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POSITIVITY CRITERIA FOR A GENERALIZED SCHRÖDINGER OPERATOR

Abstract. We determine a lower bound corresponding for a generalized Schrödinger operator with Dirichlet boundary conditions, besides with a meaningful assumption on the potential, this lower is provided explicitly.

1. Introduction

The authors V.Maz'ya and M.Shubin studied the positivity criteria for the Schrödinger operator $-\Delta+V$ with Dirichlet condition [28]. The present paper, is focused on the positivity criteria for the generalized Schrödinger operator $H_{m,V}=(-\Delta)^m+V$. More precisely, we show that for a locally real potential with an additional hypothesis on the potential, we obtain an explicit lower bound of $H_{m,V}$ defined on $L^2(\Omega)$, with Ω is an open set of $\mathbb{R}^d$, and $d>2m$. The boundary conditions are generalized Dirichlet conditions, i.e., each outward normal derivative of order less than $m-1$ is equal to zero. Regarding the latest headway on this topics, as far as I know, e.g., the authors Y.Karpeshina and Y.R.Lee [18] studied the spectrum of $H_{m,V}$ from a certain values of m and with a limit-periodic potential, i.e., the potential is stood for by a series of a periodic potentials, and they showed that the spectrum is continue and contains a semi axis, besides the lower bound corresponding to this semi axis is an eigenvalue of $H_{m,0}$, their works are inspired from [15], [35], and [36]. On the other side the authors A.Boccuto and R.Filippucci [3] studied the eigenvalues of $\sigma^{-1}H_{m,0}$ with σ is a positive density, and they looked into on a bounded set, the variational characterization of the first eigenvalue.

The author Z.Chen [4] worked on the spectrum of $H_{m,0}$ and showed that the eigenvalues hinge on the first one and the authors J.Jost, X. Li-Jost, Q.Wang, and C.Xia extended his work on a compact manifold [17]. Meanwhile, this paper improves and evolves the statement of the [11, Théorème 3], which also provides a sufficient condition on the potential for a lower bound of $H_{m,V}$, but this lower bound was not given explicitly. For the case $m=1$, lately in [12] the author M.El Aïdi provides with accuracy a lower bound for a hyperbolic Schrödinger operator defined in a proper open set of $\mathbb{H}^d$—the hyperbolic space.

Before to state the significant theorem of the present article, we need the following definitions *negligible set* and *capacitary interior diameter* which are extracted from the book [26].

DEFINITION 1. *([26, §14.1.1]) Let* Ω *be an open set of* $\mathbb{R}^d$, Q_δ *a Euclidean cube of size* $\delta > 0$, *that is* $Q_\delta := \{x = (x_l)_{1\le l\le d} \in \mathbb{R}^d \ s.t.\ |x_l - x_{0l}|_{\mathbb{R}} < \frac{\delta}{2}\}$, $\overline{Q_\delta}$ *its closure,* $|\cdot|_{\mathbb{R}}$ *is the metric distance defined on* $\mathbb{R}$ *and* $x_0 \in \mathbb{R}^d$. *Let* $\gamma \in (0,1)$, *the negligibility set* $\mathcal{N}_\gamma(\overline{Q_\delta},\Omega)$ *consists of all compact sets* $F \subset \overline{Q_\delta}$ *satisfying the following conditions*

$$\overline{Q_\delta} \setminus \Omega \subset F \subset \overline{Q_\delta} \text{ and } cap_m(F) \le \gamma cap_m\left(\overline{Q_\delta}\right).$$

$cap_m(F) := \inf_{u\in C_0^\infty(\mathbb{R}^d)} \left\{ \int_{\mathbb{R}^d} |\nabla_m u(x)|^2 dx,\ s.t.\ u_{|F} = 1,\ F \Subset \mathbb{R}^d \right\}$ *called the polyharmonic capacity where*

$\int_{\mathbb{R}^d} |\nabla_m u(x)|^2 dx := \sum_{|\alpha|=m} \int_{\mathbb{R}^d} |D^\alpha u(x)|^2 dx,$

and $D^\alpha u(x) := \dfrac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}$, $\alpha = (\alpha_l)_{1\le l\le d} \in \mathbb{N}^d$, *its length is defined by* $|\alpha| = \sum_{l=1}^{d} \alpha_l$.

An element of $\mathcal{N}_\gamma(\overline{Q_\delta},\Omega)$ is called γ-*negligible set* in $\overline{Q_\delta}$.

The motivation of this definition is that when Ω is a proper open set of $\mathbb{R}^d$, it does not contain a cube of maximal size; when we cover the whole space $\mathbb{R}^d$ with a family of cubes, then Ω is also covered by this family but some cubes do not lie completely inside of Ω, for otherwise $\Omega = \mathbb{R}^d$. Thus we get the following remark.

