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NOTES ON THE OPEN DOOR LEMMA

Abstract. Applying the open door function which maps the open unit disk $\mathbb{U}$ onto a slit domain, a certain method of the proof involving a special differential subordination which is referred to as the open door lemma was discussed by some mathematicians. In the present paper, by discussing a certain univalent function in $\mathbb{U}$ which maps $\mathbb{U}$ onto a slit domain, a new open door lemma is discussed.

1. Introduction

Let $\mathcal{H}$ denote the class of functions $p(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and a complex number c, let $\mathcal{H}[c,n]$ be the class of functions $p(z) \in \mathcal{H}$ of the form

$$p(z) = c + \sum_{k=n}^{\infty} c_k z^k.$$

Let $p(z)$ and $q(z)$ be members of the class $\mathcal{H}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}), \tag{1.1}$$

if there exists a function $w(z) \in \mathcal{H}$ with $w(0) = 0$, $|w(z)| < 1 \quad (z \in \mathbb{U})$, and such that $p(z) = q\big(w(z)\big) \quad (z \in \mathbb{U})$. From the definition of the subordinations, it is easy to show that the subordination (1.1) implies that

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.2}$$

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then we see that the subordination (1.1) is equivalent to the condition (1.2) by considering the function

$$w(z) = q^{-1}\big(p(z)\big) \qquad (z \in \mathbb{U}).$$

For $0 < r_0 \leqq 1$, we let

$$\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}, \quad \partial\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| = r_0\}$$

and $\overline{\mathbb{U}_{r_0}} = \mathbb{U}_{r_0} \cup \partial\mathbb{U}_{r_0}$. In particular, we write $\mathbb{U}_1 = \mathbb{U}$.

Miller and Mocanu [1] derived some lemma which is related to the subordination of two functions as follows.

LEMMA 1.1 *Let* $p(z) \in \mathcal{H}[c,n]$ *with* $p(z) \not\equiv c$. *Also, let* $q(z)$ *be analytic and univalent on the closed unit disk* $\overline{\mathbb{U}}$ *except for at most one pole on* $\partial\mathbb{U}$ *with* $q(0) = c$. *If* $p(z)$ *is not subordinate to* $q(z)$ *in* $\mathbb{U}$, *then there exist two points* $z_0 \in \partial\mathbb{U}_r$ *with* $0 < r < 1$ *and* $\zeta_0 \in \partial\mathbb{U}$, *and a real number* k *with* $k \geqq n$ *for which* $p(\mathbb{U}_r) \subset q(\mathbb{U})$,

(i) $\quad p(z_0) = q(\zeta_0)$

and

(ii) $\ z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$

Applying Lemma 1.1, Miller and Mocanu [2] discussed some lemma which is referred to as the open door lemma. By using a certain method which was discussed by Miller and Mocanu [2], we consider the sharp result for the open door lemma.

LEMMA 1.2 *Let* c *be a complex number with* $\operatorname{Re} c > 0$. *Also, let* $P(z) \in \mathcal{H}[c,n]$, *and suppose that*

$$P(\mathbb{U}) \subset \mathbb{C} \setminus \left\{ \ell^+_{c,n} \cup \ell^-_{c,n} \right\}, \tag{1.3}$$

where

$$\ell^+_{c,n} = \left\{ w \in \mathbb{C} : \operatorname{Re} w = 0 \ \textit{and} \ \operatorname{Im} w \geqq \frac{n}{\operatorname{Re} c} \left(|c|\sqrt{\frac{2\operatorname{Re} c}{n} + 1} - \operatorname{Im} c \right) \right\} \tag{1.4}$$

and

$$\ell^-_{c,n} = \left\{ w \in \mathbb{C} : \operatorname{Re} w = 0 \ \textit{and} \ \operatorname{Im} w \leqq -\frac{n}{\operatorname{Re} c} \left(|c|\sqrt{\frac{2\operatorname{Re} c}{n} + 1} + \operatorname{Im} c \right) \right\}. \tag{1.5}$$

If $p(z) \in \mathcal{H}[\frac{1}{c}, n]$ *satisfies the following differential equation*

$$zp'(z) + P(z)p(z) = 1 \qquad (z \in \mathbb{U}), \tag{1.6}$$

then $\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{U})$.

