{g+1})(1 - x^{g+2})(1 - x^{g+3})}{(1 - x^2)(1 - x^3)}
\]

and we write it as

\[
\left\{ \frac{(1 - x^{g+1})(1 - x^{g+2})(1 - x^{g+3})}{(1 - x^2)(1 - x^3)} \right\}_{\frac{3g}{2}}.
\]

We can remove the terms which do not change the coefficient of \(x^{\frac{3g}{2}}\), so getting

\[
\left\{ \frac{(1 - x^{g+1} - x^{g+2} - x^{g+3})}{(1 - x^2)(1 - x^3)} \right\}_{\frac{3g}{2}} = \left\{ \frac{1}{(1 - x^2)(1 - x^3)} \right\}_{\frac{3g}{2}} - \left\{ \frac{x(1 + x + x^2)}{(1 - x^2)(1 - x^3)} \right\}_{\frac{3g}{2}} = \left\{ \frac{1}{(1 - x^2)(1 - x^3)} \right\}_{\frac{3g}{2}} - \left\{ \frac{x^3}{(1 - x^2)(1 - x^3)} \right\}_{\frac{3g}{2}}
\]

(by using that \((1 - x^3) = (1 + x + x^2)(1 - x)\))

\[
= \left\{ \frac{1}{(1 - x^2)(1 - x^3)} \right\}_{\frac{3g}{2}} - \left\{ \frac{x}{(1 - x)(1 - x^2)} \right\}_{\frac{3g}{2}}
\]

\[
= \left\{ \frac{1}{(1 - x^2)(1 - x^3)} \right\}_{\frac{3g}{2}} - \left\{ \frac{x^3}{(1 - x^2)(1 - x^3)} \right\}_{\frac{3g}{2}}
\]

\[
= \left\{ \frac{1}{(1 - x^5)} \right\}_{\frac{3g}{2}} + \left\{ \frac{x^2(1 + x^2)}{(1 - x^2)(1 - x^5)} \right\}_{\frac{3g}{2}}
\]

The second summand contains only terms of the form \(x^{4m+3n}\) with \(m = 1\) or \(2\), hence none of the form \(x^{3g}\), and so does not contribute anything.

\[\square\]

3.5. Counting partitions and symmetric functions

The following combinatorial interpretation is interesting.

Proposition 13. The number \(H_{g, p, d}\) is the number of partitions of \(p\) as a sum of at most \(g\) summands from \(\{1, \ldots, d\}\), which is equal to the number of Young diagrams with \(p\) boxes, at most \(d\) rows and at most \(g\) columns.
Proof. $H_{g,p,d}$ counts monomials $a_{i_0}^{i_1} \cdots a_{i_d}^{i_d}$ with $i_0 + \ldots + i_d = g$ and $i_1 + 2i_2 + \ldots + di_d = p$ which correspond to the partition $1_{i_1} + 2_{i_2} + \ldots + d_{i_d} = p$.

\[ \square \]

**Corollary 4 (Hermite reciprocity).**

\[ H_{g,p,d} = H_{d,g,p}, \quad S^g(S^dC^2) = S^d(S^gC^2). \]

In equivalent way the number $H_{g,p,d}$ counts also the number of partitions of $p$ as a sum of at most $d$ summands from $\{1, \ldots, g\}$.

**Proof.** Taking transposition of Young diagrams.

For $d \geq p$, $H_{g,p,d}$ stabilizes to $H_{g,p}$ which is the number of partitions of $p$ as a sum of at most $g$ summands, or also the number of Young diagrams with $p$ boxes and at most $g$ rows. The coefficient $H_{g,p}$ is equal also to the number of partitions of $p$ with summands from $\{1, \ldots, g\}$, indeed it is the number of Young diagrams with $p$ boxes and at most $g$ columns, which are the transpose of the previous ones.

For $d \geq p$, the numerator of (13) has no role in the computation and we get

\[ \sum_{p=0}^{\infty} H_{g,p,d} x^p = \prod_{i=1}^{g} \frac{1}{1 - x^i}. \]

This formula has a more elementary interpretation, indeed we have

\[ \prod_{i=1}^{g} \frac{1}{1 - x^i} = \left( \sum_{i \geq 0} x^i \right) \left( \sum_{i \geq 0} x^{2i} \right) \cdots \left( \sum_{i \geq 0} x^{gi} \right) = \sum_{i_1 \geq 0, \ldots, i_g} x^{i_1 + 2i_2 + \ldots + gi_g} \]

and the coefficient of $x^p$ counts all indexes $i_j$ which make $i_1 + 2i_2 + \ldots + gi_g = p$. Cayley-Sylvester argument can be interpreted as a refinement of this elementary computation. The formula (15) can be considered as the Hilbert series of the ring of symmetric functions polynomials in $g$ variables, having Schur polynomials $\sigma_{i_j}$ as a basis.

Note from the Proposition 8 also the identity

\[ \sum_{p \geq 0} dim H_{g,p,d} x^p y^g = \prod_{i=1}^{g} \frac{1}{1 - x^i y} \]

which is meaningful because for each power of $x$ only finitely many factors are taken into account in the right-hand side.

For any fixed $p$, for $g \geq p$ $H_{g,p}$ stabilizes to $H_p$, the number of partitions of $p$, and from (15) we can write the basic formula
\[
\sum_{p=0}^{\infty} H_p x^p = \prod_{i=1}^{\infty} \frac{1}{1-x^i}
\]

which is again meaningful because for each power of \(x\) only finitely many factors are taken into account in the right-hand side. The formula (16) can be considered as the Hilbert series of the ring of symmetric functions polynomials in infinitely many variables.

Hardy and Ramanujam proved the remarkable asymptotic formula

\[
H_p \sim \frac{1}{4p\sqrt{3}} \exp \frac{2p}{3} \quad \text{for} \quad p \to \infty
\]

which is one of the many wonderful and “not expected” appearances of \(\pi\) in combinatorics. For an interesting proof in the probabilistic setting, see [5].

3.6. Generating formula for the number of invariants of ternary forms

We saw in Theorem 17 that the dimension of the space of invariants in \(S^d S^d \mathbb{C}^3\) can be obtained as \(\dim H_{g,p,d} - \dim H_{g,p-1,d}\). The following Theorem perform an analogous computation for \(SL(3)\)-invariants, and gives a clue how to perform such computations in general.

**THEOREM 18** (Sturmfels [53] Algorithm 4.7.5, Bedratyuk [7, 8]). Let \(dg = p_0 + p_1 + p_2\). Let \(h_{g,d,p_0,p_1,p_2} = \dim H_{g,d,p_0,p_1,p_2}\). Let \(I_{g,d,p,p,p}\) be the space of invariants in \(S^d S^d \mathbb{C}^3\).

\[
\text{dim } I_{g,d,p,p,p} = h_{g,d,p,p,p} - h_{g,d,p+1,p-1,p} - h_{g,d,p-1,p,p+1} + h_{g,d,p+1,p-2,p+1} + h_{g,d,p-1,p-1,p+2} - h_{g,d,p,p-2,p+2}.
\]

(17)

So by (14) we get that \(\dim I_{g,d,p,p,p}\) is the coefficient of \(x_1^{d+1} x_2^p y^g\) in

\[
(x_2 - x_1)(x_2 - 1)(x_1 - 1) \prod_{i=0}^{d-i} \prod_{j=0}^{d-i} \frac{1}{1-x_1^{d-i} x_2^j y}.
\]

**Proof.** \(S^d S^d \mathbb{C}^3\) splits as the sum of irreducible representations. Representing their weights, they span hexagons or triangles like in §2.5, all centered around \((\frac{dg}{3}, \frac{dg}{3}, \frac{dg}{3})\).

The right hand side counts 1 on every trivial summand in \(S^d S^d \mathbb{C}^3\) and it counts 0 on every nontrivial summand in \(S^d S^d \mathbb{C}^3\). The details are in [7, 8].

The six summands in (17) correspond to the action of the Weyl group \(\Sigma_3\) for \(SL(3)\), and are identified as vertices of a hexagon (see next table (18)). In [8] the formula is extended to any \(SL(n+1)\).

The following M2 script shows how to get the whole decomposition of \(h_{g,d,s}\) in cases of cubic invariant of plane quartics, other cases can be obtained by changing the values of \(d\) and \(m\). For larger values, the size of the computation can be reduced by contracting separately with \(y^m x_1^d x_2^d\) and bounding correspondingly the upper limit of the sum.
The output of previous script can be recorded in the following table

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 3 & 4 & 6 & 4 & 5 \\
2 & 4 & 11 & 15 & 9 & 4 & 1 & 2 \\
4 & 15 & 19 & 6 & 2 & 1 & 4 & 1 \\
9 & 19 & 15 & 6 & 2 & 1 & 4 & 1 \\
15 & 16 & 19 & 11 & 4 & 1 & 4 & 1 \\
19 & 11 & 9 & 4 & 1 & 4 & 1 & 1 \\
23 & 16 & 8 & 2 & 1 & 4 & 1 & 1 \\
25 & 15 & 8 & 3 & 1 & 4 & 1 & 1 \\
2 & 4 & 6 & 5 & 1 & 2 & 3 & 1 \\
1 & 1 & & & & & & \\
\end{array}
\]

The sum of all the numbers in the triangle is 680 = \( \dim S^3(S^4 \mathbb{C}^3) \). The rounded entries correspond to the six summands in Bedratyuk formula (17). According to the table (18) we compute from Theorem 18 \( \dim I_{3,4,4,4,4} = 23 - 19 - 19 + 16 + 15 - 15 = 1 \).

Note that in (18) it is enough to record the following “one third” part, and the others can be filled by symmetry.
In Sturmfels’ book [53] there is the computation of Hilbert series for plane quartics up to degree 21. Shioda got in [52] the complete series, from where one can guess possible generators. There is an unpublished paper by Ohno, claiming a system of generators for the invariant subring of plane quartics (see [2]).

**Remark 6.** L. Bedratyuk gives in [7,8] other formulas which extend Theorem 17 to $n$-ary forms. On his web page is available some software packages implementing these formulas.

### 3.7. The Reynolds operator and how to compute it. Hilbert finiteness theorem

When $G = \text{SL}(n+1)$ acts on a vector space $V$, we recall from §2.1 the notation

$$V^G = \{v \in V | g \cdot v = v \forall g \in G\}.$$ 

Since $\text{SL}(n+1)$ is reductive we have the direct sum $V = V^G \oplus (V^G)^\perp$, where $(V^G)^\perp$ is the sum of all irreducible invariant subspaces of $V$ which are nontrivial (see Theorem 14).

The Reynolds operator is the projection

$$R: V \to V^G.$$ 

If $G$ is a finite group, such a projection can be obtained just averaging over the group, that is $R(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v$. In the case of $G = \text{SL}(n+1)$, Weyl replaced the sum with the integration over a maximal compact real subgroup, which is $\text{SU}/(n+1)$, this is the “Weyl unitary trick” [19]. It is difficult to compute the integral directly from the definition. We want to show, in an example, how the Reynolds operator can be explicitly computed by the Lie algebra techniques.
\section*{Projective Invariants}

**Proposition 14.** Let \( f = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4 \in S^4 \mathbb{C}^2 \).

We consider \( R: S^2(S^4 \mathbb{C}^2) \to S^2(S^4 \mathbb{C}^2)^G \). Then

\[
R(a_0 a_4) = \frac{2}{5} I, \quad R(a_1 a_3) = -\frac{1}{10} I, \quad R(a_2^2) = \frac{1}{15} I,
\]

where \( I = a_0 a_4 - 4a_1 a_3 + 3a_2^2 \in S^2(S^4 \mathbb{C}^2)^G \), while \( R \) vanishes on all other monomials in \( S^2(S^4 \mathbb{C}^2) \).

**Proof.** The key is to consider the differential operator \( D = \sum_{i=0}^{4}(i+1)a_i \frac{\partial}{\partial a_{i+1}} \) acting on \( S^2(S^4 \mathbb{C}^2) \). The torus action has the eigenspaces \( S^2(S^4 \mathbb{C}^2) = \oplus H_i \) where the weight of \( a_0 a_j \) is \( 2(i + j) - 8 \). We write dimensions in superscripts

\[
H_{-8}^{1}, H_{-6}^{1}, H_{-4}^{2}, H_{-2}^{2}, H_{0}^{2}, H_{2}^{2}, H_{4}^{2}, H_{6}^{2}, H_{8}^{2}
\]

which come from the three representations

\[
S^2(S^4 \mathbb{C}^2) = S^8 \mathbb{C}^2 \oplus S^4 \mathbb{C}^2 \oplus S^0 \mathbb{C}^2
\]

namely

\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

We refer to the three rows above. We have

\[
a_0^2 \leftarrow a_0 a_1 \leftarrow a_0 a_2 \leftarrow a_3 a_2 \leftarrow a_1 a_3 \leftarrow a_1 a_4 \leftarrow a_2 a_4 \leftarrow a_3 a_4 \leftarrow a_4
\]

and the goal is to split these monomial spaces into the three previous rows. We compute the splitting as follows.

Since \( D^5(a_1^3) = 720 a_0 a_1 \), \( D^5(a_2 a_4) = 720 a_0 a_1 \), then \( D^5(a_2 a_4 - a_1^3) = 0 \) and \( a_2 a_4 - a_1^3 \) belongs to the second row.

Hence \( D^5(a_2 a_4 - a_1^3) = 2(a_0 a_4 + 2a_1 a_3 - 3a_2^2) \) belongs again to the second row.

Moreover \( D^5(a_3^2) = 48(a_0 a_4 + 16a_1 a_3 + 18a_2^2) \) belongs to the first row. Hence we have the decomposition

\[
H_0^3 = < I > \oplus < a_0 a_4 + 2a_1 a_3 - 3a_2^2, a_0 a_4 + 16a_1 a_3 + 18a_2^2 >
\]

and \( R \) is the projection over \(< I > \). We get the scalars stated by inverting the matrix

\[
\begin{bmatrix}
1 & -4 & 3 \\
1 & 2 & -3 \\
1 & 16 & 18
\end{bmatrix}^{-1} = \begin{bmatrix}
\frac{2}{5} & * & * \\
\frac{1}{10} & * & * \\
\frac{1}{15} & * & *
\end{bmatrix}.
\]

\( \square \)

From the above proof it should be clear that the same technique can be applied in order to compute the Reynolds operator in other cases. For an alternative approach, by using Casimir operator, see [13] 4.5.2. The computations performed by Derksen and Kemper are essentially equivalent to ours.
PROPOSITION 15. Let $R = \oplus_i R_i$ be a graded ring where $G = SL(n)$ acts. If $f \in R^G_i$, $g \in R^G_j$, then

$$R(fg) = fR(g).$$

Proof. Decompose $g = g_1 + g_2$ where $g_1 \in R^G_i$ and $g_2$ belongs to its complement $(R^G_i)^\perp$.

