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ON GENERALIZED JORDAN LEFT $*$-DERIVATIONS IN RINGS

Abstract. In this paper first we define the notion of Jordan left $*$-derivation and generalized Jordan left $*$-derivation on a $*$-ring R and then proved the following: Let $n \geq 1$ be a fixed integer and R be an $(n+1)!$-torsion free $*$-ring with identity element e. If $F, d : R \to R$ are two additive mappings satisfying $F(x^{n+1}) = (x^*)^n F(x) + \sum_{i=1}^{n} (x^*)^{n-i} x^i d(x)$ for all $x \in R$, then d is a Jordan left $*$-derivation and F is a generalized Jordan left $*$-derivation on R.

1. Introduction

Throughout R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0, x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. Recall that R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in R$. Every derivation is a Jordan derivation but the converse need not be true in general. A classical result of Herstein [8, Theorem 3.3] states that every Jordan derivation on a prime ring of characteristic different from two is a derivation. Bresar and Vukman proved this result briefly in [7]. Further, Cusack [6] has generalized this result for semiprime ring stating that every Jordan derivation on a 2-torsion free semiprime ring is a derivation (see also [6] for an alternate proof). In 1991 [5] Bresar has introduced the concept of generalized derivation as follows: an additive mapping $F : R \to R$ is said to be generalized derivation if there exists an associated derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all pairs $x, y \in R$. An additive mapping $F : R \to R$ is said to be generalized Jordan derivation if there exists a Jordan derivation $d : R \to R$ such that $F(x^2) = F(x)x + xd(x)$ for all $x \in R$. In 2000 [1], Ashraf and the first author showed that in a 2-torsion free ring, which has a commutator nonzero divisor, every generalized Jordan derivation on R is generalized derivation. Recently, in 2007 Vukman [11] has proved that every generalized Jordan derivation on a 2-torsion free semiprime ring is a generalized derivation.

Following [4], an additive mapping $d : R \to R$ is called Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$. One can easily prove that any Jordan derivation on a 2-torsion free ring is a Jordan triple derivation (see example [4], where further reference can be found). Bresar [4] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Motivated by the definition of generalized Jordan derivation, Jing and Lu [12] introduced the concept of generalized Jordan triple derivation as follows: an additive mapping $F : R \to R$ is said to be a generalized Jordan triple derivation on R if there exists a Jordan triple derivation $d: R \to R$ such that $F(xyx) = F(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$. In 1991 Bresar defined the notion of left derivation which is as follows: An additive mapping $D: R \to R$ is said to be a left derivation (resp. Jordan left derivation) if $D(xy) = xD(y) + yD(x)$ (resp. $D(x^2) = 2xD(x)$) holds for all $x, y \in R$. An additive mapping $D: R \to R$ is said to be a right derivation (resp. Jordan right derivation) if $D(xy) = D(x)y + D(y)x$ (resp. $D(x^2) = 2D(x)x$) holds for all $x, y \in R$. Clearly, every left (resp. right) derivation on a ring R is a Jordan left (resp. Jordan right) derivation but the converse need not be true in general. In [2] Ashraf et. al. proved that a Jordan left derivation on a 2-torsion free prime ring is a left derivation. Very recently, Ashraf et. al. got the following result:

THEOREM 1 ([3, Theorem 2.1]). *Let $n \geq 1$ be any fixed integer and let R be any $(n+1)!$-torsion free ring with identity element e. If $F, D: R \to R$ are the additive mappings such that $F(x^{n+1}) = x^n F(x) + nx^n D(x)$ for all $x \in R$, then D is Jordan left derivation and F is generalized Jordan left derivation on R.*