Proof. If we define the function $q(z)$ by

$$q(z) = \frac{\frac{1}{c} + \frac{1}{\bar{c}}z}{1-z} \qquad (z \in \mathbb{U}),$$

then $q(z)$ is analytic and univalent in $\mathbb{U}$ with $q(0) = \frac{1}{c}$, and $q(\mathbb{U}) = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$. In addition, we remark that $\lim_{z \to 1} q(z) = \infty$ and $\operatorname{Re} q(\zeta) = 0 \quad (\zeta \in \partial\mathbb{U} \setminus \{1\})$. If we assume that $p(z)$ is not subordinate to $q(z)$ in $\mathbb{U}$, then by Lemma 1.1, there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash\{1\}$, and a real number k with $k \geqq n$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$. We now put $si = q(\zeta_0)$, where s is real number. Then since

$$\zeta_0 = -\frac{\overline{c} - |c|^2 q(\zeta_0)}{c + |c|^2 q(\zeta_0)} = -\frac{\overline{c} - |c|^2 si}{c + |c|^2 si},$$

we have

$$k\zeta_0 q'(\zeta_0) = -\frac{k\left|c + |c|^2 si\right|^2}{2|c|^2 \operatorname{Re} c} \leqq -\frac{n\left|c + |c|^2 si\right|^2}{2|c|^2 \operatorname{Re} c} < 0.$$

Therefore, we find that

$$p(z_0) = si \quad \text{and} \quad z_0 p'(z_0) = t,$$

where s and t are real numbers with

$$t \leqq -\frac{n\left|c + |c|^2 si\right|^2}{2|c|^2 \operatorname{Re} c} = -\frac{n}{2\operatorname{Re} c}\left(1 + 2s \operatorname{Im} c + |c|^2 s^2\right) < 0. \tag{1.7}$$

If we take $z = z_0$ in the equality (1.6), then

$$z_0 p'(z_0) + P(z_0) p(z_0) = \left\{t - s \operatorname{Im} P(z_0)\right\} + i\left\{s \operatorname{Re} P(z_0)\right\} = 1,$$

which implies that

$$t - s \operatorname{Im} P(z_0) = 1 \quad \text{and} \quad s \operatorname{Re} P(z_0) = 0.$$

Since $t < 0$, it is easy to see that $s \neq 0$. Thus, we obtain

$$\operatorname{Re} P(z_0) = 0 \quad \text{and} \quad \operatorname{Im} P(z_0) = \frac{t-1}{s}. \tag{1.8}$$

It follows from (1.7) and (1.8) that

$$\operatorname{Im} P(z_0) \begin{cases} \leqq -\dfrac{n \operatorname{Im} c}{\operatorname{Re} c} + \dfrac{1}{2\operatorname{Re} c} F(s) & (s > 0) \\[2ex] \geqq -\dfrac{n \operatorname{Im} c}{\operatorname{Re} c} + \dfrac{1}{2\operatorname{Re} c} F(s) & (s < 0), \end{cases}$$

where

$$F(s) = -\frac{2\operatorname{Re} c + n + n|c|^2 s^2}{s}.$$

By observing the fluctuation of $F(s)$, we obtain that

$$\max_{s>0} F(s) = F\left(\frac{1}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n} + 1}\right) = -2n|c|\sqrt{\frac{2\operatorname{Re} c}{n} + 1}$$

and

$$\min_{s<0} F(s) = F\left(-\frac{1}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}\right) = 2n|c|\sqrt{\frac{2\operatorname{Re} c}{n}+1}.$$