Then $fR^G_i \subset R^G_i + R^G_j$ and $f(R^G_i)^\perp \subset (R^G_j)^\perp$, indeed if $f$ is nonzero, $fR^G_i$ is a $G$-module isomorphic to $R^G_i$ and $f(R^G_i)^\perp$ is a $G$-module isomorphic to $(R^G_i)^\perp$.

We close this section by recalling the wonderful proof about the finite generation of the invariant ring, proved first by Hilbert in 1890. Hilbert result, together with the 1FT (which we will see in §3.10) implies that the ring of invariants is generated by finitely many products of tableau like in §3.8.

THEOREM 19 (Hilbert). Let $G = SL(n+1)$ (although any reductive group works as well). Let $W$ be a finite dimensional $G$-module. Then the invariant subring $\mathbb{C}[W]^G$ is finitely generated.

Proof. Let $I$ be the ideal in $\mathbb{C}[W]$ generated by all the homogeneous invariants of positive degree. Since $\mathbb{C}[W]$ is Noetherian we get that $I$ is generated by homogeneous invariants $f_1, \ldots, f_s$.

We will prove that for any degree $d \mathbb{C}[W]^G_d = \mathbb{C}[f_1, \ldots, f_s]_d$ by induction on $d$. The case $d = 0$ is obvious. Let $f \in \mathbb{C}[W]^G_d \subset I_d$. Then there exist $a_i \in R$ such that $f \in \sum_i a_i f_i$.

We get $f = R(f) = \sum_i R(a_i f_i) = \sum_i R(a_i) f_i$, the last equality by Proposition 15. By the inductive assumption, each $R(a_i)$ is a polynomial in $f_i$, hence the same is true for $f$.

3.8. Tableau functions. Comparison among different applications of Young diagrams

Let $V = \mathbb{C}^{n+1}$, more precisely for the following it is enough to consider a $(n+1)$-dimensional vector space with a fixed isomorphism $\wedge^{n+1}V \cong \mathbb{C}$.

Then for every $v_1, \ldots, v_{n+1} \in V$ the determinant $v_1 \wedge \ldots \wedge v_{n+1} \in \mathbb{C}$ is well defined and it is $SL(n+1)$-invariant for the natural action of $SL(n+1)$ on $V$. Every rectangular tableau $T$ over a Young diagram of size $(n+1) \times g$ gives a tableau function which is constructed by taking the product of the determinant arising from each column.

So represents $x_1 \wedge x_2 \wedge x_3$, and represents $(x_1 \wedge x_2 \wedge x_3) (x_1 \wedge x_3 \wedge x_4)$.

To define formally this notion, we set $[a] = \{1, \ldots, a\}$ for any natural number $a$ and we notice that a tableau $T$ is encoded in a function $t: [n+1] \times [g] \rightarrow [m]$ , where $t(i,j)$ corresponds to the entry at the place $(i,j)$ of the tableau.
**Projective Invariants**

**Definition 4** (From tableau to multilinear invariants). For any tableau $T$ over a Young diagram of size $(n + 1) \times g$ filled with numbers from 1 appearing $h_1$ times until $m$ appearing $h_m$ times, so that $h_1 + \ldots + h_m = g(n + 1)$, we denote by $G_T$ the multilinear function $S^{h_1}V^\vee \times \ldots \times S^{h_m}V^\vee \to \mathbb{C}$ defined by

$$G_T(x_1^{h_1}, \ldots, x_m^{h_m}) = \prod_{j=1}^{m} (x_{i(1,j)} \land \ldots \land x_{i(n+1,j)}).$$

$G_T$ is well defined by Theorem 1, and it is called a tableau function.

**Proposition 16.** Every tableau function $G_T$ is $SL(V)$-invariant.

**Proof.** Immediate by the properties of the determinant.

Every Young diagram $\lambda$ defines (at least) four interesting objects, which give four similar theories, that is

- a representation $S^3V$ of $SL(V)$ (see Theor. 9)
- a representation $V_\lambda$ of the symmetric group (see Def. 2)
- a symmetric polynomial $s_\lambda$ (see Def. 3)
- a Schubert cell $X_\lambda$ in the Grassmannian (it will be defined in a while)

This comparison is carefully studied in [36]. The Schur functions $s_\lambda$ give an additive basis of the ring of symmetric polynomials. There is an algorithm to perform, given a symmetric function $f$, the decomposition $f = \sum c_\lambda s_\lambda$, described in the Algorithm 4.1.16 in [53]. Essentially, let $t^{\gamma_1 \ldots \gamma_m}_\lambda$ be the leading term in $f$, let $\lambda$ be the unique partition of $d$ such that the corresponding highest weight of $s_\lambda$ is $t^{\gamma_1 \ldots \gamma_m}_\lambda$. Then we output the summand $c_\lambda s_\lambda$ and we continue with $f - c_\lambda s_\lambda$.

From the computational point of view, among the four similar theories listed above, the symmetric functions are the ones that can be better understood. For example, to compute $S^2(S^3C^3)$ consider first the 10 monomials $t_1^3, t_1^2 t_2, \ldots, t_3^3$. Then consider the sum of all product of two of these monomials, which is $t_1^6 + t_1^5 t_2 + \ldots + t_3^6$. This last polynomial can be decomposed as $s_6 + s_4 2$. It follows that $S^2(S^3C^3) = S^6C^3 + S^4C^3$.

Often it is convenient, in practical computations, to consider Newton sum of powers, which behave better according to plethysm.

The Schubert cell $X_\lambda$ is defined as

$$X_\lambda = \{ m \in Gr(B^p, B^p) | \dim m \cap V_{n-k+i} = \gamma_i \geq i, \text{ for } i = 1, \ldots, k+1 \}$$

where $e_0, \ldots, e_n$ is a basis of $V$ and $V_i = \langle e_0, \ldots, e_i \rangle$.

We have the following formulas which show the strong similarities among different theories

$$S^3V \otimes S^6V = \sum_\mu c_{\mu \lambda} S^\mu V$$
for some integer coefficients $c_{\mu
u}$, which repeat in the following

$$s_\lambda \cdot s_\mu = \sum c_{\lambda\mu\nu} s_\nu,$$

$$X_\lambda \cap X_\mu = \sum c_{\lambda\mu\nu} X_\nu.$$

The tensor product of $\Sigma_d$-modules $V_\lambda$ behaves in a different way.

### 3.9. The symbolic representation of invariants

The symbolic representation is an economic way to encode and write down invariants. It works both for invariants of forms, that we consider here, and for invariants of points, that we consider in §5.

It was called by Weyl “the great war-horse of nineteenth century invariant theory”. The reader should not lose the historical article [49]. The “symbolic calculus” is essential to understand the classical sources. In the words of Enriques and Chisini ([17] pag. 37, chap. 1):

“Ma a supplire calcoli laboriosi determinandone a priori il risultato, si può anche far uso del procedimento di notazione simbolica di Cayley-Aronhold, che risponde a questa esigenza economica porgendo un modo sistematico di formazione. L’idea fondamentale contenuta nella rappresentazione simbolica costituisce un fecondo principio di conservazione formale rispetto alle degenerazioni.”

Since every invariant corresponds to a one dimensional representation in $S^m S^d V$, they are spanned by tableau as in Theorem 6.

Every $F \in S^m S^d V$ corresponds to a multilinear function

$$F: S^d V \times \cdots \times S^d V \rightarrow \mathbb{C}$$

and this last is determined by Theor. 1 by $F(x_1^d, \ldots, x_m^d)$ for any linear forms $x_i$, $i=1,\ldots,m$, which is symmetric in the $m$ entries corresponding to our label numbers.

Let $md = g(n+1)$. In the symbolic representation we start from a tableau $T$ filling the Young diagram $g^{n+1}$ with the numbers 1 repeated $d$ times, 2 repeated $d$ times and so on until $m$ repeated $d$ times. The main construction of the symbolic representation is to define an invariant $F_T \in S^m (g_d^{(n+1)})^{SL(n+1)}$, by using the above idea.

**Definition 5.** [From tableau to polynomial invariants] Let $md = g(n+1)$. Let $T$ be a tableau filling the Young diagram of rectangular size $(n+1) \times g$ with the numbers 1 repeated $d$ times, 2 repeated $d$ times and so on until $m$ repeated $d$ times. Let $G_T: \underbrace{S^d V \wedge \cdots S^d V}_m \rightarrow \mathbb{C}$ be the function introduced in Definition 4.

---

1In order to avoid messy computations, by determining in advance their result, it may be used Cayley-Aronhold symbolic representation. It gives a systematic way to construct the invariants by answering the need of simplicity. The fundamental idea of symbolic representation gives a fruitful invariance principle with respect to degeneration.
We denote by $F_T \in S^m(S^dV)$ the polynomial obtained by symmetrizing $G_T$, that is $F_T(h) = G_T(h, \ldots, h)$ for any $h \in S^dV$. $F_T$ is called a symmetrized tableau function.

**Theorem 20.** Any $F_T$ as in Definition 5 is $SL(n+1)$-invariant.

**Proof.** Let $g \in SL(n+1)$ and $h \in S^dV$. We have $F_T(g \cdot h) = G_T(g \cdot h, \ldots, g \cdot h) = G_T(h, \ldots, h) = F_T(h)$, the second equality by Proposition 16.

By the description of representations of $GL(V)$, we get an element $F_T$ in the representation of dimension one $S^1V \subset S^m(S^dV)$, possibly zero. We emphasize that $F_T(x^d) = G_T(x^d, \ldots, x^d) = 0$ for every $x \in V^\vee$, because we get a determinant with equal rows. Nevertheless the symmetrization of Def. 5 is meaningful for general $h \in S^dV^\vee$. It is strongly recommended to practice with some examples in order to understand how the construction of Def. 5 is powerful. Let’s start with the example of the invariant $J$ for binary quartics (compare also with the different example 4.5.7 in [53]).

Fill the Young diagram $\lambda = (6,6)$ with respectively four copies of each 1, 2, 3, obtaining the following

$$T = \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 & 3 & 3 \end{array}$$

It is defined the function $F_T(x^4, y^4, z^4) = (x \wedge y)^2(x \wedge z)^2(y \wedge z)^2$. In the classical literature, this representation was denoted sometimes as


By developing $(x_0y_1 - x_1y_0)^2(x_0z_1 - x_1z_0)^2(y_0z_1 - y_1z_0)^2$, we get 19 monomials. The first monomial is $x_0^2y_1^2z_1^2$ and, according to the correspondence seen in (1), we get $x_0^2 \mapsto a_0, y_1^2 \mapsto a_2, z_1^2 \mapsto a_4$, so that the first monomial corresponds to $a_0a_2a_4$. The following M2 script does automatically this job and it can be adapted to other symbolic expressions.

```m2
S=QQ[x_0,x_1,y_0,y_1,z_0,z_1,a_0..a_4]
f=(x_0*y_1-x_1*y_0)^2*(x_0*z_1-x_1*z_0)^2*(y_0*z_1-y_1*z_0)^2
symb=(x,h)->(contract(x,h)*transpose matrix{{a_0..a_4}})_(0,0)
fxy=symb(symmetricPower(4,matrix{{x_0,x_1}}),f)
fxyz=symb(symmetricPower(4,matrix{{y_0,y_1}}),fxy)
```


The result we get for $F_T$ is (up to scalar multiples) the well known expression of $J$ (compare with (5))

$$F_T = -a_3^2 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 + a_0a_2a_4.$$  

In formula (19), the bracket $[12]$ appears in two among the columns. If $[12]$ appears in three columns, one is forced to repeat 3 on the same column, getting zero. By similar elementary arguments, the reader can easily convince himself that (19) gives the only nonzero invariant in $S^3S^4C^2$.

For higher degree invariants, one meets quickly very large expressions. A computational trick, to reduce the size of the memory used, is to introduce the expression $f$ step by step and to manage the symbolic reduction of any single variable correspondingly. In the following example, the variable $x$ appears already in the first two square factors of $f$, and the symbolic reduction of $x$ can be done just at this step.

$$S=QQ[x_0,x_1,y_0,y_1,z_0,z_1,a_0..a_4]$$
$$ff=(x_0*y_1-x_1*y_0)^2*(x_0*z_1-x_1*z_0)^2$$
$$symb=(x,h)->(contract(x,h)*transpose matrix{a_0..a_4})_(0,0)$$
$$fx=symb(symmetricPower(4,matrix{x_0,x_1}),ff)$$
$$--now we introduce the third square factor$$
$$fxy=symb(symmetricPower(4,matrix{y_0,y_1}),fx*(y_0*z_1-y_1*z_0)^2)$$
$$fxyz=symb(symmetricPower(4,matrix{z_0,z_1}),fxy)$$

Note that $(C^2)^6$ contains for $\lambda=(3,3)$ a $SL(2)$-invariant subspace of dimension equal to $\dim V_{3,3}=5$. The dimension of $V_{m,m}$ is equal to $\frac{1}{m+1}\binom{2m}{m}$, the $m$-th Catalan number.

The main hidden difficulty in the application of the method of "symbolic representation" is that it is hard to detect in advance if a given symbolic expression gives the zero invariant.

3.10. The two Fundamental Theorems for invariants of forms

The First Fundamental Theorem for invariants of forms says that any invariant of forms is a linear combination of the invariants $F_T$ in Definition 5.

**Theorem 21 (First Fundamental Theorem (1FT) for forms).**

Let $md = (n+1)g$. The space of invariants of degree $m$ for $S^dV$ is generated by symmetrized tableau functions $F_T$, constructed by tableau $T$ as in Definition 5.

We postpone the proof until §5, after 1FT for invariants of points (Theorem 28) will be proved.

The theorem extends in a natural way to covariants. For example the “symbolic representation” of the Hessian $H$ defined in 3 is

$$
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 1 \\
\end{array}$$

or $H = [12]^2[1y][2y]$. 

The second fundamental theorems describes the relations between these invariants.