REMARK 1. The set $\mathcal{N}_1(\overline{Q_\delta},\Omega)$ consists of all compact of the closed cube $\overline{Q_\delta}$. If $\mathcal{N}_\gamma(\overline{Q_\delta},\Omega)$ is not an empty set, then Q_δ is almost completely inside Ω by allowing some parts of Q_δ to be out of Ω, i.e., $\overline{Q_\delta} \setminus \Omega \ne \emptyset$, and these parts have a very small capacity.

In the sequel, we need the following definition extracted from [26, §14.2.2, §16.2] (see also [29, p.207]).

DEFINITION 2. *The capacitary interior (or inner) diameter of* Ω *is the following number*

$$\delta_{\mathcal{V},\gamma,\Omega} := \sup_{Q_\delta \in Q_\Omega} \left\{ \delta,\ such\ that\ \delta^{d-2m} \ge \inf_{F\in\mathcal{N}_\gamma(\overline{Q_\delta},\Omega)} \int_{\overline{Q_\delta}\setminus F} \mathcal{V}(x)dx \right\},$$

with $\mathcal{V}$ *is a positive locally integrable function.*

$Q_\Omega := \left\{ \overline{Q_\delta} :\ cap_m(\overline{Q_\delta} \setminus \Omega) \le \gamma cap_m(\overline{Q_\delta}) \right\}.$

Precisely, Q_Ω is the set of all cubes Q_δ owning a negligible intersection with the complement of Ω.

The case when $\mathcal{V} \equiv 0$ then the capacitary interior diameter becomes equal to the number $\sup\{\delta,$ there is a cube $\overline{Q_\delta}$ $s.t.\ cap_m(\overline{Q_\delta} \setminus \Omega) \leq \gamma cap_m(\overline{Q_\delta})\}$, i.e., $\delta_{0,\gamma,\Omega}$ is the maximal size of a cube belonging to Q_Ω. The authors V.Maz'ya and M.Shubin [27] used Q_Ω to look for a lower bound and an upper bound corresponding to the Laplacian operator with Dirichlet boundary condition.

REMARKS 1. We observe that from the definition of $\mathcal{N}_\gamma(\overline{Q_\delta}, \Omega)$ it follows that for $\mathcal{V}$ fixed, the function $\delta_{\mathcal{V},\gamma,\Omega}$ is increasing w.r.t. Ω (and respectively w.r.t. γ), but in this paper we take both Ω and γ fixed. The number $\delta_{\mathcal{V},\gamma,\Omega}$ can be equal to $+\infty$, e.g., for $\Omega = \mathbb{R}^d$ and $\mathcal{V} \equiv 0$ then $\delta_{0,\gamma,\mathbb{R}^d} = +\infty$. From the definition we have $\delta_{\mathcal{V},\gamma,\Omega} > 0$.

Right now, we state our significant theorem for this paper.

THEOREM 1. *Let $V \in L^1_{loc}(\Omega)$ such that*
$\delta^{d-2m} \leq \int_{\overline{Q_\delta} \setminus F} V(x)dx$ *for all γ-negligible set F in $\overline{Q_\delta}$. Then we have*

$$H_{m,V} \geq K^{-1}_{d,V,\gamma,\Omega} \ on \ L^2(\Omega),$$

with

$$K_{d,V,\gamma,\Omega} = \frac{2^{2d-1}3^{d-2m}(3^{2(m+1)}-1)\delta^{2m}_{V,\gamma,\Omega}}{\gamma c_{d,m}} > 0, for\ \gamma \to 0^+.$$

The constant $c_{d,m}$ comes from $cap_m(\overline{Q_\delta}) \geq c_{d,m}\delta^{d-2m}$, for more detail on this constant, see the Lemma 3 in below. By applying the spectral theorem, we can provide a similar version of the Theorem 1 for a real potential semi-bounded from below.

The operator $(-\Delta)^m$ is elliptic, i.e., there is a strictly positive constant a_0 such that

$$\langle(-\Delta)^m u, u\rangle_{L^2(\Omega)} \geq a_0 \int_\Omega \sum_{|\alpha|=m} |D^\alpha u(x)|^2 dx,$$

with $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ is the inner product defined on the Hilbert space $L^2(\Omega)$.
Without loss of generality, we take $a_0 \geq 1$, and we work with the quadratic form Q_V associated to the operator $H_{m,V}$, and defined as

$$Q_V(u) := \int_\Omega \sum_{|\alpha|=m} |D^\alpha u(x)|^2 dx + \int_\Omega V(x)u^2(x)dx \quad \text{for} \quad u \in C_0^\infty(\Omega),$$

and Q_V is closable w.r.t. the following norm

$$||u||_{Q_V} = \left(Q_V(u) + ||u||^2_{L^2(\Omega)}\right)^{1/2},$$

then Q_V corresponds to an unique self-adjoint operator namely $H_{m,V}$ [32, §VIII.6].