From the above-mentioned calculations, we conclude that

$$\operatorname{Re} P(z_0) = 0 \quad \text{and} \quad \operatorname{Im} P(z_0) \begin{cases} \leqq -\dfrac{n\operatorname{Im} c}{\operatorname{Re} c} - \dfrac{n|c|}{\operatorname{Re} c}\sqrt{\dfrac{2\operatorname{Re} c}{n}+1} & (s>0) \\ \\ \geqq -\dfrac{n\operatorname{Im} c}{\operatorname{Re} c} + \dfrac{n|c|}{\operatorname{Re} c}\sqrt{\dfrac{2\operatorname{Re} c}{n}+1} & (s<0), \end{cases}$$

which means that $P(z_0) \notin \mathbb{C} \setminus \{\ell^+_{c,n} \cup \ell^-_{c,n}\}$. This contradicts the assumption, and hence we must have $p(z) \prec q(z) \quad (z \in \mathbb{U})$, which implies that $\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{U})$. □

REMARK 1.3 In the open door lemma, Miller and Mocanu [1] assumed that $P(\mathbb{U}) \subset \mathbb{C} \backslash \ell_{c,n}$, where

$$\ell_{c,n} = \left\{ w \in \mathbb{C} : \operatorname{Re} w = 0 \ \text{ and } \ |\operatorname{Im} w| \geqq \frac{n}{\operatorname{Re} c}\left(|c|\sqrt{\frac{2\operatorname{Re} c}{n}+1} + \operatorname{Im} c\right)\right\}.$$

In Lemma 1.2, we supposed that $P(\mathbb{U})$ belongs to the slit domain $\mathbb{C} \setminus \{\ell^+_{c,n} \cup \ell^-_{c,n}\}$ which is not symmetric with respect to the real axis.

We next introduce a certain univalent function $R(z)$ in $\mathbb{U}$ such that $R(\mathbb{U}) = \mathbb{C} \setminus \{\ell^+_{c,n} \cup \ell^-_{c,n}\}$.

REMARK 1.4 Let b be a complex number with $|b| < 1$ such that

$$\frac{n|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n}+1}\ \frac{2b}{1-b^2} - \frac{n\operatorname{Im} c}{\operatorname{Re} c} i = c. \tag{1.9}$$

If we set

$$R_1(z) = \frac{b-z}{1-\bar{b}z} \qquad (z \in \mathbb{U}),$$

$$R_2(z) = \frac{n|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n}+1}\ \frac{2z}{1-z^2} \qquad (z \in \mathbb{U})$$

and

$$R_3(w) = w - \frac{n\operatorname{Im} c}{\operatorname{Re} c} i \qquad \big(w \in R_2(\mathbb{U})\big),$$

then the function $R(z)$ defined by

$$R(z) = \big(R_3 \circ R_2 \circ R_1\big)(z) \tag{1.10}$$

$$= \frac{2n|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n}+1}\;\frac{(b-z)(1-\overline{b}z)}{(1-\overline{b}z)^2-(b-z)^2} - \frac{n\operatorname{Im} c}{\operatorname{Re} c}i \qquad (z \in \mathbb{U})$$

is analytic and univalent in $\mathbb{U}$ with $R(0) = c$. Moreover, since $R_2(z)$ maps $\mathbb{U}$ onto the complex plane w with slits along the half-lines

$$|\operatorname{Im} w| \geqq \frac{n|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n}+1},$$

we easily see that $R(\mathbb{U}) = \mathbb{C} \setminus \left\{\ell_{c,n}^{+} \cup \ell_{c,n}^{-}\right\}$. The function $R(z)$ defined by (1.10) is called the open door function (cf. [2]).

REMARK 1.5 Let us consider the complex number b with $|b| < 1$ and the relation (1.9). From the relation (1.9), we have

$$b^2 + \frac{\frac{4c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}}{\frac{c^2}{|c|^2}\left(\frac{2\operatorname{Re} c}{n}+1\right)-1}b - 1 = 0. \tag{1.11}$$