An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$-ring. An additive mapping $d: R \to R$ is called a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) for all $x, y \in R$. An additive mapping $F: R \to R$ is said to be a generalized $*$-derivation (resp. generalized Jordan $*$-derivation) if there exists a $*$-derivation (Jordan $*$-derivation) such that $F(xy) = F(x)y^* + xd(y)$ (resp. $F(x^2) = F(x)x^* + xd(x)$) for all $x, y \in R$. Motivated by the definition of $*$-derivation and generalized $*$-derivation, Rehman et. al. in [10] introduced the notions of left $*$-derivation and generalized left $*$-derivation as follows. Let R be a $*$-ring. An additive mapping $d: R \to R$ is said to be left $*$-derivation if $d(xy) = x^*d(y) + yd(x)$ for all $x, y \in R$. An additive mapping $F: R \to R$ is said to be generalized left $*$-derivation if there exists a left $*$-derivation d such that $F(xy) = x^*F(y) + yd(x)$ for all $x, y \in R$. The concept of generalized left $*$-derivations cover the concept of left $*$-derivations. Moreover, a generalized left $*$-derivation with $d = 0$ includes the concept of right $*$-centralizer i.e., an additive mapping $T: R \to R$ satisfying $T(xy) = x^*T(y)$ for all $x, y \in R$. Motivated by the definitions of left $*$-derivation and generalized left $*$-derivation on a $*$-ring, it is very obvious to introduce the concepts of Jordan left $*$-derivation and generalized Jordan left $*$-derivation on a $*$ ring R, which is as follows: An additive mapping $d: R \to R$ is said to be Jordan left $*$-derivation if $d(x^2) = x^*d(x) + xd(x)$ for all $x \in R$. An additive mapping $F: R \to R$ is said to be generalized Jordan left $*$-derivation if there exists a Jordan left $*$-derivation d such that $F(x^2) = x^*F(x) + xd(x)$, for all $x \in R$.

Now, let $F: R \to R$ and $d: R \to R$ two additive mappings; following the goals of the previous results and, in particolar, of the Theorem 1, here we will find a condition that leads F and d to be, respectively, a generalized Jordan left $*$-derivation and a Jordan left $*$-derivation.

2. Main Theorem

In this paper we prove the following result:

THEOREM 2. *Let $n \geq 1$ be a fixed integer and let R be an $(n+1)!$-torsion free $*$-ring with identity element e. If $F, d : R \to R$ are two additive mappings satisfying*

$$F(x^{n+1}) = (x^*)^n F(x) + \sum_{i=1}^{n} (x^*)^{n-i} x^i d(x) \tag{1}$$

for all $x \in R$, then d is a Jordan left $*$-derivation and F is a generalized Jordan left $*$-derivation on R.

Proof. Replacing x by e, the identity element of R, in (1), we get $F(e) = (e^*)^n F(e) + \sum_{i=1}^{n} (e^*)^{n-i} e^i d(e)$. Recall that, by [[9], Lemma 2.1], $e^* = e$, then we obtain $F(e) = F(e) + nd(e)$. This implies that $nd(e) = 0$ and since R is n-torsion free, we get $d(e) = 0$. Replacing x by $x + ke$ in (1), where k be a positive integer, we obtain

(2)

$$F((x+ke)^{n+1}) = ((x+ke)^*)^n F(x+ke) + \sum_{i=1}^{n} ((x+ke)^*)^{n-i} (x+ke)^i d(x+ke) =$$

$$= (x^* + ke)^n (F(x) + kF(e)) + \sum_{i=1}^{n} (x^* + ke)^{n-i} (x+ke)^i d(x).$$

On expanding, we find that

$$F\Big[x^{n+1} + \binom{n+1}{1} x^n k + \binom{n+1}{2} x^{n-1} k^2 + \cdots + k^{n+1} e\Big] =$$

$$= \Big[(x^*)^n + \binom{n}{1} (x^*)^{n-1} k + \binom{n}{2} (x^*)^{n-2} k^2 + \cdots + k^n e\Big] (F(x) + kF(e)) +$$

$$+ \sum_{i=1}^{n} \Big[(x^*)^{n-i} + \cdots + \binom{n-i}{n-i-2} (x^*)^2 k^{n-i-2} + \binom{n-i}{n-i-1} x^* k^{n-i-1} + k^{n-i} e\Big] \cdot$$

$$\cdot \Big[x^i + \cdots + \binom{i}{i-2} x^2 k^{i-2} + \binom{i}{i-1} x k^{i-1} + k^i e\Big] d(x)$$