There is an alternative way to show that Q_V is closable. Precisely, Q_V is closable in

$L^2(\Omega)$ if and only the measure $d\mu(x) := V(x)dx$ is $(2,m)$—*absolutely continuous*, i.e., if $cap_m(B) = 0$ then $\mu(B) = 0$ **,** for a Borel set B [26, §16.4 Theorem1].

The proof of the Theorem 1 is done through Q_V.

We recall that the authors Y.Egorov, V.Kondratiev and M.El Aïdi provided for a class of potentials, an upper bound of the number of all negative eigenvalue associated to the operator $H_{m,-V}$ with more general boundary conditions, i.e., Robin conditions [9], [10], [13].

Some applications on macroscopic and microscopic objects.

We consider the following problem

$$(\mathrm{GDP})_{m,V} : \begin{cases} ((-\Delta)^m + V)\, u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \ldots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\frac{\partial u}{\partial \nu}$ is the outward normal derivatives, ν is the outward normal vector on the boundary $\partial\Omega$, and λ is an eigenvalue.

As an application of the $(\mathrm{GDP})_{m,V}$, we have the special case $(\mathrm{GDP})_{2,0}$, which is called *the clamped plate problem* physically its eigenvalues stand for the properties to the trembling of a campled plate. Thereby, the authors Q.M. Cheng, T.Ichikawa, and S.Mametsuka studied this problem in a compact Riemannian immersed in Euclidean space this is by using the Nash's theorem [5], [6], [37], *et al.*

The study of the eigenvalues corresponding to the problem $(\mathrm{GDP})_{1,0}$ permits us to look into the properties of a fixed membrane. Concerning a microscopic application, we have an interesting case which is the study of the properties of the first eigenvalue, corresponding to a Schrödinger Dirichlet problem $(\mathrm{GDP})_{1,V}$ for $d = 3$, in quantum mechanics it is called the *stability of the first kind*, i.e., the first finite energy level of a particle in motion trapped in an electric field, besides knowing its lower bound it is called *stability of the second kind* or *stability of matter states*. The author J.P.Solovej [34] made an insightful review on stability of matter of the E.Lieb and R.Seirenger' book [22], for more bibliography on this area see [2, 7, 23, 24, 25, 33], *et al.*

Scheme for showing the Theorem 1.

The second section, is focused on a generalization of [20, Lemma 2.1], for getting a lower bound of Q_V when restricted to smooth function. In the third section is reserved to look for a lower bound of Q_V defined for bump function, more precisely we cover its support by congruent cubes and use a partition of unity. It follows that we get a sufficient condition for a continuous embedding between two Sobolev spaces, with an explicit bound for the norm. Finally, effortlessly, we afford an upper bound corresponding to the spectrum's bottom associated to $H_{m,V}$.

2. Miscellaneous Lemmas

We recall the complete version of [20, Lemma 2.1] showed by the authors V.Kondratiev, V.Maz'ya, and M.Shubin.

LEMMA 1. *For every function $u \in Lip(\overline{Q_\delta})$ vanishing on a compact set $F \subset \overline{Q_\delta}$, i.e., the Euclidean distance between the support of u and F is strictly positive, u is not identically null on $\overline{Q_\delta}$, we have*

$$cap_1(F) \int_{\overline{Q_\delta}} |u(x)|^2 dx \le \frac{C_d}{\delta^{-d}} \int_{\overline{Q_\delta}} |\nabla u(x)|^2 dx, \tag{1}$$

with $C_d = 2.3^d \left(1 + \frac{16}{\pi^2}\right)$.

Thereby, our goal is to give a generalized version of this result, thus without loss of generality we take a *standard normalization* function, i.e., we work with $u \in C^\infty(\overline{Q_\delta})$ such that $\delta^{-d} \int_{\overline{Q_\delta}} |u(x)|^2 dx = 1$. And we show the following lemma

LEMMA 2. *Let $u \in C^\infty(\overline{Q_\delta})$ and for all $\alpha \in \mathbb{N}^n$ with $1 \le |\alpha| \le m-1$ we suppose that for $z \in \partial\overline{Q_\delta}$ we have $D^\alpha u(z) = 0$, i.e., the trace of u at z is null, besides $u_{|F} = 0$ for F a compact set in $\overline{Q_\delta}$. Then we have*

$$cap_m(F) \le \left(\frac{3^{d-2m}(3^{2(m+1)}-1)}{2^3}\right) \int_{\overline{Q_\delta}} |\nabla_m u(x)|^2 dx.$$