Noting that

$$\left(\frac{\frac{2c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}}{\frac{c^2}{|c|^2}\left(\frac{2\operatorname{Re} c}{n}+1\right)-1}\right)^2 + 1 = \left(\frac{\frac{c^2}{|c|^2}\left(\frac{2\operatorname{Re} c}{n}+1\right)+1}{\frac{c^2}{|c|^2}\left(\frac{2\operatorname{Re} c}{n}+1\right)-1}\right)^2,$$

it follows from the relation (1.11) that $b = b^{+}, b^{-}$, where

$$b^{+} = \frac{-\frac{2c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1} + \left\{\frac{c^2}{|c|^2}\left(\frac{2\operatorname{Re} c}{n}+1\right)+1\right\}}{\frac{c^2}{|c|^2}\left(\frac{2\operatorname{Re} c}{n}+1\right)-1} = \frac{\frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}-1}{\frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}+1}$$

and

$$b^{-} = \frac{-\frac{2c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1} - \left\{\frac{c^2}{|c|^2}\left(\frac{2\operatorname{Re} c}{n}+1\right)+1\right\}}{\frac{c^2}{|c|^2}\left(\frac{2\operatorname{Re} c}{n}+1\right)-1} = -\frac{\frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}+1}{\frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}-1}.$$

Since

$$\left|\frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}+1\right|^2 - \left|\frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}-1\right|^2 = \frac{4\operatorname{Re} c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1} > 0,$$

we find that

$$|b^{+}| = \frac{1}{|b^{-}|} = \left|\frac{\frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}-1}{\frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}+1}\right| < 1.$$

Therefore, we see that

$$b = 1 - \frac{2}{\frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n}+1}+1} \qquad (0 < |b| < 1). \tag{1.12}$$

In particular, if $c > 0$ in the equality (1.12), we obtain

$$b = 1 - \frac{2}{\sqrt{\frac{2c}{n}+1}+1} = \frac{n\left(\sqrt{\frac{2c}{n}+1}-1\right)^2}{2c} \qquad (0 < b < 1).$$

Since the open door function $R(z)$ given in (1.10) is univalent in $\mathbb{U}$, we find that the assumption (1.3) in Lemma 1.2 is equivalent to the subordination

$$P(z) \prec R(z) \qquad (z \in \mathbb{U}) \tag{1.13}$$

from the definition of the subordinations. Also, it is easy to see that

$$\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{U}) \quad \text{if and only if} \quad p(z) \prec \frac{\frac{1}{c}+\frac{1}{\bar{c}}z}{1-z} \quad (z \in \mathbb{U})$$

for $p(z) \in \mathcal{H}[\frac{1}{c}, n]$. Hence by Lemma 1.2, we derive the open door lemma concerned with the subordinations bellow.

LEMMA 1.6 *Let c be a complex number with $\operatorname{Re} c > 0$, and let $P(z) \in \mathcal{H}[c,n]$ satisfy the subordination (1.13). If $p(z) \in \mathcal{H}[\frac{1}{c}, n]$ satisfies the differential equation (1.6), then*

$$p(z) \prec \frac{\frac{1}{c}+\frac{1}{\bar{c}}z}{1-z} \qquad (z \in \mathbb{U}).$$

2. Notes on new open door function

Since the open door function $R(z)$ given in (1.10) is complicated, we will provide a simpler version of the open door function by using another method.

In order to discuss our problem, we first notice the differential equation

$$zp'(z) + P(z)p(z) = 1 \qquad (z \in \mathbb{U}). \tag{2.1}$$

It follows from the equation (2.1) that

$$P(z) = \frac{1}{p(z)} - \frac{zp'(z)}{p(z)} \qquad (z \in \mathbb{U}).$$

Hence, the subordination relation in Lemma 1.6 can be written as follows:

$$\frac{1}{p(z)} - \frac{zp'(z)}{p(z)} \prec R(z) \quad (z \in \mathbb{U}) \quad \text{implies} \quad p(z) \prec \frac{\frac{1}{c}+\frac{1}{\bar{c}}z}{1-z} \quad (z \in \mathbb{U})$$

for $p(z) \in [\frac{1}{c}, n]$, where $R(z)$ is the open door function given in (1.10). We now set

$$q(z) = \frac{\frac{1}{c} + \frac{1}{\overline{c}}z}{1-z} \qquad (z \in \mathbb{U}).$$