**Theorem 22 (Second Fundamental Theorem (2FT) for forms).**

The relations among the generating invariants \( F_T \) of Theorem 21 are generated by Plücker relations like in Remark 2.

More precisely, fix a subset of \( k + 2 \) elements \( i_0 \ldots i_{k+1} \) and a set of \( k \) elements \( j_0 \ldots j_{k-1} \). Let \( T_s = [i_0 \ldots i_s \ldots i_{k+1}][i_s j_0 \ldots j_{k-1}] \) a \( 2 \times (k + 1) \) tableau and assume the numbers appearing are as in Definition 5 (this does not depend on \( s \) because every \( T_s \) permutes the same numbers). Then the Plücker relations are

\[
\sum_{s=0}^{k+1} (-1)^s F_{T_s} = 0,
\]

which hold for any subsets of respectively \( k + 2 \) and \( k \) elements.

This description gives unfortunately cumbersome computations.

4. Hilbert series of invariant rings. Some more examples of invariants

4.1. Hilbert series

In all the examples where a complete system of invariants (or covariants) is known, the following steps can be performed

- (i) compute the Hilbert series of the invariant ring.
- (ii) guess generators of the corrected degree.
- (iii) compute the syzygies among the generators of step (ii), hence compute the subalgebra generated by these generators.
- (iv) check if the subalgebra coincides at any degree with the algebra by comparing the two Hilbert series.

The algebra of covariants has the bigraduation

\[
\text{Cov}(S^d \mathbb{C}^2) = \bigoplus_{n,e} \text{Hom}^{SL(2)}(S^n(S^d \mathbb{C}^2), S^e \mathbb{C}^2)
\]

and its Hilbert series depends correspondingly on two variables \( z, w \)

\[
F_d(z,w) = \dim \text{Cov}(S^d \mathbb{C}^2)_{n,e} z^n w^e
\]

in such a way that the coefficient of \( w^b z^b \) denotes the dimension of

\[
\text{Hom}^{SL(2)}(S^b \mathbb{C}^2, S^n(S^d \mathbb{C}^2)).
\]
The exponent of \( z \) is the degree, the exponent of \( w \) is called the order. The covariants of order zero coincide with the invariants.

In 1980 Springer has found an efficient algorithm for the computation of Hilbert series for binary forms, by using residues. Variations of this algorithm may be found in Procesi’s book \[47\] chap. 15, §3 or in Brion’s notes \[3\]. In the following we follow \[3\].

Denote \( \Phi_j : \mathbb{C}[z] \to \mathbb{C}[z] \) the linear map given by

\[
\Phi_j(z^n) = \begin{cases} 
  z^{n/j} & \text{if } n \equiv 0 \mod j \\
  0 & \text{otherwise}
\end{cases}
\]

\[
\Phi_1(z^n) = z^n \\
\Phi_2(z^n) = \begin{cases} 
  z^{n/2} & \text{if } n \text{ is even} \\
  0 & \text{otherwise}
\end{cases} \\
\Phi_3(z^n) = \begin{cases} 
  z^{n/3} & \text{if } n \equiv 0 \mod 3 \\
  0 & \text{otherwise}
\end{cases}
\]

We have the equality \( \Phi_j(ab) = \Phi_j(a) \Phi_j(b) \), if \( a \in \mathbb{C}[z] \) (or \( b \in \mathbb{C}[z'] \)). The map \( \Phi_j \) extends to a unique linear map \( \Phi_j : \mathbb{C}((z)) \to \mathbb{C}((z)) \) which again satisfies the above equality.

**Theorem 23.** Let \( M = \oplus_{r} \text{Hom}_{SL(2)}^{\text{SL}(2)}(S^n(S^d \mathbb{C}^2), S^e \mathbb{C}^2) \). The Hilbert series is

\[
F_M(z) = \sum_{0 \leq j < d/2} (-1)^j \Phi_{d-2j} \left( (1 - z^2) \gamma_{d,j}(z) \right)
\]

where

\[
\gamma_{d,j}(z) = \frac{z^{(j+1)}}{\prod_{k=1}^{d-j} (1 - z^{2k}) \prod_{l=1}^{d-j} (1 - z^{2l})}.
\]

**Proof.** \[3\]

**Corollary 5.**

\[
F_d(z,w) = \sum_{0 \leq j < d/2} (-1)^j \Phi_{d-2j} \left( \frac{1 - z^2}{1 - zw} \gamma_{d,j}(z) \right).
\]

The formula of Corollary 5 can be used to compute the Hilbert series of covariant rings of binary forms of small degree.

We reproduce for completeness some of the results, although in the following we will reprove some of them with tools from representation theory (like in §5.3), when available.

We have

\[
F_3(z,0) = \Phi_3 \left( (1 - z^2) \gamma_{3,0}(z) \right) - \Phi_1 \left( (1 - z^2) \gamma_{3,1}(z) \right) = \Phi_3 \left( \frac{1}{(1 - z^2)(1 - z^6)} \right) - \Phi_1 \left( \frac{z^2}{(1 - z^4)(1 - z^6)} \right)
\]
and get

\[ F_3(z, 0) = \frac{1}{1-z^2} \Phi_3 \left( \frac{1}{1-z^2} \right) - \frac{z^2}{(1-z^2)(1-z^4)} = \frac{1}{1-z^2} \frac{z^2}{(1-z^2)(1-z^4)} = \frac{1}{1-z^4}. \]

Moreover

\[ F_4(z, 0) = \Phi_4 \left( (1-z^2)\gamma_{4,0}(z) \right) - \Phi_2 \left( (1-z^2)\gamma_{4,1}(z) \right) = \phi_4 \left( \frac{1}{(1-z^2)(1-z^6)} \right) - \phi_2 \left( \frac{z^2}{(1-z^2)(1-z^4)(1-z^8)} \right) \]

and get

\[ F_4(z, 0) = \frac{1}{(1-z)(1-z^2)} \Phi_4 \left( \frac{1}{1-z^6} \right) - \frac{z}{(1-z)(1-z^2)(1-z^4)} = \frac{1}{(1-z)(1-z^2)} \frac{z}{(1-z)(1-z^2)(1-z^4)} = \frac{1}{(1-z^2)(1-z^4)}. \]

For the Hilbert series of covariants we get

\[(21)\]

\[ F_3(z, w) = \frac{1+z^3w^3}{(1-z^3)(1-zw^3)(1-z^2w^2)}, \]

\[ F_4(z, w) = \frac{1+z^3w^6}{(1-z^2)(1-z^2)(1-zw^4)(1-z^2w^4)}. \]

**4.2. Covariant ring of binary cubics**

Let \( f = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3 \). From the series (21), computed also in [53] (4.2.17), we can guess the table of covariants, namely

<table>
<thead>
<tr>
<th>order</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
Indeed we know some covariants exactly of the expected degrees, namely

<table>
<thead>
<tr>
<th>degree</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>f</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>H</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Q</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Δ</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $\Delta = [12]^2[13][24][34]^2$ is the discriminant, $H = (f, f)_2 = [12]^2[1x][2x]$ is the Hessian, $Q = (f, H)_1$. The Hessian vanishes identically on the twisted cubic, indeed its plain expression is (we divide by 36 for convenience)

\[ H = \frac{1}{36} (f_{00}f_{22} - f_{12}^2) = (a_1a_3 - a_2^2)x^2 + (a_0a_3 - a_1a_2)xy + (a_0a_2 - a_1^2)y^2. \]

The condition $H = 0$ represents the two points such that, together with $f = 0$, make a equianharmonic 4ple (see §4.4).

The condition $Q = 0$ represents the three points such that, together with $f = 0$, make a harmonic 4ple (see §4.4).

To fix the scalars, we pose

\[ \Delta = 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2) - (a_0a_3 - a_1a_2)^2. \]

Then there is a unique syzygy which is $36H^3 + 9\Delta f^2 + Q^2 = 0$, note that $Q^2$ is expressed rationally by the others. Indeed the Hilbert series is

\[ F_3(z, w) = \frac{1 + z^3w^3}{(1 - z^4)(1 - zw)(1 - z^2w^2)}. \]

The three factors at the denominator correspond respectively to $\Delta, f, H$. The Hilbert series says that the subalgebra generated by these covariants coincides with the covariant ring. In particular the invariant ring is free and it is generated by $\Delta$. We have proved

**Theorem 24.** The covariants of a binary cubic $f$ are generated by $\Delta, H, f$ and $Q$, satisfying the single relation

\[ 36H^3 + 9\Delta f^2 + Q^2 = 0. \]

We recommend reading lecture XXI in [24] 1.8, where the solution of the cubic equation is obtained from the relation of Theorem 24.
4.3. Apolarity and transvectants

Let $V$ be a vector space with basis $x_0, \ldots, x_n$ and let

$$R = K[x_0, \ldots, x_n] = \bigoplus_{m=0}^{\infty} S^m V$$

be the polynomial ring. Let’s recall that the dual ring

$$R^\vee = K[\partial_0, \ldots, \partial_n] = \bigoplus_{m=0}^{\infty} S^m V^\vee$$

can be identified with ring of polynomial differential operators, where $\partial_i = \frac{\partial}{\partial x_i}$.

The action of $R^\vee$ over $R$ was classically called as apolarity. In particular for any integers $e \geq d \geq 0$ we have the apolar linear maps

$$S^d V^\vee \otimes S^e V \to S^{e-d} V.$$

When dim $U = 2$, that is for polynomials over a projective line, the apolarity is well defined for $f, g$ both in $S^d U$. This is due to the canonical isomorphism $U \simeq U^\vee \otimes \wedge^2 U$.

This allows to make explicit the $SL(2)$-decompositions

$$S^d C^2 \otimes S^e C^2 = \bigoplus_{i=0}^{\min(d,e)} S^{d+e-2i} C^2, \quad S^2 (S^d C^2) = \bigoplus_{i=0}^{\lfloor d/2 \rfloor} S^{2d-4i} C^2.$$

Let $(x_0, x_1)$ be coordinates on $U$. If $f = (a_0 x_0 + a_1 x_1)^d$ and $g = (b_0 x_0 + b_1 x_1)^d$, then the contraction between $f$ and $g$ is seen to be proportional to $(a_0 b_1 - a_1 b_0)^d$. This computation extends by linearity to any pair $f, g \in S^d U$, because any polynomial can be expressed as a sum of $d$-th powers. The resulting formula for $f = \sum_{i=0}^{d} \binom{d}{i} f_i x^{d-i} x^i$ and $g = \sum_{i=0}^{d} \binom{d}{i} g_i x^{d-i} x^i$ is that $f$ is apolar to $g$ if and only if

$$\sum_{i=0}^{d} (-1)^i \binom{d}{i} f_i g_{d-i} = 0.$$

In order to prove this formula, by linearity and by Theorem 1 it is again sufficient to assume $u^d$ and $v^d$. In particular

**Proposition 17.** Let $p, t^d \in S^d U$. $p$ is apolar to $t^d$ if and only if $l$ divides $p$.

**Proposition 18.** If $d$ is odd, any $f \in S^d U$ is apolar to itself. Apolarity defines a skew nondegenerate form in $\wedge^2 (S^d C^2)$.

If $d$ is even, the condition that $f$ is apolar to itself defines a smooth quadric in $\mathbb{P} S^d U$. Apolarity defines a symmetric nondegenerate form in $S^2 (S^d C^2)$.

From the geometric point of view, let $f = l_1 \ldots l_d \in S^d C^2$ be the decomposition in linear factors and denote by $[f]$ the corresponding point in $\mathbb{P} S^d C^2$. Let $P(f)$ be the hyperplane spanned by $\{l_1^d, \ldots, l_d^d\}$. $f$ is apolar to itself if and only if $[f] \in P(f)$.

The natural algebraic generalization of the apolarity is given by transvection.
If $u^d \in S^d U$, $v^e \in S^e U$ we define the $n$-th transvectant to be\[ (u^d, v^e)_n = u^{d-n} v^{e-n} [uv]^n \in S^{d+e-2n} \mathbb{C}^2. \]

If $f \in S^d U$, $g \in S^e U$ and $0 \leq n \leq \min(d, e)$, the general $\text{SL}(2)$-invariant formula is
\[
(f, g)_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\partial f}{\partial x^{d-i} y^i} \frac{\partial g}{\partial x^i y^{e-n-i}}.
\]
Note that $(f, g)_1$ is the Jacobian, while $(f, f)_2$ is the Hessian.

For $f \in S^4 \mathbb{C}^2$ we can express the invariants $I$ and $J$ introduced in §1.2 in terms of transvectants. Indeed it is easy to check that $I = (f, f)_2$, $J = (f, (f, f)_2)_4$. This gives a recipe to compute the expressions of $I$ and $J$, that can be extended to other situations. A Theorem of Gordan states that all invariants of binary forms can be expressed by the iterate application of transvectants. We will not pursue this approach here. Transvectants are close to symbolic representation of §3.9, see [41] example 6.26.

4.4. Invariant ring of binary quartics

A polynomial $f = f_0 x^4 + 4 f_1 x^3 y + 6 f_2 x^2 y^2 + 4 f_3 x y^3 + f_4 y^4 \in S^4 U$ is called equianharmonic if its apolar to itself. So $f$ is equianharmonic if and only if
\[ f_0 f_4 - 4 f_1 f_3 + 3 f_2^2 = 0, \]
which is the expression for the classical invariant $I$ of binary quartics, see (4).

The symbolic expression is $[12]^4$ or
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2
\end{bmatrix}
\]

The invariant $J$ (see (5)) has the symbolic expression $[12]^2[13]^2[23]^2$ or
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 \\
2 & 2 & 3 & 3 & 3 & 3
\end{bmatrix}
\]

It is equal to the determinant
\[
J = \begin{vmatrix} f_0 & f_1 & f_2 \\ f_1 & f_2 & f_3 \\ f_2 & f_3 & f_4 \end{vmatrix}.
\]

A binary quartic with vanishing $J$ is called harmonic. A binary quartic $f$ is harmonic if and only if $f$ has an apolar quadratic form, if and only if $f$ is sum of two 4th powers, instead of the three which are needed for the general $f$.\footnote{translated “scorrimento” in [17], from the original German “ueber-schiebung.”}
Theorem 25. Let \( d = 4 \), let \( I_{g,2g,4} \) be the dimension of the space of invariants of degree \( g \) of the binary quartic. Then
\[
\sum_{g=0}^{\infty} I_{g,2g,4} x^g = \frac{1}{(1-x^2)(1-x^4)}.
\]
The ring of invariants is freely generated by \( I,J \), that is \( \oplus_{m} S^m(S^4(C^2))_{SL(2)}^{SL(2)} = \mathbb{C}[I,J] \).