Proof. We recall that $C^\infty(\overline{Q_\delta})$ is the set of complex-valued smooth functions, and $|u(x)|^2 = |\Re u(x)|^2 + |\mathrm{Im}\, u(x)|^2$ where $\Re$ is the real part of a complex number and and Im its imaginary part, thus in the sequel it is enough to use real-valued functions. The Cauchy-Schwarz inequality applied between the functions 1 and $|u|$ gives us

$$\overline{|u|} := \frac{1}{|Q_\delta|} \int_{Q_\delta} |u(x)| dx \le \left(|Q_\delta|^{-1} \int_{\overline{Q_\delta}} |u(x)|^2 dx\right)^{1/2} = 1,$$

the volume of Q_δ is presented by $|Q_\delta| := \delta^d$. For our study, we consider the function $\psi = 1 - u$, its average is well positive and $\psi_{|F} = 1$ We define a standard cutoff function $\phi \in [0,1]$ defined by $\phi_{|\overline{Q_\delta}} = 1$, $\phi_{\mathbb{R}^d \setminus Q_{2\delta}} = 0$. Furthermore, we take $|D^\zeta \phi| \le k_\zeta \delta^{-|\zeta|}$ where k_ζ is a constant depending on $\zeta = (\zeta_l)_{1 \le l \le d} \in \mathbb{N}^d$.

By definition, the capacity is defined through a smooth function with a compact support, thus let ψ_1 be the extension function defined on $\overline{Q_{3\delta}}$ by $\psi_1(x) = \psi(x/3)$. Consequently, the function $\psi_1\phi$ belongs to $C_0^\infty(\overline{Q_{3\delta}}) \subset C_0^\infty(\mathbb{R}^d)$ and $(\psi_1\phi)_{|F} = 1$. By using an elementary calculus, we have

$$\int_{\overline{Q_{3\delta}}} |D^\alpha \psi_1(x)|^2 dx = 3^{d-2|\alpha|} \int_{\overline{Q_\delta}} |D^\alpha \psi(x)|^2 dx,$$

for all $\alpha \in \mathbb{N}^d$.
Since that the function $\psi \in C^\infty(\overline{Q_\delta})$ does not vanish then, first of all, we apply Neumann Poincaré inequality to ψ, precisely an easy calculus, we obtain

$$\overline{\psi} \leq \delta^{-d/2} \int_{Q_\delta} |u(x) - \overline{u}|^2 dx.$$

Whence by applying the Neumann Poincaré inequality to the function u we get

$$\overline{\psi}^2 \leq \frac{\delta^{2-d}}{\pi^2} \int_{Q_\delta} |\nabla\psi(x)|^2 dx.$$

Accordingly, we have

$$\begin{aligned} \int_{Q_\delta} |\psi(x)|^2 dx &\leq 2\int_{Q_\delta} |\psi(x) - \overline{\psi(x)}|^2 dx + 2\int_{Q_\delta} |\overline{\psi(x)}|^2 dx \\ &\leq \delta^2 \int_{Q_\delta} |\nabla\psi(x)|^2 dx. \end{aligned}$$

Secondly, we apply quasi-generalized Friedrichs'inequality[1] to $\nabla_k\psi = \nabla_k u$ for $k \in \{1, \ldots, m-1\}$ and we obtain

$$\begin{aligned} cap_m(F) &\leq \int_{\overline{Q_{2\delta}}} \left|\nabla_m\big((\psi_1\phi)(x)\big)\right|^2 dx \\ &= \int_{\overline{Q_{2\delta}}} \sum_{|\alpha|=m} \left|D^\alpha(\psi_1\phi)(x)\right|^2 dx \\ &\leq \int_{\overline{Q_{2\delta}}} \sum_{\substack{\beta\leq\alpha \\ |\alpha|=m}} c_{\beta,\alpha} \left|D^\beta\psi_1(x) D^{\alpha-\beta}\phi(x)\right|^2 dx, \\ &\leq \int_{\overline{Q_{3\delta}}} \sum_{\substack{\beta\leq\alpha \\ |\alpha|=m}} \delta^{-2|\alpha-\beta|} \left|D^\beta\psi_1(x)\right|^2 dx \qquad (*) \\ &= \int_{\overline{Q_\delta}} \sum_{\substack{\beta\leq\alpha \\ |\alpha|=m}} \delta^{-2|\alpha-\beta|} 3^{d-2|\beta|} \left|D^\beta\psi(x)\right|^2 dx \\ &= 3^d \int_{\overline{Q_\delta}} \sum_{k=0}^{m} \delta^{-2m+2k} 3^{-2k} |\nabla_k\psi(x)|^2 dx \\ &\leq 3^d ||\nabla_m\psi(x)||^2_{L^2(\overline{Q_\delta})} \sum_{k=0}^{m} 3^{-2k} \\ &\leq \frac{3^{d-2m}(3^{2(m+1)}-1)}{2^3} ||\nabla_m\psi(x)||^2_{L^2(\overline{Q_\delta})}. \end{aligned}$$