From the theory of the differential subordinations (cf. [1]), we see that $p(z) \in [\frac{1}{c}, n]$ satisfies the following implication:

$$\frac{1}{p(z)} - \frac{zp'(z)}{p(z)} \prec \frac{1}{q(z)} - \frac{nzq'(z)}{q(z)} \quad (z \in \mathbb{U}) \quad \text{implies} \quad p(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Then, a simple calculation yields to

$$\frac{1}{q(z)} - \frac{nzq'(z)}{q(z)} = \frac{1-z}{\frac{1}{c} + \frac{1}{\overline{c}}z} - n\left(-\frac{1}{1+\frac{c}{\overline{c}}z} + \frac{1}{1-z}\right)$$

$$= -\overline{c} - \frac{n}{1-z} + \frac{2\operatorname{Re} c + n}{1+\frac{c}{\overline{c}}z} \qquad (z \in \mathbb{U}).$$

From the above facts, we expect that the function $R_{c,n}(z)$ defined by

$$R_{c,n}(z) = -\overline{c} - \frac{n}{1-z} + \frac{2\operatorname{Re} c + n}{1+\frac{c}{\overline{c}}z} \qquad (z \in \mathbb{U}) \tag{2.2}$$

is a new open door function.

We discussed some properties for the function $R_{c,n}(z)$ defined by (2.2) as follows.

THEOREM 2.1 *Let n be a positive integer, and let c be a complex number with* $\operatorname{Re} c > 0$. *Then the function $R_{c,n}(z)$ defined by* (2.2) *is analytic and univalent in $\mathbb{U}$ with* $R_{c,n}(0) = c$.
In addition, the function $R_{c,n}(z)$ maps $\mathbb{U}$ onto the complex plane with the slits along the half-lines $\ell^{+}_{c,n}$ and $\ell^{-}_{c,n}$, where $\ell^{+}_{c,n}$ and $\ell^{-}_{c,n}$ are defined by (1.4) *and* (1.5) *respectively.*

Proof. It is easy to see that $R_{c,n}(z)$ is analytic in $\mathbb{U}$ with $R_{c,n}(0) = c$. Thus, we first show that $R_{c,n}(z)$ is univalent in $\mathbb{U}$. For $z_1, z_2 \in \mathbb{U}$ with $z_1 \neq z_2$, we calculate that

$$R_{c,n}(z_1) - R_{c,n}(z_2) = \left(-\overline{c} + 2\operatorname{Re} c\, \frac{\overline{c} - (\overline{c}+n)z_1}{(1-z_1)(\overline{c}+cz_1)}\right) - \left(-\overline{c} + 2\operatorname{Re} c\, \frac{\overline{c} - (\overline{c}+n)z_2}{(1-z_2)(\overline{c}+cz_2)}\right)$$

$$= \frac{2\operatorname{Re} c \left\{ \big(\overline{c} - (\overline{c}+n)z_1\big)(\overline{c}+cz_2)(1-z_2) - \big(\overline{c} - (\overline{c}+n)z_2\big)(\overline{c}+cz_1)(1-z_1) \right\}}{(1-z_1)(\overline{c}+cz_1)(1-z_2)(\overline{c}+cz_2)}$$

$$= \frac{2\operatorname{Re} c\,(z_2 - z_1)\left\{ |c|^2 + n\overline{c} - |c|^2(z_1+z_2) + (|c|^2 + nc)z_1 z_2 \right\}}{(1-z_1)(\overline{c}+cz_1)(1-z_2)(\overline{c}+cz_2)}.$$

We now suppose that

$$|c|^2+n\overline{c}-|c|^2(z_1+z_2)+(|c|^2+nc)z_1z_2=0 \qquad (z_1,z_2\in\mathbb{U}). \tag{2.3}$$

Then, it follows from the equality (2.3) that

$$|z_1|=\left|\frac{|c|^2+n\overline{c}-|c|^2z_2}{|c|^2-(|c|^2+nc)z_2}\right|<1,$$

which implies that

$$\left(\left||c|^2+nc\right|^2-|c|^4\right)|z_2|^2>\left||c|^2+nc\right|^2-|c|^4.$$