Proof. The series has been shown (without proof) in §4.1. We sketch the proof by following again [24], in a way similar to the proof of Corollary 3. We will see a different proof in Theorem 36.

We have from Corollary 17 that \( I_{g,2g,4} \) is the degree \( 2g \) coefficient of
\[
\frac{(1-x^4+1) \ldots (1-x^{4g})}{(1-x^2) \ldots (1-x^g)} = \frac{(1-x^{g+1})(1-x^{g+2})(1-x^{g+3})(1-x^{g+4})}{(1-x^2)(1-x^3)(1-x^4)}
\]
and we write it as
\[
\left\{ \frac{(1-x^{g+1})(1-x^{g+2})(1-x^{g+3})(1-x^{g+4})}{(1-x^2)(1-x^3)(1-x^4)} \right\}_{2g}.
\]

We can remove the terms which do not change the coefficient of \( x^{2g} \), so getting
\[
\left\{ \frac{1}{(1-x^2)(1-x^3)(1-x^4)} \right\}_{2g} = \left\{ x(1+x+x^2+x^3) \right\}_{2g} = \left\{ x(1+x+x^2+x^3)(1-x) \right\}_{2g}
\]
(by using that \( (1-x^4) = (1+x+x^2+x^3)(1-x) \))
\[
= \left\{ \frac{1}{(1-x^2)(1-x^3)(1-x^4)} \right\}_{2g} = \left\{ \frac{x}{(1-x^2)(1-x^3)(1-x^4)} \right\}_{2g}
\]
\[
= \left\{ \frac{x^2}{(1-x^2)(1-x^3)(1-x^4)} \right\}_{2g} = \left\{ \frac{1+x^3-x^2}{(1-x^3)(1-x^4)} \right\}_{2g}
\]
Since in the denominator only even powers appear, we can remove \( x^3 \) from the numerator and we get
\[
= \left\{ \frac{1}{(1-x^3)(1-x^4)} \right\}_{2g} = \left\{ \frac{1}{(1-x^2)(1-x^4)} \right\}_{2g},
\]
as we wanted.

For the last assertion, consider the subring
\( \mathbb{C}[I,J] \subset \oplus_{m} S^m(S^4(C^2))_{SL(2)}^{SL(2)} \).

In order to prove the equality, it is enough to show that the two rings have the same Hilbert series. We have proved that the Hilbert series of the invariant ring
\( G. \) Ottaviani

\( \odot_m S^0(S^1(C^2))^ {SL(2)} \) is \( \frac{1}{(1-x^2)(1-x^3)} \), which is the Hilbert series of a ring with two algebraically independent generators of degree 2 and 3. The invariants \( I \) and \( J \) have respectively degree 2 and 3. So it is enough to prove that \( I, J \) are algebraically independent.

Assume we have a relation in degree \( g \) between \( I, J \), that is a relation
\[
\sum_{k+l=g} c_{kl} I^k J^l = 0
\]
which hold identically for any \( a_0, \ldots, a_4 \). Since
\[
I(a_0,0,0,a_3,a_4) = a_0 a_4, \quad J(a_0,0,0,a_3,a_4) = -a_0 a_3^2
\]
we get
\[
\sum_{k+l=g} c_{kl} (-1)^k a_0^{k+l} a_3^l a_4^k = 0.
\]
All \( k \), as well as all \( l \), are distinct, because every \( k \) determines uniquely \( l \) and conversely. It follows get \( c_{kl} = 0 \) \( \forall k, l \), as we wanted.

Note that in degree 6 there are two independent invariants, \( I^3 \) and \( J^2 \) given respectively by

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
4 & 4 & 4 & 4 & 5 & 5 & 5 & 6 \\
\end{array}
\]
and

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 4 & 4 \\
2 & 2 & 3 & 3 & 3 & 5 & 5 & 6 \\
\end{array}
\]

It is not trivial to show directly that all the semistandard \( 2 \times 12 \) tableau functions give, under the Plücker relations, a linear combination of these two.

Indeed the Hilbert series we computed in §4.1 is
\[
F_4(z,w) = \frac{1 + z^3 w^6}{(1-z^2)(1-z^3)(1-z^4)(1-z^5 w^4)}
\]
The factors at the denominator correspond respectively at \( I, J, f, H \).

The syzygy represents \( Q^2 \) as a combination of \( J f^3, I f^2 H, H^3 \).

\( I^3/J^2 \) is an absolute invariant, \( \Delta = I^3 - 27 J^2 \) is the discriminant.

The table of covariants is the following, where in the first column we read \( I, J, f, H \), in the column labeled with 4 we find respectively \( f, H \) and in the last column we find \( Q = (f,H) \).

<table>
<thead>
<tr>
<th>order</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The vanishing of \( Q \) as covariant (meaning that its seven cubic generators all vanish, express the fact that the quartic is a square). So they give the equations of a
classically well known surface, which is the projection in $\mathbb{P}^4$ of the Veronese surface from a general point in $\mathbb{P}^5$.

For an extension to binary forms of any degree see [1].

$Q = 0$ represents the three pairs of double points for the three involutions which leave $f$ invariant.

4.5. SL(2) as symplectic group. Symplectic construction of invariants for binary quartics

The content of this subsection was suggested by Francesco Ghéraldelli several years ago. I report here with the hope that somebody could be interested and take this idea further.

The starting point is that $SL(2)$ can be seen as the symplectic group, preserving $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Many invariants for binary forms of even degree can be computed in a symplectic setting. In the example of binary quartics we have $a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4 = \begin{bmatrix} x^2 & 2 x y & y^2 \end{bmatrix} \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} x^2 \\ 2 x y \\ y^2 \end{bmatrix}$

Define for $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, $x = \begin{bmatrix} x' \\ y' \end{bmatrix}$ and note that, setting

$f(g) = S^2 g = \begin{bmatrix} 2 \alpha \delta & 2 \alpha \gamma & \beta \gamma & \beta \delta \\ \alpha \delta & \alpha \gamma & 2 \beta \gamma & 2 \beta \delta \\ \beta \gamma & \beta \delta & \gamma \delta & \delta^2 \\ \beta \delta & \beta \gamma & \gamma \delta & \gamma^2 \end{bmatrix}$, we get

$\begin{bmatrix} x'^2 \\ 2 x'y' \\ y'^2 \end{bmatrix} = f(g) \begin{bmatrix} x^2 \\ 2 x y \\ y^2 \end{bmatrix}.$

Hence $f(g)^T \begin{bmatrix} 0 & 0 & 1 \\ - \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} f(g) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & - \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and we get

$$\det \begin{bmatrix} a_0 & a_1 & a_2 + t \\ a_1 & a_2 - \frac{t}{2} & a_3 \\ a_2 + t & a_3 & a_4 \end{bmatrix} = \frac{t^3}{2} + t \cdot I(a_1) + J(a_1).$$

The beauty of this formula is that the two invariants $I$ and $J$, coming respectively from (4) and (5), are defined at once. What happens for higher degree?

Remark 7. The reader can find something similar at the end of Procesi’s book [47]. In [47], $S^d \mathbb{C}^2$ is considered inside $S^{d-2} \mathbb{C}^2 \otimes S^{d-2} \mathbb{C}^2 \simeq End(S^{d-2} \mathbb{C}^2)$, in the case $d = 4k$. The coefficients of this characteristic polynomial are conjecturally the generators of the invariant ring for $S^d \mathbb{C}^2$. 
4.6. The cubic invariant for plane quartics

This is the easiest example of invariant of ternary forms defined by the symbolic representation of §3.9. Let \((x_0, x_1, x_2)\) be coordinates on a 3-dimensional complex space \(V\) and \((y_0, y_1, y_2)\) be coordinates on \(V^\vee\). Let

\[
f(x_0, x_1, x_2) = \sum_{i+j+k=4} \frac{4!}{i!j!k!} f_{ijkl} x_0^i x_1^j x_2^k \in S^4 V
\]

be the equation of a plane quartic curve on \(\mathcal{P}(V)\). By Corollary 2, all invariants of \(f\) have degree which is multiple of 3. The invariant of smallest degree has degree 3 and it is defined by the tableau

\[
T = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3
\end{array}
\]

We denote \(A = F_T\). The trilinear form \(A(f, g, h)\), for \(f, g, h \in S^4 V\) satisfies

\[
A(a^4, b^4, c^4) = (a \wedge b \wedge c)^4,
\]

or, expanding the linear forms

\[
A((a_0 x_0 + a_1 x_1 + a_2 x_2)^4, (b_0 x_0 + b_1 x_1 + b_2 x_2)^4, (c_0 x_0 + c_1 x_1 + c_2 x_2)^4) = \begin{vmatrix}
a_0 & a_1 & a_2 \\
b_0 & b_1 & b_2 \\
c_0 & c_1 & c_2
\end{vmatrix}^4.
\]

The explicit expression of the cubic invariant \(A\) can be found at art. 293 of Salmon’s book [50], it can be checked with the M2 script of 3.9 and it is the sum of the following 23 summands (we denote \(A(f)\) for \(A(f, f, f)\))

\[
A(f) = f_{000} f_{00} f_{0} + 3 f_{000} f_{00} f_{0} + 4 f_{00} f_{0} f_{00} + 12 (f_{00} f_{0} f_{000} + f_{00} f_{00} f_{0} + f_{000} f_{0} f_{0}) + 6 f_{00} f_{0} f_{00} f_{0} - 4 (f_{00} f_{0} f_{0} f_{00} + f_{00} f_{00} f_{0} f_{0} + f_{00} f_{000} f_{0}) + 4 f_{00} f_{00} f_{0} f_{00} f_{000} - 12 (f_{00} f_{0} f_{000} f_{0} f_{0} + f_{000} f_{0} f_{0} f_{00} f_{0} + f_{0} f_{000} f_{00} f_{0} f_{000} + f_{0} f_{0} f_{000} f_{00} f_{0} f_{000} + f_{0} f_{0} f_{000} f_{000} f_{0} f_{0} + f_{0} f_{000} f_{000} f_{0} f_{0} f_{0} + f_{0} f_{0} f_{000} f_{000} f_{0} f_{0} + f_{0} f_{0} f_{000} f_{000} f_{0} f_{0}) + 12 (f_{00} f_{0} f_{000} f_{0} f_{0} f_{0} + f_{000} f_{0} f_{0} f_{00} f_{0} f_{0} + f_{00} f_{0} f_{0} f_{000} f_{0} f_{0} f_{0} + f_{0} f_{000} f_{0} f_{0} f_{0} f_{000} + f_{0} f_{0} f_{000} f_{0} f_{0} f_{000} + f_{0} f_{0} f_{000} f_{0} f_{0} f_{00} f_{0} f_{0} + f_{0} f_{0} f_{000} f_{0} f_{0} f_{000} f_{0} f_{0}).
\]

This expression for the cubic invariant can be found also by applying the differential operators \(D_1, D_2\) defined in §3.3 to the space of isobaric monomials of degree 3 and total weight \([4, 4, 4]\). Indeed there are 23 such monomials and the only linear combination of these monomials which is killed by the differential operators is a scalar multiple of the cubic invariant.

The differential operators are analogous to (12) and are

\[
\sum_{i_0 + \ldots + i_2 = 4} i_1 f_{i_0, i_1}^{i_1} f_{i_2}^{i_2} \frac{\partial I}{\partial f_{i_0}^{i_0}} = 0,
\]

\[
\sum_{i_0 + \ldots + i_2 = 4} i_2 f_{i_0, i_1}^{i_1} f_{i_2}^{i_2} \frac{\partial I}{\partial f_{i_0}^{i_0}} = 0.
\]
It is enough to impose $19 + 19 = 38$ conditions to the 23-dimensional space. Compare with [53] example 4.5.3 where similar computations were performed for the ternary cubic.

Note that given $f, g \in S^4 V$, the equation $A(f, g, l^4) = 0$ defines an element in the dual space $S^4 V^\vee$, possibly vanishing.

**Proposition 19.** (i) $A(f, g, l^4) = 0$ if and only if the restrictions $f|_l, g|_l$ to the line $l = 0$ are apolar.

(ii) Let $A(f, g, *) = H$. We have $A(f, g, l^4) = 0$ if and only if $H(l) = 0$.

**Proof.** To prove (i), consider $f = (\sum_{i=0}^2 a_i x_i)^4, g = (\sum_{i=0}^2 b_i x_i)^4, l = x_2$. Then

$$A(f, g, l^4) = \left| \begin{array}{ccc} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ 0 & 0 & 1 \end{array} \right|^4 = \left| \begin{array}{cc} a_0 & a_1 \\ b_0 & b_1 \end{array} \right|^4 = f|_l \cdot g|_l.$$

This formula extends by linearity to any $f, g$.

(ii) follows because $H(l) = H \cdot l^4$.

**Remark 8.** Let $l_1, l_2$ be two lines. $A(l_1^4, l_2^4, f) = 0$ gives the condition that $f$ passes through the intersection point $l_1 = l_2 = 0$.

Note also from Prop. 19 that $A(f, f, l^4) = 0$ if and only if $f$ cuts $l$ in an equianharmonic 4-tuple. The quartic curve $A(f, f, *)$ in the dual space is called the **equianharmonic envelope** of $f$. It is sometimes called a “contravariant”. The “transfer principle of Clebsch” says that from the symbolic expression $(ab)(cd) \ldots$ for a invariant it follows $(ab^*)(cd^*) \ldots$ for the (envelope) contravariant.

This gives the classical geometric interpretation of the cubic invariant for plane quartics. The condition $A(f, f, f) = 0$ means that $f$ is apolar with its own equianharmonic envelope (see [12]), note that it gives a solution to Exercise (1) in the last page of [53].