[1] Quasi-generalized Friedrichs' inequality: For $u \in C^\infty(\overline{Q_\delta})$ and $z \in \partial\overline{Q_\delta}$ such that $D^\alpha u(z) = 0$ for $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq m-1$, we have

$$||\nabla_k u||_{L^2(\overline{Q_\delta})} \leq \delta^{m-k} ||\nabla_m u||_{L^2(\overline{Q_\delta})},\ k \in \{1, \ldots, m-1\},$$

the proof is based on the classical Friedrichs' inequality.

The constant $c_{\beta,\alpha}$ depends on the multi-indices β,α. Precisely, $c_{\beta,\alpha}$ comes from the following Leibniz Formula

$$\begin{aligned}|D^{\alpha}(\psi_1\phi)(x)|^2 &= \left|\sum_{\beta\leq\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}D^{\beta}\psi_1(x)D^{\alpha-\beta}\phi(x)\right|^2\\ &\leq \sum_{\beta\leq\alpha}c_{\beta,\alpha}|D^{\beta}\psi_1(x)|^2|D^{\alpha-\beta}\phi(x)|^2,\end{aligned}$$

$\alpha! := \prod_{l=1}^{d}\alpha_l$.

Regarding the inequality $(*)$, we used $|D^{\alpha-\beta}\phi(x)| \leq k_{\alpha-\beta}\delta^{-|\alpha-\beta|}$, and we chose $k_{\alpha-\beta} \leq c_{\beta,\alpha}^{-1/2}$. □

Thereby, we have, with the same assumption as Lemma 2,

$$cap_m(F)\int_{\overline{Q_\delta}}|u(x)|^2dx \leq \frac{C_{d,m}}{\delta^{-d}}\int_{\overline{Q_\delta}}|\nabla_m u(x)|^2dx, \tag{2}$$

with $C_{d,m} = \frac{3^{d-2m}(3^{2(m+1)}-1)}{2^3}$.

Now we show the following crucial Lemma

LEMMA 3. *Let $V \in L^1_{loc}(\Omega)$ with $\delta^{d-2m} \leq \int_{\overline{Q_\delta}\setminus F} V(x)dx$ for all γ-negligible set F in $\overline{Q_\delta}$, then for $u \in C^\infty(\overline{Q_\delta})$ we have*

$$K^{-1}_{d,\gamma,\delta}\int_{\overline{Q_\delta}}|u(x)|^2dx \leq \int_{\overline{Q_\delta}}|\nabla_m u(x)|^2dx + \int_{\overline{Q_\delta}}|u(x)|^2V(x)dx, \tag{3}$$

with $K_{d,\gamma,\delta} := \dfrac{3^{d-2m}(3^{2(m+1)}-1)}{2\gamma c_{d,m}\delta^{-2m}} > 0$, *for* $\gamma \to 0^+$,

$c_{d,m}$ is a constant relying on d,m and defined through the polyharmonic capacity of $\overline{Q_\delta}$, precisely $cap_m(\overline{Q_\delta}) \geq c_{d,m}\delta^{d-2m}$ [26, §10.4.3] (see also [1, Theorem 5.2]). For the case of a ball of radius δ, B_δ we have $cap_1(B_\delta) = (d-2)\omega_{d-1}\delta^{d-2}$ [26, §2.2.4], where ω_{d-1} is the area measure of the unit sphere $\mathbb{S}^{d-1}$. The authors M.Shubin and V.Kondratiev [21] gave an equivalent definition of the capacity, and they determined in another manner the value of $cap_1(B_\delta)$, this is by looking for on a ball the fundamental solution corresponding to the Laplacian operator.