Since

$$\left||c|^2+nc\right|^2-|c|^4=n|c|^2\left(2\operatorname{Re}c+n\right)>0,$$

we find that $|z_2|>1$. This contradicts the fact $z_2\in\mathbb{U}$, and hence we must have

$$|c|^2+n\overline{c}-|c|^2(z_1+z_2)+(|c|^2+nc)z_1z_2\neq 0 \qquad (z_1,z_2\in\mathbb{U}).$$

Therefore, we conclude that

$$R_{c,n}(z_1)-R_{c,n}(z_2)=\frac{2\operatorname{Re}c\,(z_2-z_1)\left\{|c|^2+n\overline{c}-|c|^2(z_1+z_2)+(|c|^2+nc)z_1z_2\right\}}{(1-z_1)(\overline{c}+cz_1)(1-z_2)(\overline{c}+cz_2)}\neq 0$$

for $z_1,z_2\in\mathbb{U}$ with $z_1\neq z_2$, which proves that $R_{c,n}(z)$ is univalent in $\mathbb{U}$.

We next consider the image of $\mathbb{U}$ by the function $R_{c,n}(z)$. Letting

$$z=e^{i\theta}\ (0\leqq\theta<2\pi)\ \text{ and }\ c=|c|e^{i\varphi}\ \left(|\varphi|<\frac{\pi}{2}\right),$$

we obtain

$$\begin{aligned}R_{c,n}(e^{i\theta})&=-\overline{c}-\frac{n}{1-e^{i\theta}}+\frac{2\operatorname{Re}c+n}{1+e^{2i\varphi}e^{i\theta}}\\&=-\left(\operatorname{Re}c-i\operatorname{Im}c\right)-n\left(\frac{1}{2}+\frac{i}{2}\cot\frac{\theta}{2}\right)+\left(2\operatorname{Re}c+n\right)\left\{\frac{1}{2}-\frac{i}{2}\tan\left(\frac{\theta}{2}+\varphi\right)\right\}\\&=i\left[\operatorname{Im}c-\frac{n}{2}\left\{\left(\frac{2\operatorname{Re}c}{n}+1\right)\tan\left(\frac{\theta}{2}+\varphi\right)+\cot\frac{\theta}{2}\right\}\right].\end{aligned}$$

Therefore, we have

$$\begin{cases}\operatorname{Re}R_{c,n}(e^{i\theta})=0\\[2ex]\operatorname{Im}R_{c,n}(e^{i\theta})=\operatorname{Im}c-\dfrac{n}{2}\left\{\left(\dfrac{2\operatorname{Re}c}{n}+1\right)\tan\left(\dfrac{\theta}{2}+\varphi\right)+\cot\dfrac{\theta}{2}\right\}.\end{cases}$$

Here, we observe the fluctuation of $\operatorname{Im} R_{c,n}(e^{i\theta})$ for $0 < \theta < 2\pi$ $(\theta \neq \pi - 2\varphi)$. If we let

$$F(\theta) = \operatorname{Im} c - \frac{n}{2}\left\{\left(\frac{2\operatorname{Re} c}{n} + 1\right)\tan\left(\frac{\theta}{2} + \varphi\right) + \cot\frac{\theta}{2}\right\},$$

then, a simple calculation yields that

$$F'(\theta) = -\frac{n}{2}\left\{\left(\frac{2\operatorname{Re} c}{n} + 1\right)\frac{1}{2\cos^2\left(\frac{\theta}{2} + \varphi\right)} - \frac{1}{2\sin^2\frac{\theta}{2}}\right\} \tag{2.4}$$

$$= \frac{n}{4\cos^2\left(\frac{\theta}{2} + \varphi\right)}\left\{\frac{\cos^2\left(\frac{\theta}{2} + \varphi\right)}{\sin^2\frac{\theta}{2}} - \left(\frac{2\operatorname{Re} c}{n} + 1\right)\right\}$$