Now, using (1) we obtain

$$F\Big[\binom{n+1}{1}x^nk+\binom{n+1}{2}x^{n-1}k^2+\cdots+k^{n+1}e\Big]=k(x^*)^nF(e)+\Big[\binom{n}{1}(x^*)^{n-1}k+$$

$$+\binom{n}{2}(x^*)^{n-2}k^2+\cdots+\binom{n}{n-2}(x^*)^2k^{n-2}+\binom{n}{n-1}x^*k^{n-1}+k^ne\Big](F(x)+kF(e))+$$

$$+\sum_{i=1}^{n}\Big[\binom{n-i}{1}(x^*)^{n-i-1}k+\cdots+\binom{n-i}{n-i-2}(x^*)^2k^{n-i-2}+\binom{n-i}{n-i-1}x^*k^{n-i-1}+$$

$$+k^{n-i}e\Big]x^id(x)+\sum_{i=1}^{n}\cdot\Big[(x^*)^{n-i}+\binom{n-i}{1}(x^*)^{n-i-1}k+\ldots+\binom{n-i}{n-i-2}(x^*)^2k^{n-i-2}+$$

$$+\binom{n-i}{n-i-1}x^*k^{n-i-1}+k^{n-i}e\Big]\cdot\Big[\binom{i}{1}x^{i-1}k+\binom{i}{2}x^{i-2}k^2+\cdots+\binom{i}{i-2}x^2k^{i-2}+$$

$$\binom{i}{i-1}xk^{i-1}+k^ie\Big]\cdot d(x)$$

Since k is different from zero, it can be written as

$$f_0(x^*,e)+kf_1(x^*,e)+\cdots+k^nf_n(x^*,e)=0$$

for all $x\in R$, where $f_i(x^*,e)$ are the coefficients of $k^i{}'s$, for all $i=0,2,\cdots,n$. Now, replacing k by $1,2,\cdots,n+1$ in turn and considering the resulting system of n homogeneous equations, we get that the resulting matrix of the system is a Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1\\ 1 & 2 & 2^2 & \ldots & 2^n\\ 1 & 3 & 3^2 & \ldots & 3^n\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 1 & n+1 & (n+1)^2 & \ldots & (n+1)^n \end{pmatrix}$$

Since the determinant of the matrix is equal to the product of positive integers, each of which is less then $n+1$, and since R is $(n+1)!$-torsion free, it follows immediately that $f_i(x^*,e)=0$, for all $x\in R$ and $i=0,2,\cdots,n$. Now, $f_{n-1}(x^*,e)=0$ implies that

$$(n+1)F(x)=F(x)+nx^*F(e)+nd(x)$$

for all $x\in R$. This yields that $nF(x)=nx^*F(e)+nd(x)$, that is

$$F(x)=x^*F(e)+d(x) \tag{3}$$

for all $x\in R$.

Again, $f_{n-2}(x^*,e)=0$ gives that

$$n(n+1)F(x^2)=2nx^*F(x)+n(n-1)(x^*)^2F(e)+n(n+1)xd(x)+n(n-1)x^*d(x)$$

for all $x\in R$. Since R is n-torsion free, then we obtain

$$(n+1)F(x^2)=2x^*F(x)+(n-1)(x^*)^2F(e)+(n+1)xd(x)+(n-1)x^*d(x)$$

for all $x\in R$. Thanks to (3), it becomes

$$(n+1)F(x^2)=(n+1)(x^*)^2F(e)+(n+1)x^*d(x)+(n+1)xd(x)$$

Since R is $(n+1)$-torsion free, then

$$F(x^2)=(x^*)^2F(e)+x^*d(x)+xd(x) \tag{4}$$

for all $x\in R$. Replacing x by x^2 in (3), we get

$$F(x^2)=(x^*)^2F(e)+d(x^2) \tag{5}$$

Comparing (4) and (5), we find that

$$d(x^2)=x^*d(x)+xd(x) \tag{6}$$

for all $x\in R$. Now, by (4), we can write

$$F(x^2)=(x^*)^2F(e)+x^*d(x)+xd(x)=$$

$$=x^*(x^*F(e)+d(x))+xd(x)$$

for all $x\in R$. Again, by (3), it follows

$$F(x^2)=x^*F(x)+xd(x) \tag{7}$$

for all $x\in R$. The relations (6) and (7) lead to our required conclusions.

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