Regarding the proof, we work again with real- valued function and we follow the techniques used for the proof of the [20, Lemma 2.2]. The inequality (3) of the Lemma 3 is satisfied for $u \in C^\infty(\overline{Q_\delta})$, then $u \neq 0$ in its support, so we consider the following relative open set

$$\mathcal{M}_\tau := \left\{x \in \overline{Q_\delta} \text{ such that } |u(x)| > \tau \geq 0\right\} \subset \overline{Q_\delta},$$

with $\tau^2 := \frac{1}{4\delta^d}\int_{\overline{Q_\delta}} |u(x)|^2 dx$. From the monotonicity of the capacity, see the definition, we have

$$cap_m\left(\overline{Q_\delta}\setminus \mathcal{M}_\tau\right) \leq cap_m\left(\overline{Q_\delta}\right).$$

So we need to deal with two cases, i.e., we have either:
$cap_m\left(\overline{Q_\delta}\setminus \mathcal{M}_\tau\right) > \gamma cap_m\left(\overline{Q_\delta}\right)$ or $cap_m\left(\overline{Q_\delta}\setminus \mathcal{M}_\tau\right) \leq \gamma cap_m\left(\overline{Q_\delta}\right)$.

For the case $cap_m\left(\overline{Q_\delta}\setminus \mathcal{M}_\tau\right) > \gamma cap_m\left(\overline{Q_\delta}\right)$, we have the following lemma

LEMMA 4. *Let $u \in C^\infty(\overline{Q_\delta})$, we have*

$$\delta^{-2m}\int_{\overline{Q_\delta}} |u(x)|^2 dx \leq b_{d,\gamma}\int_{\overline{Q_\delta}} |\nabla_m u(x)|^2 dx,$$

with $b_{d,\gamma} = \frac{3^{d-2m}(3^{2(m+1)}-1)}{2\gamma c_{d,m}}$ *and* $\gamma \to 0^+$.

Proof. Let $x \in \mathcal{M}_\tau$, then $|u(x)|^2 \leq 2\tau^2 + 2(|u(x)|-\tau)^2$, whence
for all $\tau \geq 0$, we have

$$\begin{aligned}\int_{\overline{Q_\delta}} |u(x)|^2 dx &\leq 2\tau^2\delta^d + 2\int_{\mathcal{M}_\tau}(|u(x)|-\tau)^2 dx\\ &\leq \frac{1}{2}\int_{\overline{Q_\delta}} |u(x)|^2 dx + 2\int_{\mathcal{M}_\tau}(|u(x)|-\tau)^2 dx.\end{aligned}$$

Otherwise

$$\int_{\overline{Q_\delta}} |u(x)|^2 dx \leq 4\int_{\mathcal{M}_\tau}(|u(x)|-\tau)^2 dx. \tag{4}$$

Right now, we use (2) with the compact $F = \overline{Q_\delta}\setminus \mathcal{M}_\tau$, and with the following function

$$(|u(x)|-\tau)_+ = \begin{cases} |u(x)|-\tau & \text{if } x \in \mathcal{M}_\tau,\\ 0 & \text{if } x \in \overline{Q_\delta}\setminus\mathcal{M}_\tau,\end{cases}$$

we have

$$cap_m\left(\overline{Q_\delta}\setminus \mathcal{M}_\tau\right) \leq \frac{C_{d,m}\int_{\overline{Q_\delta}} |\nabla_m(|u(x)|-\tau)_+|^2 dx}{\delta^{-d}\int_{\overline{Q_\delta}} |(|u(x)|-\tau)_+|^2 dx}.$$

By using the definition of the function $(|u(x)|-\tau)_+$, the fact that for real-valued function u we have $|\nabla_m|u|| = |\nabla_m u|$ and the inequality (4) we get

$$\begin{aligned} cap_m\left(\overline{Q_\delta}\setminus\mathcal{M}_\tau\right) &\leq \frac{C_{d,m}\int_{\mathcal{M}_\tau}|\nabla_m(|u(x)|-\tau)|^2dx}{\delta^{-d}\int_{\mathcal{M}_\tau}|u(x)-\tau)|^2dx} \\ &\leq \frac{4C_{d,m}\int_{\overline{Q_\delta}}|\nabla_m u(x)|^2dx}{\delta^{-d}\int_{\overline{Q_\delta}}|u(x)|^2dx}, \end{aligned}$$

with $C_{d,m} = \frac{3^{d-2m}(3^{2(m+1)}-1)}{2^3}\cdot$ Then the inequality of the Lemma 4 is deduced by using that $cap_m\left(\overline{Q_\delta}\setminus\mathcal{M}_\tau\right) > \gamma cap_m\left(\overline{Q_\delta}\right) \geq \gamma c_{d,m}\delta^{d-2m}$. □

Concerning the case $cap_m\left(\overline{Q_\delta}\setminus\mathcal{M}_\tau\right) \leq \gamma cap_m\left(\overline{Q_\delta}\right)$, we have the following lemma