$$= \frac{n}{4\cos^2\left(\frac{\theta}{2} + \varphi\right)}\left\{\left(\cos\varphi\cot\frac{\theta}{2} - \sin\varphi\right)^2 - \left(\frac{2\operatorname{Re} c}{n} + 1\right)\right\}$$

$$= \frac{n}{4\cos^2\left(\frac{\theta}{2} + \varphi\right)}\left\{\left(\frac{\operatorname{Re} c}{|c|}\cot\frac{\theta}{2} - \frac{\operatorname{Im} c}{|c|}\right)^2 - \left(\frac{2\operatorname{Re} c}{n} + 1\right)\right\},$$

where $\theta \neq 0, \pi - 2\varphi$. Thus, we see that $F'(\theta) = 0$ for $\theta = \theta_1, \theta_2$, where

$$\theta_1 = 2\cot^{-1}\left(\frac{\operatorname{Im} c}{\operatorname{Re} c} + \frac{|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n} + 1}\right)$$

and

$$\theta_2 = 2\cot^{-1}\left(\frac{\operatorname{Im} c}{\operatorname{Re} c} - \frac{|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n} + 1}\right).$$

Then since

$$\frac{\operatorname{Im} c}{\operatorname{Re} c} - \frac{|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n} + 1} < \frac{\operatorname{Im} c}{\operatorname{Re} c} < \frac{\operatorname{Im} c}{\operatorname{Re} c} + \frac{|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n} + 1}$$

and

$$2\cot^{-1}\frac{\operatorname{Im} c}{\operatorname{Re} c} = \pi - 2\varphi,$$

a simple check gives us that

$$0 < \theta_1 < \pi - 2\varphi < \theta_2 < 2\pi.$$

Moreover, it is easy to see that $F'(\theta)$ is positive for $0<\theta<\theta_1$ and $\theta_2<\theta<2\pi$, and $F'(\theta)$ is negative for $\theta_1<\theta<\theta_2$ $(\theta\neq\pi-2\varphi)$. Also, it follows from (2.4) that

$$\lim_{\theta\to+0}F(\theta)=\lim_{\theta\to\psi-0}F(\theta)=-\infty$$

and

$$\lim_{\theta\to\psi+0}F(\theta)=\lim_{\theta\to2\pi-0}F(\theta)=+\infty,$$

where $\psi=\pi-2\varphi$.

Noting that

$$\cot\frac{\theta_1}{2}=\frac{\operatorname{Im}c}{\operatorname{Re}c}+\frac{|c|}{\operatorname{Re}c}\sqrt{\frac{2\operatorname{Re}c}{n}+1}\quad\text{and}\quad\cot\frac{\theta_2}{2}=\frac{\operatorname{Im}c}{\operatorname{Re}c}-\frac{|c|}{\operatorname{Re}c}\sqrt{\frac{2\operatorname{Re}c}{n}+1},$$

we find that

$$\begin{aligned}F(\theta_1)&=\operatorname{Im}c-\frac{n}{2}\left\{\left(\frac{2\operatorname{Re}c}{n}+1\right)\left(\frac{\operatorname{Im}c}{\operatorname{Re}c}+\frac{|c|^2}{(\operatorname{Re}c)^2}\frac{1}{\cot\frac{\theta_1}{2}-\frac{\operatorname{Im}c}{\operatorname{Re}c}}\right)+\cot\frac{\theta_1}{2}\right\}\\&=-\frac{n}{\operatorname{Re}c}\left(|c|\sqrt{\frac{2\operatorname{Re}c}{n}+1}+\operatorname{Im}c\right)<0\end{aligned}$$

and

$$\begin{aligned}F(\theta_2)&=\operatorname{Im}c-\frac{n}{2}\left\{\left(\frac{2\operatorname{Re}c}{n}+1\right)\left(\frac{\operatorname{Im}c}{\operatorname{Re}c}+\frac{|c|^2}{(\operatorname{Re}c)^2}\frac{1}{\cot\frac{\theta_2}{2}-\frac{\operatorname{Im}c}{\operatorname{Re}c}}\right)+\cot\frac{\theta_2}{2}\right\}\\&=\frac{n}{\operatorname{Re}c}\left(|c|\sqrt{\frac{2\operatorname{Re}c}{n}+1}-\operatorname{Im}c\right)>0.\end{aligned}$$