**4.7. The Aronhold invariant as a pfaffian**

Another classical invariant of ternary forms is the Aronhold invariant for plane cubics, it is defined by the tableau

$$T = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{bmatrix}$$

We denote by $Ar$ the corresponding multilinear form $G_T$ and also its symmetrization $F_T$. We get

$$Ar(x^3, y^3, z^3, w^3) = (x \wedge y \wedge z)(x \wedge y \wedge w)(x \wedge z \wedge w)(y \wedge z \wedge w).$$
The expression of the Aronhold invariant $Ar$ has 25 monomials and it can be found in [53] Prop. 4.4.7 or in [16] (5.13.1). The Aronhold invariant is a “lucky” case, were the geometric interpretation follows easily from the symbolic notation. It is not a surprise that was one of the first examples leading Aronhold to the symbolic notation. If a plane cubic $f$ is sum of three cubes (namely, it is $SL(3)$-equivalent to the Fermat cubic $f = x_0^3 + x_1^3 + x_2^3$) we have $Ar(f) = 0$. Indeed $Ar(f, f, f)$ splits as a sum of $Ar(x_0^3, x_1^3, x_2^3)$ where $i_k \in \{0, 1, 2\}$, so that $\{i_0, i_1, i_2, i_3\}$ contains at least a repetition, in such a way that all summands contributing to $Ar(f, f, f)$ vanish.

Let $W$ be a vector space of dimension 3. In particular $\Gamma^{2, 1}W = ad W$ is self-dual and it has dimension 8. We refer to [42] Theor. 1.2 for the proof of the following result, see also [33] example 1.2.1.

**Theorem 26.** For any $\phi \in S^3W$, let $A_\phi: \Gamma^{2, 1}W \to \Gamma^{2, 1}W$ be the $SL(V)$-invariant contraction operator. Then $A_\phi$ is skew-symmetric and the pfaffian $Pf A_\phi$ is the equation of $\Omega(3)$, i.e. it is the Aronhold invariant $Ar$.

The corresponding picture is

\[ \begin{array}{ccc} \otimes & * & * \\ & * \\ \end{array} \rightarrow \begin{array}{ccc} & * \\ & * \end{array} \cong \begin{array}{ccc} \end{array} \]

The $SL(W)$-module $\text{End}_0 W$ consists of the subspace of endomorphisms of $W$ with zero trace. The contraction

$$A_\phi: \text{End}_0 W \to \text{End}_0 W$$

in the case $\phi = v^3$ satisfies

$$A_{\phi^3}(M)(w) = (M(v) \wedge v \wedge w) v$$

where $M \in \text{End} W, w \in W$.

**Remark 9.** We recall from [16] the definition of the Scorza map. For any plane quartic $F$ and any point $x \in \mathbf{P}(W)$ we consider the polar cubic $P_x(F)$. Then $Ar(P_x(F))$ is a quartic in the variable $x$ which we denote by $S(F)$. The rational map $S: \mathbf{P}(S^4W) \dashrightarrow \mathbf{P}(S^4W)$ is called the Scorza map. Our description of the Aronhold invariant shows that $S(F)$ is defined as the degeneracy locus of a skew-symmetric morphism on $\mathbf{P}(W)$

$$\Omega(-2)^8 \xrightarrow{f} \Omega(-1)^8.$$ 

It is easy to check (see [6]) that $\text{Coker } f = E$ is a rank two vector bundle over $S(F)$ such that $c_1(E) = K_{S(F)}$.

I owe to A. Buckley the claim that from $E$ it is possible to recover the even theta characteristic $\theta$ on $S(F)$ defined in [16, (7.7)] (see also next section), by following a construction by C. Pauly [44]. There are exactly eight maximal line subbundles $P_i$ of
These eight line bundles are related by the equality (Lemma 4.2 in [44])

$$\otimes_{i=1}^8 P_i = K_{S(F)}^2.$$

The construction in [44] §4.2 gives a net of quadrics in the following way. For the general stable $E$ as ours, there exists a unique stable bundle $E'$ with $c_1(E') = \Omega$ such that $h^0(E \otimes E')$ has the maximal value 4. The extensions

$$0 \rightarrow P_i \rightarrow E \rightarrow K_{S(F)} \otimes P_i^{-1} \rightarrow 0$$

$$0 \rightarrow P_i^{-1} \rightarrow E' \rightarrow P_i \rightarrow 0$$

define eight sections in $Hom(E', E) \simeq E \otimes E'$ as the compositions

$$E' \rightarrow P_i \rightarrow E$$

which give eight points in $\mathcal{P}H^0(E \otimes E')$. These eight points are the base locus for a net of quadrics, which gives a symmetric representation of the original quartic curve $S(F)$ and then defines an even theta characteristic.

### 4.8. Clebsch and Lüroth quartics. Theta characteristics

A plane quartic $f \in S^4V$ is called Clebsch if it has an apolar conic, that is if there exists a nonzero $q \in S^2V^\vee$ such that $q \cdot f = 0$.

One defines, for any $f \in S^4V$, the catalecticant map $C_f: S^2V^\vee \rightarrow S^2V$ which is the contraction by $f$. The equation of the Clebsch invariant is easily seen as the determinant of $C_f$, that is we have (cf. [16], example (2.7))

**Theorem 27 (Clebsch).** A plane quartic $f$ is Clebsch if and only if $\det C_f = 0$. The conics which are apolar to $f$ are the elements of $ker \ C_f$.

It follows (cf. [14], Lemma 6.3.22) that the general Clebsch quartic can be expressed as a sum of five 4-th powers, that is

$$f = \sum_{i=0}^4 l_i^4.$$

A general Clebsch quartic $f$ can be expressed as a sum of five 4-th powers in $\infty!$ many ways. Precisely the 5 lines $l_i$ belong to a unique smooth conic $Q$ in the dual plane, which is apolar to $f$ and it is found as the generator of $ker \ C_f$. Equivalently, the 5 lines $l_i$ are tangent to a unique conic, which is the dual conic of $Q$.

We recall that a theta characteristic on a general plane quartic $f$ is a line bundle $\Theta$ on $f$ such that $\Theta^2$ is the canonical bundle. Hence deg $\Theta = 2$. There are 64 theta characteristic on $f$. If the curve is general, every bitangent is tangent in two distinct points $P_1$ and $P_2$, and the divisor $P_1 + P_2$ defines a theta characteristic $\Theta$ such that $h^0(\Theta) = 1$, as
these are called odd theta characteristic and there are 28 of them. The remaining 36 theta characteristic $\theta$ are called even and they satisfy $h^0(\theta) = 0$ (for any curve, even theta characteristics have even $h^0(\theta)$).

The Scorza map is the rational map from $\mathbb{P}^4 = \mathbb{P}(S^4V)$ to itself which associates to a quartic $f$ the quartic $S(f) = \{x \in \mathbb{P}(V) | Ar(P_x(f)) = 0\}$, where $P_x(f)$ is the cubic polar to $f$ at $x$ and $Ar$ is the Aronhold invariant [14,16,51]. It is well known that it is a 36 : 1 map. Indeed the curve $S(f)$ is equipped with an even theta characteristic. For a general quartic curve, its 36 inverse images through the Scorza map give all the 36 even theta characteristic.

A Lüroth quartic is a plane quartic containing the ten vertices of a complete pentalateral, or the limit of such curves.

If $l_i$ for $i = 0, \ldots, 4$ are the lines of the pentalateral, we may consider as divisor (of degree 4) over the curve. Then $l_0 + \ldots + l_4$ consists of 10 double points, the meeting points of the 5 lines. Let $P_1 + \ldots + P_0$ the corresponding reduced divisor of degree 10. Then $P_1 + \ldots + P_0 = 2H + \theta$ where $H$ is the hyperplane divisor and $\theta$ is a even theta characteristic, which is called the pentalateral theta. The pentalateral theta was called pentagonal theta in [16], and it coincides with [14], Definition 6.3.30 (see the comments thereafter).

The following result is classical [51], for a modern proof see [14,16].

**Proposition 20.** Let $f$ be a Clebsch quartic with apolar conic $Q$, then $S(f)$ is a Lüroth quartic equipped with the pentalateral theta corresponding to $Q$.

**Proof.** Let $f = \sum_{i=1}^{5} l_i^4$. Let $l_p$ and $l_q$ be two lines in the pentalateral and let $x_{pq} = l_p \cap l_q$. Then

$$P_{x_{pq}}(f) = \sum_{i=1}^{5} P_{x_{pq}}(l_i^4) = \sum_{i=1}^{5} 4l_i^3 P_{x_{pq}}(l_i).$$

In the above sum at most three summands survive, because the ones with $i = p, q$ are killed. Then $P_{x_{pq}}(f)$ is a Fermat cubic and $Ar(P_{x_{pq}}(f)) = 0$, hence $x_{pq} \in S(f)$. It follows that $S(f)$ is inscribed in the pentalateral and it is Lüroth.

The number of pentalateral theta on a general Lüroth quartic, called $\delta$, is equal to the degree of the Scorza map when restricted to the hypersurface of Clebsch quartics.

Explicitly, if $f$ is Clebsch with equation

$$l_0^4 + \ldots + l_4^4,$$

then $S(f)$ has equation

$$\sum_{j=0}^{4} k_i \prod_{j \neq i} l_j,$$

where $k_i = \prod_{p < q < r | \{p, q, r\} \neq \varnothing} [l_p l_q l_r]$ (see [14], Lemma 6.3.26) so that $l_0, \ldots , l_4$ is a pentalateral inscribed in $S(f)$. Note that the conic where the five lines which are the summands of $f$ are tangent, is the same conic where the pentalateral inscribed in $S(f)$ is tangent.
Remark 10. The degree of Lüroth invariant is 54. This has been proved by Morley in 1919 [38], see [43] for a review of his nice proof. To have the flavour of the complexity, think that the space of monomials of degree 54 in the 15 variables \( a_{ijk} \) with \( i+j+k = 4 \) has dimension \( \binom{58}{14} \approx 10^{14} \), while the space of isobaric ones has dimension 62,422,531,333 \( \approx 10^{11} \).

The dimension of the space of invariants was computed first by Shioda [52], it is 1165. Ohno computes this space as dimension 1380 with 215 relations. This is reviewed by Basson, Lercier, Ritzenthaler, Sijsling in [2], where the Lüroth invariant has been described as linear combination of these monomials. This computation allows to detect if a given plane quartic is Lüroth. Moreover it disproves a guess by Morley at the end of [38] about the explicit form of the Lüroth invariant. Still the existence of a determinantal formula or other simple descriptions for the Lüroth invariant is sought.

Recently, a determinantal description for the undulation invariant of degree 60 has been found [45]. It vanishes on quartic curves that have an undulation point, that is a line meeting the quartic in a single point with multiplicity 4. A beautiful classical source about plane quartics is Ciani monograph [12].

Proposition 21. Let \( Y_{10} = \sigma_{9}(\mathcal{V}_{6}(\mathbb{P}^2)) \) be the determinantal hypersurface in \( \mathbb{P}S^6_{\mathbb{C}^3} \) of sextics having a apolar cubic. The dual variety \( Y_{10}^{\vee} \) is the variety of double cubics.

Proof. By Remark 1, the dual variety corresponds to the sextics which are singular in 9 points, hence they are double cubics.

Proposition 22. The 3-secant variety \( \sigma_{3}(Y_{10}^{\vee}) \) is the theta divisor, that is the locus of sextic curves which admit an effective even theta-characteristic. Its degree is 83200.

Proof. The sextics in the variety of 3-secant to \( Y_{10} \) can be written as \( A^2 + B^2 + C^2 \) where \( A, B, C \) are cubics. Since all plane conics are projectively equivalent, they can be written as \( AC - B^2 = 0 \), that is as a \( 2 \times 2 \) symmetric determinant with cubic entries. Write

\[
\Omega(-3)^2 \xrightarrow{M} \Omega^2
\]

with \( M \) symmetric. The cokernel is a effective theta-characteristic and conversely every effective theta-characteristic arises in this way (see [6] and remark 4 in [9]). The computation of the degree is a nontrivial result proved in [9].

Question What is the degree of the theta divisor for plane curves of degree \( d \), that is the locus of plane curves of degree \( d \) which admit an effective even theta-characteristic ?
5. Invariants of points. Cremona equations for the cubic surface and invariants of six points

5.1. The two Fundamental Theorems for invariants of points

Given \( p_1, \ldots, p_d \in \mathbb{P}^n \), we can write their coordinates in a \((n+1) \times d\) matrix, writing the coordinates of \( p_i \) in the \( i \)-th column. The ring of polynomials over these coordinates \( \mathbb{C}[V \otimes \mathbb{C}^d] = \oplus_m S^m (V \otimes \mathbb{C}^d) \) has a natural multigraduation

\[
\oplus_{m_1, \ldots, m_d} (S^{m_1} V \otimes \cdots \otimes S^{m_d} V),
\]

where the coordinates of \( p_i \) appear with total degree \( m_i \). The group \( SL(n+1) \) acts on \( I PV \), then it acts on the multigraded ring. Classically, these rings have been studied in the “democratic” case when all \( n_i \) are equal. So the invariant ring to be studied was \( \oplus_m S^m V \otimes \cdots \otimes S^m V \) \( SL(n+1) \). In these cases, there is the additional action of \( \Sigma_d \) on the points and then on the invariant ring. The \( SL(n+1) \times \Sigma_d \)-invariants were called “rational”, while the ones invariant just for the smaller subgroup \( SL(n+1) \times Alt(d) \) were called “irrational”.

After GIT has been developed, it has been understood that it is convenient to fix a weight (polarization) \( h = (h_1, \ldots, h_d) \), so getting \( \mathbb{C}[V \otimes \mathbb{C}^d]_{(h)} = \oplus_p S^{ph_1} V \otimes \cdots \otimes S^{ph_d} V \).

The invariant subring \( \mathbb{C}[V \otimes \mathbb{C}^d]_{(h)} \) is called the invariant ring of \( d \) ordered points on \( \mathbb{P}^n \) with respect to the weight \( h \). When the weight \( h \) is not specified, it is understood that it is \( h = 1^d \).

The invariant subring \( \mathbb{C}[V \otimes \mathbb{C}^d]_{(h)}^{SL(n) \times \Sigma_d} \) is called the invariant ring of \( d \) unordered points on \( \mathbb{P}^n \).

In the case \( n = 1 \), the invariant ring of \( d \) unordered points on the line coincides with the invariant ring of binary forms of degree \( d \). This is clear associating to any binary form its scheme of roots.

For the convenience of the reader, we repeat with slight changes the construction of Definition 4.