LEMMA 5. *Let $u \in C^\infty(\overline{Q_\delta})$, $V \in L^1_{loc}(\Omega)$ with $\delta^{d-2m} \leq \int_{\overline{Q_\delta}\setminus F} V(x)dx$ for all $F \in \mathcal{N}_\gamma(\overline{Q_\delta},\Omega)$, i.e., $\delta > \delta_{V,\gamma,\Omega}$, then we have*

$$\left(\frac{\delta^{-2m}}{4}\right)\int_{\overline{Q_\delta}}|u(x)|^2dx \leq \int_{\overline{Q_\delta}}|u(x)|^2V(x)dx.$$

Proof. Since that we are in the case $cap_m\left(\overline{Q_\delta}\setminus\mathcal{M}_\tau\right) \leq \gamma cap_m\left(\overline{Q_\delta}\right)$ and as $\mathcal{M}_\tau$ is relatively open set, i.e., $\mathcal{M}_\tau \subset \Omega\cap\overline{Q_\delta}$, thus it is obvious that $\overline{Q_\delta}\setminus\ \Omega \subset \overline{Q_\delta}\setminus\ \mathcal{M}_\tau$, then $\mathcal{N}_\gamma(\overline{Q_\delta},\Omega)$ is not an empty set. Now by using the following unfolding $\mathcal{M}_\tau = \overline{Q_\delta}\setminus\left(\overline{Q_\delta}\setminus\mathcal{M}_\tau\right)$, we have

$$\begin{aligned} \int_{\overline{Q_\delta}}|u(x)|^2V(x)dx &\geq \int_{\mathcal{M}_\tau}|u(x)|^2V(x)dx \\ &\geq \tau^2\int_{\mathcal{M}_\tau}V(x)dx \\ &= \left(\frac{1}{4\delta^d}\int_{\overline{Q_\delta}}|u(x)|^2dx\right)\int_{\overline{Q_\delta}\setminus\left(\overline{Q_\delta}\setminus\mathcal{M}_\tau\right)}V(x)dx \\ &\geq \left(\frac{1}{4\delta^d}\int_{\overline{Q_\delta}}|u(x)|^2dx\right)\inf_{F\in\mathcal{N}_\gamma(\overline{Q_\delta},\Omega)}\int_{\overline{Q_\delta}\ \setminus\ F}V(x)dx \\ &\geq \left(\frac{\delta^{-2m}}{4}\right)\int_{\overline{Q_\delta}}|u(x)|^2dx. \end{aligned}$$

□

The proof of Lemma 3 is the synthesis of the Lemma 4 and the Lemma 5, more precisely for $u \in C^\infty(\overline{Q_\delta})$ and for $\gamma \to 0^+$, we obtain

$$\left(\frac{3^{d-2m}(3^{2(m+1)}-1)}{2\gamma c_{d,m}\delta^{-2m}}\right)^{-1}\int_{\overline{Q_\delta}}|u(x)|^2dx \leq \int_{\overline{Q_\delta}}|\nabla_m u(x)|^2dx + \int_{\overline{Q_\delta}}|u(x)|^2V(x)dx.$$

3. Method of proof of Theorem 1.

To show that the Lemma 3 is true for a bump function $u \in C_0^\infty(\Omega)$, we cover the support of u by a finite family of cubes $\left(Q_{(\delta)}^{(k)}\right)_{k\in J\subset\mathbb{N}}$ with the same side δ, J is a finite set. By assumption of the Theorem 1, we have $\delta > \delta_{V,\gamma,\Omega}$, then the inequality (3) of the Lemma 3 is satisfied over each cube $\overline{Q_\delta^{(k)}}$. Therefore, we pick an adequate locally finite partition of unity subordinated to the covering $\left(Q_{(\delta)}^{(k)}\right)_{k\in J\subset\mathbb{N}}$, such that the multiplicity of this covering—the maximal number of times of cubes overlap—is bounded by 2^{2d}, we applied the result [21, Lemma 2.5] (see also [16]) for a flat manifold. Accordingly, for $u \in C_0^\infty(\Omega)$ we obtain

$$\begin{aligned}\int_\Omega |u(x)|^2dx &\leq \sum_{1\leq k\leq 2^{2d}}\int_{\overline{Q_\delta^{(k)}}}|u(x)|^2dx\\ &\leq \sum_{1\leq k\leq 2^{2d}}K_{d,\gamma,\delta}\left(\int_{\overline{Q_\delta^{(k)}}}|\nabla_m u(x)|^2dx+\int_{\overline{Q_\delta^{(k)}}}|u(x)|^2V(x)dx\right)\\ &\leq 2^{2d}K_{d,\gamma,\delta}\left(\int_\Omega|\nabla_m u(x)|^2dx+\int_\Omega|u(x)|^2V(x)dx\right).\end{aligned}$$