Therefore, we conclude that $R_{c,n}(z)$ maps $\mathbb{U}$ onto the complex plane with the slits along the half-lines $\ell_{c,n}^{+}$ and $\ell_{c,n}^{-}$, where $\ell_{c,n}^{+}$ and $\ell_{c,n}^{-}$ are defined by (1.4) and (1.5) respectively.

This completes the proof of the assertions of Theorem 2.1. □

EXAMPLE 2.2 Taking $n=1$ and $c=1+i$, we have

$$R_{1+i,1}(z)=-1+i-\frac{1}{1-z}+\frac{3}{1+iz}\qquad(z\in\mathbb{U}).$$

The function $R_{1+i,1}(z)$ maps $\mathbb{U}$ onto the complex plane with the slits along the half-lines $\ell^+_{1+i,1}$ and $\ell^-_{1+i,1}$, where

$$\ell^+_{1+i,1} = \left\{ w \in \mathbb{C} : \operatorname{Re} w = 0 \ \text{ and } \ \operatorname{Im} w \geqq \sqrt{6} - 1 \right\}$$

and

$$\ell^-_{1+i,1} = \left\{ w \in \mathbb{C} : \operatorname{Re} w = 0 \ \text{ and } \ \operatorname{Im} w \leqq -\left(\sqrt{6} + 1\right) \right\}.$$

REMARK 2.3 By Remark 1.4 and Theorem 2.1, we find that

$$R(0) = R_{c,n}(0) \quad \text{and} \quad R(\mathbb{U}) = R_{c,n}(\mathbb{U}),$$

where $R(z)$ and $R_{c,n}(z)$ are given in (1.10) and (2.2) respectively. Hence, the open door function $R(z)$ can be also defined in terms of the function $R_{c,n}(z)$.

Applying the new open door function $R_{c,n}(z)$ defined by (2.2) in Lemma 1.2, we obtain the simpler following version of the open door lemma as follows.

THEOREM 2.4 *Let c be a complex number with* $\operatorname{Re} c > 0$, *and let* $P(z) \in \mathcal{H}[c, n]$ *satisfy*

$$P(z) \prec -\bar{c} - \frac{n}{1-z} + \frac{2\operatorname{Re} c + n}{1 + \frac{c}{\bar{c}} z} \qquad (z \in \mathbb{U}).$$

If $p(z) \in \mathcal{H}[\frac{1}{c}, n]$ *satisfies the differential equation* (2.1), *then*

$$p(z) \prec \frac{\frac{1}{c} + \frac{1}{\bar{c}} z}{1-z} \qquad (z \in \mathbb{U}),$$

which means that $\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{U})$.

For two open door functions $R(z)$ and $R_{c,n}(z)$, it is difficult to find that

$$R(z) = R_{c,n}(z) \qquad (z \in \mathbb{U}) \tag{2.5}$$

by the calculation, because $R(z)$ given in (1.10) is complicated. But, if we consider the special case $n = 1$ and $c = 4$, then since $b = \frac{1}{2}$ in the equality (1.12), we see that

$$R(z) = \frac{6\left(\frac{1}{2} - z\right)\left(1 - \frac{1}{2}z\right)}{\left(1 - \frac{1}{2}z\right)^2 - \left(\frac{1}{2} - z\right)^2} = \frac{2(1-2z)(2-z)}{1-z^2} \qquad (z \in \mathbb{U}).$$

On the other hand, we find that

$$R_{4,1}(z) = -4 - \frac{1}{1-z} + \frac{9}{1+z} = \frac{2(1-2z)(2-z)}{1-z^2} \qquad (z \in \mathbb{U}).$$

Thus, we can observe the equality (2.5) for $n = 1$ and $c = 4$ from this calculation.

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