**Definition 6** (From tableau to invariants of points).

Let \( h_1 + \ldots + h_d = (n+1)g \). For any tableau \( T \) over a Young diagram of size \((n+1) \times g\), filled with numbers from 1 appearing \( h_1 \) times, until \( d \) appearing \( h_d \) times, we denote by \( G_T \) the multilinear function \( G_T : S^{h_1} V \otimes \cdots \otimes S^{h_d} V \to \mathbb{C} \) defined by (compare with Definition 4)

\[
G_T (x_1^{h_1}, \ldots, x_d^{h_d}) = \prod_{j=1}^d (x_{(1,j)} \wedge \cdots \wedge x_{(n+1,j)}).
\]

\( G_T \) is well defined by Theorem 1 and it is \( SL(V) \)-invariant by Proposition 16.
The geometric meaning of the vanishing of $G_T$, where

$$T = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

is that the corresponding points $x_1, \ldots, x_4 \in V^\vee$ are dependent.

**Theorem 28 (1FT for ordered points).**

The invariant ring $\mathbb{C}[V \otimes \mathbb{C}^d]_{SL(n+1)}$ of $d$ ordered points on $\mathbb{P}^V$ with respect to $h$ is generated by functions $G_T$ like in Definition 6 for tableau $T$ having weight multiple of $h$.

**Proof.** The decomposition (see Theorem 11) $S^p(V \otimes \mathbb{C}^d) = \oplus_{\lambda} \mathbb{C}^S \otimes \mathbb{C}^\lambda C^m$, where the sum is extended to all Young diagrams $\lambda$ with $|\lambda| = p$, shows that $S^p(V \otimes \mathbb{C}^d)_{SL(n+1)} = 0$ for $p$ which is not multiple of $(n+1)$ and $S^p(V \otimes \mathbb{C}^d)_{SL(n+1)} = S^m C^d$ if $p = (n+1)g$ and $\mu$ is the Young diagram with $(n+1)$ rows and $g$ columns.

By Theorem 9, $S^m C^d$ has a basis consisting of semistandard Young tableau $T$, where the numbers 1, $\ldots$, $d$ appear. This basis has a natural multigraduation, depending on partitions $p = m_1 + \ldots + m_d$, where in $T$ the number 1 appears $m_1$ times, 2 appears $m_2$ times, until $d$ appears $m_d$ times. Moreover, this basis fits with the other decomposition

$$S^p(V \otimes \mathbb{C}^d) = \oplus (S^{m_1} V \otimes \ldots \otimes S^{m_d} V),$$

where the sum is extended to all the partitions with $d$ summands $p = m_1 + \ldots + m_d$ which induce $S^p(V \otimes \mathbb{C}^d)_{SL(n+1)} = \oplus (S^{m_1} V \otimes \ldots \otimes S^{m_d} V)^{SL(n+1)}$.

In other words, the summand $(S^{m_1} V \otimes \ldots \otimes S^{m_d} V)^{SL(n+1)}$ has a natural basis consisting of semistandard Young tableau, consisting in the Young diagram $(n+1) \times \frac{p}{n+1}$ filled with 1 appearing $m_1$ times, 2 appearing $m_2$ times, until $d$ appearing $m_d$ times.

These semistandard tableau $T$ correspond to the multilinear function $G_T$ of Definition 6.

**Corollary 6 (1FT for unordered points).** All invariants of $d$ unordered points are polynomials in the tableau functions $G_T$, where in $T$ the numbers from 1 to $d$ appear equally, which moreover are symmetric under permutation of points.

The invariant ring of $d$ unordered points on $\mathbb{P}^V$ is isomorphic to

$$\oplus_m [S^d (S^n V)]^{SL(n+1)}.$$

**Remark 11.** It is interesting to compare the invariant ring of $d$ unordered points with the invariant ring for forms in $S^d V$, which is $\oplus m S^n (S^d V)^{SL(n+1)}$. Note the swapping between $m$ and $d$. Note also that on $\mathbb{P}^4$ the swapping makes no difference, by Hermite reciprocity (Corollary 4).
**Proof of Theorem 21, 1FT for forms** Consider, in the proof of Theorem 28, the case \( m_1 = \ldots = m_d = m \), so that \( p = md \).

We get that \((S^m V \otimes \ldots \otimes S^m V)^{\text{SL}(n+1)}\) has a natural basis consisting of \( G^T \), where \( T \) is a semistandard Young tableau, filling the diagram \((n+1) \times \frac{md}{n+1}\) with 1 appearing \( m \) times, 2 appearing \( m \) times, until \( d \) appearing \( m \) times.

Considering the subspace of \( \Sigma_d \)-invariants, we get the space of symmetric multi-linear functions \( F^T \), like in Definition 5, which indeed is \( S^d(S^m V)^{\text{SL}(n+1)} \). By swapping \( m \) with \( d \), we get exactly the construction performed in §3.9.

Note that by 1FT, \( S^{mh_1} V \otimes \ldots \otimes S^{mh_d} V \neq 0 \) if and only if \( m(h_1 + \ldots + h_d) \) is a multiple of \((n+1)\). The invariants of minimal degree are those with \( m = \text{lcm}(h_1 + \ldots + h_d, n+1) \).

If \( g(n+1) = md \) we have the weight \((1, \ldots, 1)\) and the graded ring \( \bigoplus_m S^m V \otimes \ldots \otimes S^m V \).

**Theorem 29 (2FT for points).** In the invariant ring of \( d \) (ordered or unordered) points, all the relations between the tableau functions are generated by the Plücker relations \( \sum_{s=0}^{d} (-1)^s G^s = 0 \), exactly like in Theorem 22, with the tableau \( T \) having the correct weight.

A strong improvement of 2FT for ordered points on \( \mathbb{P}^d \) is relatively recent, and it will be treated in Theorem 33.

The theory is better explained by examples.

**Proposition 23.**

(i) The conic through \( P_0, \ldots, P_3 \) has equation

\[
[014][234][02x][13x] - [024][134][01x][23x] = 0.
\]

(ii) 6 points in the plane \( \mathbb{P}^2 \) lie on a conic if and only if

\[
d_2 := [014][234][025][135] - [024][134][015][235] = 0.
\]

(it is an irrational invariant, indeed it is \( \text{Alt}(6) \)-invariant but not \( \Sigma_6 \)-invariant).

**Proof.** The singular conics between \( P_0, \ldots, P_3 \) are \([01x][23x], [02x][13x], [03x][12x]\). We ask that there exist \( A \) and \( B \) such that \( A[01x][23x] + B[02x][13x] \) vanishes for \( x = P_4 \). Hence we have \( A[014][234] + B[024][134] = 0 \) which is satisfied for \( A = -[024][134] \) and \( B = [014][234] \).

**Remark 12.** The expression in (ii) of Proposition 23 is the symbolic expression for the \( 6 \times 6 \) determinant having in the \( i \)-th row the coefficients \( x_0^2 \ldots x_2^2 \) computed at \( P_i \).
Note again that the skew-symmetry is not at all evident from the symbolic expression. It can be showed by using the Plücker relations, in this case there are 35 quadratic relations.

5.2. The graphical algebra for the invariants of \( d \) points on the line. Kempe’s Lemma

In 1894 A. Kempe [29] introduced a graphical representation of invariants.

Let fix a weight \((h_1, \ldots, h_d)\) and consider the algebra

\[
\mathbb{C}[[\mathbb{C}^2 \otimes \mathbb{C}^d]]_{(h)} = \oplus_m S^{mh_1} \mathbb{C}^2 \otimes \cdots \otimes S^{mh_d} \mathbb{C}^2.
\]

Any \( d \) points on a line \( \mathbb{P}^1 \) can be represented as \( d \) vertex of a regular polygon, numbered clockwise from 1 to \( d \).

The bracket function \((ij)\) between \( i \) and \( j \) is represented as an arrow from \( i \) to \( j \). A tableau function of weight \( mh \) is represented as a graph with valence \( mh_i \) at the vertex \( i \).

For example \((12)(34)(56)\) represents an invariant with respect to the weight \( 1^6 \) and it corresponds to the graph

```
      1
     / \   \  \\
    /   \ /   \\
   6   2
     \ / \  /
      3
      5
     \   /
       4
```

Inverting one arrow corresponds to a sign change. The relation \((ij)(kl) - (ik)(jl) + (il)(jk) = 0\) is represented graphically by

\[
(26)
\]

The graphical algebra is the same algebra generated by tableau functions. Every linear combination of tableau functions transfers to a linear combination of corresponding graphs and conversely. The product of two graphs corresponds to the union of the corresponding arrows, like in

```
      1
     / \   \  \\
    /   \ /   \\
   3   2
     \ / \  /
      3  =
```

```
      1
     / \   \  \\
    /   \ /   \\
   3   2
     \ / \  /
```

```
      1
     / \   \  \\
    /   \ /   \\
   3   2
     \ / \  /
      3  
```
This graphical algebra has been carefully studied in a series of recent papers by Howard, Millson, Snowden and Vakil. In order to review their approach and their main results, we recall the following basic result from graph theory.

**Theorem 30 (Hall Marriage Theorem).** Consider a graph $G$ with $2m$ vertices, $m$ being positive and $m$ being negative. A perfect matching is a collection of $m$ edges, each one joining one positive vertex with one negative vertex, in such a way that every vertex belongs to one edge. The necessary and sufficient condition that $G$ contains a perfect matching, is that for every subset $Y$ of positive vertices, the cardinality of the set of negative vertices which are connected to at least one member of $Y$ is bigger or equal than the cardinality of $Y$.

**Theorem 31 (Kempe’s Lemma, IFT for ordered points on $\mathbb{P}^1$).** All invariants of ordered points on $\mathbb{P}^1$ are generated by tableau functions of minimal degree

$$\frac{lcm(h_1 + \ldots + h_d, (n+1))}{h_1 + \ldots + h_d}.$$

With respect to the weight $(1, 1, \ldots, 1)$, the generating invariants have degree 1 when $d$ is even and degree 2 when $d$ is odd.

**Proof.** We follow [25], who gave a shorter proof than Kempe original one. We assume for simplicity that $d$ is even and $h = (1, \ldots, 1)$, and refer to [25] for the general case. Consider the graphical description of an invariant of weight $(m, \ldots, m)$. Divide the vertices into two subsets of equal cardinality, called positive and negative.

So the edges have three possible types: positive (both vertices positive), negative (both vertices negative) and neutral (two opposite vertices).

Since every monomial is homogeneous, every vertex has valence $m$. It follows that the number of positive edges is equal to the number of negative edges. Applying the relation (26) to a pair given by a positive and a negative edge, we get all neutral edges. Continuing in this way, we get a combination of graphs, each one with all neutral edges. Then the assumption of Hall Marriage Theorem is satisfied, because from every subset $Y$ of $p$ positive vertices start $pm$ edges. Since the valence of each vertex is $m$, $Y$ must connect to at least $p$ negative vertices. So a perfect matching exists. Note that a perfect matching corresponds to a generator of minimal degree 1. By factoring this generator we can conclude by induction on $m$.

From Theorem 9 and from the proof of Theorem 28, a basis of tableau functions $G_T$ of minimal degree is given by semistandard tableau. In the case of points of $\mathbb{P}^1$, an alternative description is possible. A graph is said to be noncrossing if no two edges cross in an interior point.

**Theorem 32 (Kempe).** A basis of the tableau functions of minimal degree is given by noncrossing complete matching of minimal degree.

**Proof.** We apply the relation (26) to a pair of noncrossing edges. We get a combination of graphs, in each of them the total euclidean length of the edges is strictly smaller. This
process must terminate, because there is a finite number of complete matching, hence a finite number of total euclidean length. When the process terminates, we have a combination of noncrossing complete matching, otherwise the process could be repeated.

This shows that the noncrossing complete matching span. For the proof of independence of noncrossing graphs with the same weight $h$, assume we have a nonzero relation involving a minimal number $n$ of vertices. Not all the graphs appearing contain the edge $(n-1)n$, otherwise we could remove it, obtaining a smaller relation. Now identify the vertices $(n-1)$ and $n$, so that the graphs containing the edge $(n-1)n$ go to zero.

This gives a bijection between the graphs on $n$ vertices with weight $h$, not containing the edge $(n-1)n$, and the graphs on $(n-1)$ vertices with weight $(h_1, \ldots, h_{n-2}, h_{n-1} + h_n)$.

We get a nonzero relation on $(n-1)$ vertices, contradicting the minimality.

Remark 13. The above proof gives a graphical version of the straightening algorithm, as pointed out in [25] Prop. 2.5.

For example, the space of invariants of 6 points on $\mathbb{P}^1$ with weight $1^6$ is generated by the noncrossing complete matching as follows. We draw all the arrows from even to odd.

\begin{align*}
(t_1) &:\quad 6 \quad 2 \quad 1 \\
      &\downarrow \quad \downarrow \quad \downarrow \\
      &\quad 5 \quad 3 \quad 4 \\
(t_2) &:\quad 6 \quad 1 \\
      &\downarrow \\
      &\quad 5 \quad 3 \quad 4 \\
(t_3) &:\quad 6 \quad 1 \\
      &\downarrow \\
      &\quad 5 \quad 3 \quad 4 \\
(t_4) &:\quad 6 \quad 1 \\
      &\downarrow \quad \downarrow \\
      &\quad 5 \quad 3 \quad 4 \\
(t_5) &:\quad 6 \quad 1 \\
      &\downarrow \\
      &\quad 5 \quad 3 \quad 4 \\
\end{align*}

**Theorem 33 (2FT for ordered points on $\mathbb{P}^1$, Howard, Millson, Snowden and Vakil [27]).** In the invariant ring for $d$ ordered points on $\mathbb{P}^1$, with any weight $w \neq 1^6$, the relation among the generators of minimal degree are generated by quadric relations.
In the case $w = 1^6$, we will see in Theorem 37 that there is a unique cubic relation among the generators of minimal degree of the invariant ring of six ordered points on $\mathbb{P}^1$ (this relation gives the Segre cubic primal, see Remark 16).

**Remark 14.** Theorem 39 will show that, in the case of $d$ unordered points, the relations are more complicated, certainly not generated by quadric relations.

### 5.3. Molien formula and elementary examples

The following Theorem by Molien shows that the Hilbert series of the invariant subring is the average of the inverse of the characteristic polynomial.

**Proposition 24.** Let $G$ be a finite group acting on $U$. The induced action on the symmetric algebra $S^i U$ has Hilbert series

$$(28) \quad \sum_{i=0}^{+\infty} \dim (S^i U)^G q^i = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1-qg)}$$

where $g$ acts on $U$.

**Proof.** [53] Theor. 2.2.1

We analyze now some invariant rings for $d$ points on $\mathbb{P}^1$, for small $d$. The simplest cases is the following.