By extending δ to $\delta_{V,\gamma,\Omega}$ and by taking the inferior on all $u \in C_0^\infty(\Omega)\setminus\{0\}$ we obtain

$$L \geq K_{d,V,\gamma,\Omega}^{-1}.$$

With

$$L := \inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} R(u) \tag{5}$$

such that

$$R(u) := \left[\frac{\left(\int_\Omega|\nabla_m u(x)|^2dx+\int_\Omega|u(x)|^2V(x)dx\right)}{\int_\Omega|u(x)|^2dx}\right]$$

and

$$K_{d,V,\gamma,\Omega} := \frac{2^{2d-1}3^{d-2m}(3^{2(m+1)}-1)\delta_{V,\gamma,\Omega}^{2m}}{\gamma c_{d,m}},\ \textit{with}\ \gamma\to 0^+.$$

So formally we have

$$H_{m,V} \geq K^{-1}_{d,V,\gamma,\Omega} \text{ on } L^2(\Omega)\cdot$$

REMARK 2. Under the assumptions on the potential V, see Theorem 1, we get a sufficient condition for a continuous embedding, precisely the Sobolev space $\mathcal{H}^m_0(\Omega)$ is continuously embedded in $L^2(\Omega)$ with an explicit constant, i.e.,

$$||u||_{L^2(\Omega)} \leq K^{1/2}_{d,V,\gamma,\Omega}||u||_{\mathcal{H}^m_0(\Omega)},$$

where $\mathcal{H}^m_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ w.r.t. the following norm

$$||u||_{\mathcal{H}^m_0(\Omega)} = \left(||u||^2_{L^2(\Omega,V(x)dx)} + ||\nabla_m u||^2_{L^2(\Omega)}\right)^{1/2}.$$

With a supplementary assumption on V, e.g., if

$$\lim_{x\to+\infty}\left[\inf_{F\in\mathcal{N}_\gamma(\overline{Q(x,\delta)},\Omega)}\int_{\overline{Q(x,\delta)}\setminus F} V(x)dx\right] = +\infty,$$

then the spectrum associated to $H_{m,V}$ is discrete [26, §16.5 Theorem 4] and the constant $K^{-1}_{d,V,\gamma,\Omega}$ stands for a lower bound of the first eigenvalue of $H_{m,V}$.

Now, when the spectre of the operator $H_{m,V}$ is totally empty of the eigenvalues, so the spectrum is reduced to its essential spectrum, i.e., the complement within the spectrum of isolated eigenvalues with finite multiplicity. E.g., we apply Persson's result [30], and we get an explicit lower bound corresponding to the bottom of the essential spectrum.

On an upper bound.

By using the inequality (5), we have

$$L \leq \frac{\left(\int_{\text{support}(u_0)} |\nabla_m u_0(x)|^2 dx + \int_{\text{support}(u_0)} |u_0(x)|^2 V(x)dx\right)}{\int_{\text{support}(u_0)} |u_0(x)|^2 dx},$$

where supp(u_0) is the compact support of u_0, e.g., supp(u_0)$\subset \overline{Q_{\delta_0}}$ such that $\delta_0 < \delta_{V,\gamma,\Omega}$, $u_0 \in [0,1]$ such that $u_{0|\overline{Q_{\delta_0/2}}} = 1$, $|\nabla_m u_0(x)| \leq 2\delta_0^{-m}$. Furthermore, we suppose that $\text{supp}(u_0) \subset \overline{Q_{\delta_0}} \setminus F$ for all $F \in \mathcal{N}_\gamma(\overline{Q_{\delta_0}},\Omega)$. Thereby we have

$$\begin{aligned} L &\leq 2^d.4\delta_0^{-2m} + 2^d\delta_0^{-d}\int_{\text{supp}(u_0)} V(x)dx \\ &\leq 5.2^d\delta_0^{-2m}. \end{aligned}$$

Regarding the last inequality, we have taken the inferior on all γ-negligible set in $\overline{Q_{\delta_0}}$, and as expected $\delta_0 < \delta_{V,\gamma,\Omega}$. Now by extending δ_0 to $\delta_{V,\gamma,\Omega}$, we get

$$\frac{\delta^{-2m}_{V,\gamma,\Omega}\gamma c_{d,m}}{2^{2d-1}3^{d-2m}(3^{2(m+1)}-1)} \leq L \leq 5.2^d\delta^{-2m}_{V,\gamma,\Omega}, \qquad \gamma \to 0^+.$$

By all means, we can improve this upper bound. We may compare our result with those obtained in [8, 14, 19, 31]. The Theorem 1, paves the way to look for a lower bound of a generalized Schrödinger operator defined in a geodesically complete Riemannian manifold with bounded geometry, i.e., its injectivity radius is strictly positive and every covariant derivative of the Riemannian curvature tensor is bounded.

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