**Theorem 34.** The Hilbert series of the invariant ring of three ordered points on $\mathbb{P}^1$ is

$$\frac{1}{1-t^2}. \quad \text{The ring is generated by the only noncrossing matching of valence 2 which is}$$

$$t_0 = \begin{array}{c}
\text{1} \\
\text{3} \\
\text{2}
\end{array}
$$

**Proof.** Immediate from Theorem 32.

We get the second promised proof of the Corollary 3.

**Corollary 7.** The Hilbert series of the invariant ring of three unordered points on $\mathbb{P}^1$ is

$$\frac{1}{1-t^4}. \quad \text{The ring is generated by the discriminant } \Delta = t_0^2.$$
**Proof.** The function $t_0$ is $\text{Alt}(3)$-invariant (for even permutations), while $t_0^2$ is $\Sigma_3$-invariant.

**Theorem 35.** The Hilbert series of the invariant ring of four ordered points on $\mathbb{P}^1$ is

$$\frac{1}{(1-t)^2}.$$  

The ring is generated by the two noncrossing matchings which are

$$j_0 = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}, \quad j_1 = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}.$$  

**Proof.** $j_0$ and $j_1$ generate the invariant ring by Theorem 32. Moreover $j_0$ and $j_1$ are algebraically independent. This can be shown directly or by observing that the geometric quotient is one dimensional.

In the point (i) of the following Theorem we get a second proof of Theorem 25.

**Theorem 36.**

(i) The Hilbert series of the invariant ring of four unordered points on $\mathbb{P}^1$ (binary quartics) is

$$\frac{1}{(1-t^2)(1-t^3)}.$$  

The ring is generated by $I, J$ defined in (4), (5).

(ii) The Hilbert series of the $\text{Alt}(4)$-invariant ring of four unordered points on $\mathbb{P}^1$ (binary quartics) is

$$\frac{1+t^3}{(1-t^2)(1-t^3)} = \frac{1-t^6}{(1-t^2)(1-t^3)^2}.$$  

The ring is generated by $I, J, \sqrt{D}$ where $\sqrt{D}$ has degree 3 and it is the product of differences of the roots of the quartic. The only relation is $(\sqrt{D})^2 = I^3 - 27J^2$.

**Proof.** The result is elementary, but we give the details of the representation theoretic approach as warming up for the more interesting case of six points (Theorem 39). The two generators $j_0, j_1$ of Theorem 35 span the unique irreducible representation $W$ of dimension 2 of $\Sigma_4$. So we have to compute the Hilbert series of $\bigoplus_p S^p(W)^{\Sigma_4}$. The proof is a straightforward computation by using Molien’s formula (28), by summing over the five conjugacy classes in $\Sigma_4$.

The result is

$$\frac{1}{24} \left[ \frac{1}{(1-t)^2} + \frac{6}{1-t^2} + \frac{8}{1+t+t^2} + \frac{6}{1-t^2} + \frac{3}{(1-t)^2} \right] = \frac{1}{(1-t^2)(1-t^3)}.$$
Let $I, J$ be the second and third elementary symmetric function of $j_0, -j_1, j_1 - j_0$. $I$ and $J$ correspond, up to scalar factor, to (4), (5) and they are the generators of the invariant ring corresponding to the two factors in the denominator of the Hilbert series. This proves (i). (ii) can be proved by restricting the sum to the even conjugacy classes, which are the ones with numerator 1, 8, and 3.

**Remark 15.** The cross ratio \( \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)} \) of four points with affine coordinates $x_i$ for $i = 1, \ldots, 4$ has the expression $\frac{j_1 - j_0}{j_1}$. It parametrizes the moduli space of 4 ordered points on $\mathbb{P}^1$, which is isomorphic to $\mathbb{P}^1$ itself.

### 5.4. Digression about the symmetric group $\Sigma_6$ and its representations

We list a representative for each of the 11 conjugacy classes of $\Sigma_6$.

<table>
<thead>
<tr>
<th></th>
<th>even</th>
<th>number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>(1)</td>
<td>*</td>
</tr>
<tr>
<td>C2</td>
<td>(12)</td>
<td></td>
</tr>
<tr>
<td>C3</td>
<td>(12)(34)</td>
<td>*</td>
</tr>
<tr>
<td>C4</td>
<td>(12)(34)(56)</td>
<td>*</td>
</tr>
<tr>
<td>C5</td>
<td>(123)</td>
<td></td>
</tr>
<tr>
<td>C6</td>
<td>(123)(45)</td>
<td></td>
</tr>
<tr>
<td>C7</td>
<td>(123)(456)</td>
<td>*</td>
</tr>
<tr>
<td>C8</td>
<td>(1234)</td>
<td></td>
</tr>
<tr>
<td>C9</td>
<td>(1234)(56)</td>
<td>*</td>
</tr>
<tr>
<td>C10</td>
<td>(12345)</td>
<td>*</td>
</tr>
<tr>
<td>C11</td>
<td>(123456)</td>
<td></td>
</tr>
</tbody>
</table>

The natural action of $\Sigma_d$ on $\mathbb{C}^d$ splits into the trivial representation $U$ (dimension one) and the standard representation $V$ (dimension $d - 1$). The subspace $V \subset \mathbb{C}^d$ is given by $\sum e_i = 0$.

The exterior representation $U' : \Sigma_d \to \mathbb{C}^*$ which sends $p \in \Sigma_d$ to its sign $\epsilon(p)$ has again dimension one. For any representation $W$, it is customary to denote $W' = W \otimes U'$. $W'$ corresponds to the transpose Young diagram of $W$. Each representation of $\Sigma_d$ is self-dual.

We list the irreducible representations in the case $d = 6$ and we put in evidence...
the transpose diagrams.

<table>
<thead>
<tr>
<th>name</th>
<th>shape</th>
<th>dimension</th>
<th>name</th>
<th>shape</th>
<th>dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = U$</td>
<td></td>
<td>1</td>
<td>$X_7 = \wedge^3 V$</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>$X_2 = V$</td>
<td></td>
<td>5</td>
<td>$X_8 = X_5'$</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>$X_3$</td>
<td></td>
<td>9</td>
<td>$X_9 = X_3'$</td>
<td></td>
<td>9</td>
</tr>
<tr>
<td>$X_4 = \wedge^2 V$</td>
<td></td>
<td>10</td>
<td>$X_{10} = \wedge^4 V$</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>$X_5$</td>
<td></td>
<td>5</td>
<td>$X_{11} = U'$</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$X_6 = X_6'$</td>
<td></td>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One peculiarity of the above list is that there are four irreducible representations of dimension $5 = d - 1$. This happens only in the case $d = 6$. $S_6$ is the only symmetric group which admits an automorphism which is not inner, which can be defined indeed by means of $X_5$ (or dually by $X_8$).

We list the character table of $\Sigma_6$, from the appendix of [20]. Note that the first column gives the dimension.
Other useful formulas are ([19] ex. 2.2)

\[ \chi_{S^w} (g) = \frac{1}{2} \left[ \chi_w (g) + \chi_w (g^2) \right] . \]

(29) \[ \chi_{S^w} (g) = \frac{1}{6} \chi_w (g)^3 + \frac{1}{2} \chi_w (g) \chi_w (g^2) + \frac{1}{3} \chi_w (g^3). \]

We have the following table explaining how the powers of elements divide among the conjugacy classes.

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
<th>C10</th>
<th>C11</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>X2</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>X3</td>
<td>9</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>X4</td>
<td>10</td>
<td>2</td>
<td>-2</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>X5</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X6</td>
<td>16</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>X7</td>
<td>10</td>
<td>-2</td>
<td>-2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>X8</td>
<td>5</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X9</td>
<td>9</td>
<td>-3</td>
<td>1</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>X10</td>
<td>5</td>
<td>-3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>X11</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

5.5. The invariant ring of six points on the line

**Theorem 37.** The Hilbert series of the invariant ring of six ordered points on \(\mathbb{P}^1\) is

\[ \frac{1 - t^3}{(1 - t)^5}, \]

The ring is generated by the five noncrossing matching \(t_1, \ldots, t_5\) as in (27). We have the unique relation in degree 3

(31) \[ t_1 t_2 (-t_1 - t_2 + t_3 + t_4 + t_5) - t_3 t_5 = 0. \]
Proof. The five noncrossing matching $t_1, \ldots, t_5$ listed in (27) generate by Theorems 31 and 32. They are easily identified as a basis for the representation $X_5$ of $\Sigma_6$. By dimensional reasons (the quotient has dimension three), only one relation is expected.

The fact that there are no relations in degree two can be proved by counting the number of semistandard tableau $2 \times 6$ of weight $2^6$, which are indeed 15. In order to understand the relation, we add a sixth tableau which is a linear combination of the first noncrossing $t_1, \ldots, t_5$ as

$$
(32) \quad t_0 = -t_1 - t_2 + t_3 + t_4 + t_5.
$$

After some computation with the straightening algorithm, as in the proof of Theorem 32, we get $t_0 = -t_1 - t_2 + t_3 + t_4 + t_5$.

Looking at the graphs there is the obvious relation $t_0 t_1 t_2 - t_3 t_4 t_5 = 0$ that we show in the following picture.

This concludes the proof.
REMARK 16. The cubic 3-fold (31) is called the Segre cubic primal. It has 10 singular points and it contains 15 planes (see [15] example 11.6). It can be expressed as the sum of six cubes, and not eight like the general cubic 3-fold (see [42, 48]). An explicit expression as a sum of cubes will be obtained in (34).

REMARK 17. In [15] example 11.6 it is reported an alternative combinatorial proof of Theorem 37, by counting the number of semistandard tableau $2 \times 3m$ of weight $m$.

Since the relation (31) is $\text{Alt}(6)$-invariant but not $\Sigma_6$-invariant, we need a preliminary step to get the $\Sigma_6$-invariants.

By following Coble [10] §3, we define the Joubert invariants (see [14] for an intrinsic way to compute these invariants).

\begin{align*}
\end{align*}

In terms of redundant $t_0, \ldots, t_5$ we have nicer expressions

\[
\begin{align*}
A/2 &= -t_0 + t_1 + t_2 \\
B/2 &= -t_3 - t_4 + t_5 \\
C/2 &= +t_3 - t_4 - t_5 \\
D/2 &= +t_0 - t_1 + t_2 \\
E/2 &= +t_0 + t_1 - t_2 \\
F/2 &= -t_3 + t_4 - t_5
\end{align*}
\]

Note that

\[A + B + C + D + E + F = 0.\]

so that any five among them give a basis of the space of invariant tableau functions for six ordered points.

A direct inspection shows that an even permutation of the points effects an even permutation of the functions and that an odd permutation of the points effects an odd permutation of the functions accompanied by a change of sign. For example after the permutation (12) we get

\[A \rightarrow -D \quad B \rightarrow -E \quad C \rightarrow -F,\]

after the permutation (13) we get

\[A \rightarrow -F \quad B \rightarrow -D \quad C \rightarrow -E.\]
Let \(b_{15} = (A - B)(A - C) \ldots (E - F)\) be the product of differences. The previous permutation rules about \(A, \ldots, F\) show that \(b_{15}\) is \(\Sigma_6\)-invariant.

We denote by \(a_i\) the \(i\)-th elementary symmetric function of \(A, \ldots, F\). It follows that \(a_i\) are \(\text{Alt}(6)\)-invariant and \(\Sigma_6\)-invariant for even \(i\).

Consider \(\prod_{i<j}(x_j - x_i) = \Delta\) which is a \(\text{Alt}(6)\)-invariant, and it is the square root of the discriminant of the polynomial having the points as roots.

Every \(\text{Alt}(6)\)-invariant can be written as \(a + b\Delta\) where \(a, b\) are \(\Sigma_6\)-invariants, see [53] Prop. 1.1.3. It follows that

\[
A^3 + B^3 + C^3 + D^3 + E^3 + F^3 = 0,
\]

which is the promised expression of the Segre cubic primal as a sum of six cubes.

**Theorem 38 (Coble).** The Hilbert series of the \(\text{Alt}(6)\)-invariant ring of six (unordered) points on \(\mathbb{P}^1\) is

\[
1 + t^{15} \left( \frac{(1 - t^2)(1 - t^4)(1 - t^6)}{1 + t^{15}} \right).
\]

The ring is generated by \(a_2, a_4, a_6, \Delta, b_{15}\) with the relation in degree 30 expressing \(b_{15}^2\) as a polynomial in the other generators.

**Proof.** \(X5\) and \(X8\) restrict to the same representation of \(\text{Alt}(6)\), that we call again \(X5\). We compute first the Hilbert series of \(\oplus_p S^p(X5)^{\text{Alt}(6)}\). The proof is a straightforward computation by using Molien formula (28). The characteristic polynomial of any \(g \in \text{Alt}(6)\) can be computed with the following trick. The trace of the action of \(g^j\) is computed by the character table and by the table (30). It gives the Newton sums \(\sum_{j=1}^5 \lambda_j^i\), where \(\lambda_j\) are the eigenvalues of the \(5 \times 5\) matrix representing the action of \(g\) on \(X5\). Then, by the Newton identities, we can compute the \(i\)-th elementary symmetric functions.

We sum in Molien formula over the six even conjugacy classes in \(\Sigma_6\), as in the proof of Theorem 36.

The result is

\[
1 + t^{15} \left[ \frac{1}{(t-1)^5} + \frac{45}{(t-1)^3(t+1)^2} + \frac{40}{(t-1)(t^2+t+1)^2} + \frac{40}{(t-1)^3(t^2+t+1)} + \frac{90}{(t-1)(t+1)^2(t^2+1)} + \frac{144}{(t-1)(t^4+t^3+t^2+t+1)} \right] =
\]

\[
\frac{1 + t^{15}}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^6)}.
\]
By considering the relation (31), the factor \((1 - t^3)\) cancels from the denominator in the Hilbert series. The remaining factors in the denominator correspond to \(a_2, a_4, \Delta, a_6\). This suggests that there is another generator of degree 15, which is identified with \(b_{15}\). By dimensional reasons, we expect a single relation. The square \(b^2_{15}\) is \(\Sigma_6\)-invariant and it corresponds to the discriminant of the polynomial having \(A, \ldots, F\) as roots, so it can be expressed as a polynomial in \(a_i\).

**Theorem 39 (Coble).** The Hilbert series of the \(\Sigma_6\)-invariant ring of six (unordered) points on \(\mathbb{P}^1\) (binary sextic) is

\[
\frac{1 + t^{15}}{(1 - t^2)(1 - t^4)(1 - t^5)(1 - t^{10})}.
\]

The ring is generated by \(a_2, a_4, a_6, \Delta, b_{15}\) with the relation in degree 30 expressing \(b^2_{15}\) as a polynomial in \(a_2, \ldots, a_6\).

**Proof.** The Hilbert series is obtained as in the proof of Theorem 38, by adding the contribution of the remaining five conjugacy classes. Among the generators of the \(\text{Alt}(6)\)-invariant ring, \(a_2, a_4, a_6, b_{15}\) are already \(\Sigma_6\)-invariant. \(\Delta^2\) is another independent invariant. The ring generated by these invariants has the claimed Hilbert series. □

The Hilbert function of \(d\) ordered points on \(\mathbb{P}^1\) has been found by Howe.

**Theorem 40.** The dimension of the space of invariants for \(d\) ordered points on \(\mathbb{P}^1\) and multidegree \(k^d\) with the weight \(\begin{cases} 1^d & \text{if } d \text{ is even} \\ 2^d & \text{if } d \text{ is odd} \end{cases}\) is

\[
\sum_{j=0}^{(d-1)/2} (-1)^j \binom{d}{j} \binom{k(d/2 - j) + d - 2 - j}{d - 2}.
\]

In this formula it is understood that a binomial coefficient \(\binom{a}{b}\) is zero if \(a < b\). In the case \(d\) even, this formula gives the degree \(k\) part of the invariant ring with respect to \(1^d\).

In the case \(d\) odd, the formula is meaningful for \(k\) even, and gives the degree \(k/2\) part of the invariant ring with respect to \(2^d\).

**Proof.** ([28] 5.4.2.3)). □

As an application, from this formula, in [18] Theorem 1.6, it has been computed the Hilbert series of the invariant ring of 8 ordered points on \(\mathbb{P}^1\), which is

\[
\frac{1 + 8t + 22t^2 + 8t^3 + t^4}{(1 - t)^6} = \frac{1 - 14t^2 + 175t^4 - 512t^5 + 700t^6 - 512t^7 + 175t^8 - 14t^{10} + t^{12}}{(1 - t)^{14}}.
\]

The second formulation is reported because the coefficients are the Betti number of the resolution of the GIT quotient \(\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^6)_{18} / \text{SL}(2)\), which is the moduli space of 8 ordered points on a line (see [18] Lemma 1.4, [26] Prop. 7.2).

Howe’s formula in Theorem 40 has been generalized to arbitrary weights in [23].
5.6. The invariant ring of six points on the plane. Cremona hexahedral equation for the cubic surface

Let \( a_1, \ldots, a_6 \) be six points on the projective plane. The aim of this section is to report in representation theoretic language the results by Coble [10] about six points on the plane, and to apply them to the construction of Cremona hexahedral equations.

We denote by \((ijk)\) the 3 \( \times \) 3 determinant of three points labeled with \(i, j, k\). In particular \((i j x)\) is the equation in the coordinate \(x\) of the line through \(a_i\) and \(a_j\). This fits particularly well with the graphical description of the previous section, indeed the line \((i j x)\) can be seen just by prolongation of the arrow between \(i\) and \(j\).

Any relation between invariants \((ij)\) on \(\mathbb{P}^1\) transfers to an analogous relation among covariants \((ijk)\) on \(\mathbb{P}^2\). This is “Clebsch transfer principle”.

In general, an invariant is a function \(F(a_1, \ldots, a_6)\) which is symmetric (classically called rational) or skew-symmetric (classically called irrational) in the points, of degree \(q\) in the coordinates of each point, such that for every \(g \in SL(3)\) \(F(ga_1, \ldots, ga_6) = (\det g)^{q/3}F(a_1, \ldots, a_6)\).

The covariants are polynomials in the invariants \((ijk)\) and in the \((i j x)\), which again are symmetric or skew-symmetric in the six points.

For example \((12x)(34x)(56x)\) represents a cubic splitting in three lines. There are 15 such cubics (as symbols), which span the 4-dimensional space of all cubics through the six points.

The six generators \(t_0, \ldots, t_5\) of (27) and (32) give six cubics through the six lines. The relation \(t_0 + t_1 + t_2 = t_3 + t_4 + t_5\) transfers to an analogous relation between cubics.

It is more convenient to consider the six invariants \(A \ldots F\) of (33) which induce the following list of cubics

\[
\begin{align*}
 a &= (25x)(13x)(46x) + (51x)(42x)(36x) + (14x)(35x)(26x) + (43x)(21x)(56x) + (32x)(54x)(16x) \\
b &= (53x)(12x)(46x) + (14x)(23x)(56x) + (25x)(34x)(16x) + (31x)(45x)(26x) + (42x)(51x)(36x) \\
c &= (53x)(41x)(26x) + (34x)(25x)(16x) + (42x)(13x)(56x) + (21x)(54x)(36x) + (15x)(32x)(46x) \\
d &= (45x)(31x)(26x) + (53x)(24x)(16x) + (41x)(25x)(36x) + (32x)(15x)(46x) + (21x)(43x)(56x) \\
e &= (31x)(24x)(56x) + (12x)(53x)(46x) + (25x)(41x)(36x) + (54x)(32x)(16x) + (43x)(15x)(26x) \\
f &= (42x)(35x)(16x) + (22x)(14x)(56x) + (31x)(52x)(46x) + (15x)(43x)(26x) + (54x)(21x)(36x)
\end{align*}
\]

For a representation-theoretic way to get these expressions, see [14] 9.4.4.

They satisfy the relation

\[ a + b + c + d + e + f = 0 \]

and indeed the six cubics span \(X_5\).

Again an even permutation of the points effects an even permutation of the cubics and an odd permutation of the points effects an odd permutation of the cubics accompanied by a change of sign. For example after the permutation \((12)\) we get

\[ a \rightarrow -d \quad b \rightarrow -e \quad c \rightarrow -f. \]

By the same argument given for invariants of points, the relation (34) transfers
to the relation \( a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = 0 \) (in equivalent way the third elementary symmetric function of \( a, \ldots, f \) vanishes).

Since the space of cubics through 6 points is four dimensional, there is a further relation

\[
\overline{a}a + \overline{b}b + \overline{c}c + \overline{d}d + \overline{e}e + \overline{f}f = 0,
\]

which is uniquely determined considering the additional condition \( \overline{a} + \overline{b} + \overline{c} + \overline{d} + \overline{e} + \overline{f} = 0 \).

In order to find the additional relation between \( a, \ldots, f \), Coble considers a second interesting formulation. Define \( (ij, kl, mn) \) to be the function which represents that the three lines \( cij, ckl, cmn \) are concurrent in a point. It is a \( 3 \times 3 \) determinant whose rows are \( i \cap j, k \cap l, m \cap n \). This function can be expressed in terms of \( (ijk) \) by the Lagrange identity

\[
(v \wedge w) \cdot (m \wedge n) = (v \cdot m)(w \cdot n) - (v \cdot n)(w \cdot m).
\]

With \( v = i \wedge j, w = k \wedge l \), we get

\[
(ij, kl, mn) = (ijm)(kln) - (ijn)(klm) = (ijl)(kjm) - (ijk)(lmn) =
\]

\[
(ikl)(jmn) - (jkl)(imn),
\]

where the last two identities are obtained by a permutation of the rows of the mixed product. They can be seen also as a consequence of the Plücker relations.

Coble considers ([10] page 170) the following expressions, obtained by formally replacing in \((33)\) \((ij)(kl)(mn)\) with \((ij, kl, mn)\).

\[
\begin{align*}
\overline{a} &= (25, 13, 46) + (51, 42, 36) + (14, 35, 26) + (43, 21, 56) + (32, 54, 16) \\
\overline{b} &= (53, 12, 46) + (14, 23, 56) + (25, 34, 16) + (31, 45, 26) + (42, 51, 36) \\
\overline{c} &= (53, 41, 26) + (34, 25, 16) + (42, 13, 56) + (21, 54, 36) + (15, 32, 46) \\
\overline{d} &= (45, 31, 26) + (53, 24, 16) + (41, 25, 36) + (32, 15, 46) + (21, 43, 56) \\
\overline{e} &= (31, 24, 56) + (12, 53, 46) + (25, 41, 36) + (54, 32, 16) + (43, 15, 26) \\
\overline{f} &= (42, 35, 16) + (23, 14, 56) + (31, 52, 46) + (15, 43, 26) + (54, 21, 36)
\end{align*}
\]

which satisfy

\[
(36) \quad \overline{a} + \overline{b} + \overline{c} + \overline{d} + \overline{e} + \overline{f} = 0.
\]

Now any permutation of the points effects a permutation of the cubics, without any change of sign. For example after the permutation \((12)\) we get

\[
\overline{a} \rightarrow \overline{d}, \quad \overline{b} \rightarrow \overline{e}, \quad \overline{c} \rightarrow \overline{f}.
\]

Indeed \( \overline{a}, \ldots, \overline{f} \) span the representation \( X^8 \).

**Proposition 25.**

\[
\overline{a}a + \overline{b}b + \overline{c}c + \overline{d}d + \overline{e}e + \overline{f}f = 0.
\]
**Proof.** Consider the point at the intersection of $(12x)$ and $(34x)$. By a direct computation, see for example [14] Theor. 9.4.13, the cubic at the left-hand side of our statement vanishes. By the invariance, the cubic vanishes at all the 45 intersection points of the lines $(ijx)$ and $(klx)$, hence it has to contain the 15 lines and must vanish.

Let $a_i$ for $i = 2, \ldots, 6$ be the $i$-th elementary symmetric function in $\overline{a}, \ldots, \overline{f}$ (remind that by (36) we have $a_1 = 0$). Remind the expression $d_2$ in (25), which is an irrational invariant expressing that the six points lie on a conic. Let $S$ be the product of the differences of $a, \ldots, f$. It is an irrational invariant of degree 15. Note that its square is the discriminant of the polynomial with roots $a, \ldots, f$ and it can be expressed as a polynomial in $a_2, \ldots, a_6$.

**THEOREM 41.** The Hilbert series of the invariant ring of six ordered points on $\mathbb{P}^2$ is

$$1 + r^2 = \frac{1 - r^4}{(1 - r)^5(1 - r^2)}.$$  

The ring is generated by $\overline{a}, \ldots, \overline{f}$ and by $d_2$. We have the relation in degree 4 $d_2^2 = a_2^2 - 4a_4$.

**Proof.** The Hilbert series reduces to a computation of semistandard tableau, that can be achieved by counting the integral points in a certain polytope, see [15] §11.2. The invariants of degree 1 are generated by $\overline{a}, \ldots, \overline{f}$ with the relation (36). In alternative, also the tableau functions $(123)(456)$, $(124)(356)$, $(125)(346)$, $(134)(256)$, $(135)(246)$ can be taken as generators. Note that $d_2$ cannot be obtained as a polynomial in $\overline{a}, \ldots, \overline{f}$ because otherwise it should be a multiple of $\Delta$. Indeed all $\text{Alt}(6)$-invariants can be written as $\Delta \cdot p$ where $p$ is a symmetric polynomial in $\overline{a}, \ldots, \overline{f}$ (see also Theorem 42).

**REMARK 18.** This description shows that Kempe’s Lemma (Theorem 31) fails for points on $\mathbb{P}^2$, indeed $d_2$ is a generator which has degree 2, while the minimal degree is 1.

**THEOREM 42 (Coble).** The Hilbert series of the $\text{Alt}(6)$-invariant ring of six (unordered) points on $\mathbb{P}^2$ is

$$1 + r^{15} = \frac{1 - r^4}{(1 - r)^2(1 - r^3)(1 - r^5)(1 - r^6)}.$$  

The ring is generated by $a_2, d_2, a_3, a_5, a_6, \Delta$ with the relation in degree 30 expressing $\Delta^2$ as a polynomial in the other generators.

**Proof.** $X_8$ is an irreducible representation of $\text{Alt}(6)$. We already computed in the proof of Theorem 38 the Hilbert series of $\oplus_p S^p(X_8)^{\text{Alt}(6)}$ by using Molien formula.

The result is

$$1 + r^{15} = \frac{1 - r^4}{(1 - r^2)(1 - r^3)(1 - r^4)(1 - r^5)(1 - r^6)}.$$  

(37)
The factors in the denominator correspond to $a_2, \ldots, a_6$. This suggests that there is another generator of degree 15, which is identified with $\Delta$. Now from Theorem 41 we have to add $d_2$ to the generators, which is already $\text{Alt}(6)$-invariant. We get that $a_4$ can be deleted by the generators.

**Theorem 43 (Coble).** The Hilbert series of the $\Sigma_6$-invariant ring of six (unordered) points on $\mathbb{P}^2$ is

$$1 + t^{17}$$

$$\frac{1}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)}$$

The ring is generated by $a_2, a_3, a_4, a_5, a_6, d_2 \Delta$ with the relation in degree 34 expressing $(d_2 \Delta)^2$ as a polynomial in $a_2, \ldots, a_6$.

**Proof.** Among the generators of the $\text{Alt}(6)$-invariant ring, $a_2, a_3, a_5, a_6$ are already $\Sigma_6$-invariant. The relation $d_2^2 = a_2^2 - 4a_4$ allows to add $a_4$ at the generators at the place of $d_2$. Also $d_2 \Delta$ is a $\Sigma_6$-invariant of degree 17.

**Remark 19.** By adding the contribution of the remaining five odd conjugacy classes to (37) we get

$$1$$

$$\frac{1}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)}.$$
These are known as Cremona hexahedral equation [11, 14]. They show that the general cubic surface is a hyperplane section of the Segre cubic primal defined in Remark 16.

The following line

\[ a + b = c + d = e + f = 0 \]

belongs to the cubic surface. By applying permutations, we get 15 such lines.

The remaining 12 lines correspond to the proper transform of the six lines and of the six conics through five among the six points. They form a so called double-six on the cubic surface.

References


AMS Subject Classification: 13A50, 14N05, 15A72

Giorgio OTTAVIANI,
Dipartimento di Matematica e Informatica “U. Dini”, Università di Firenze
viale Morgagni 67/A, 50134 Firenze, ITALY
e-mail: ottavian@math.unifi.it

Lavoro pervenuto in redazione il 18.06.2013.