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CIRIC TYPE FIXED POINT RESULTS IN 2-MENGER SPACES

Abstract. In this paper we establish two common fixed point results in 2-Menger spaces. Our results are established without any continuity assumption on the functions. In one of our theorems we have used the Hadzic type t-norm. In another theorem we have used a control function. Two illustrative examples are also given. The idea of the theorems is borrowed from a recent result of Ciric.

1. Introduction

The foundation of metric fixed point theory was laid by S. Banach [3] in 1922 in his celebrated contraction mapping principle. There are lots of results which generalize the Banach contraction mapping principle. One such generalization was attempted by Khan, Swaleh and Sessa in [22]. They introduced a new category of contraction in metric spaces. "Altering distance function" is a control function which alters the distance between two points in a metric space. This concept was further generalized in a number of works. There are several number of works in fixed point theory involving altering distance function in metric spaces, some of these may be noted in [28, 32] and [33].

The concept of metric spaces has been extended in various ways. Gähler [15] extended the concept of metric spaces to 2-metric spaces in which a positive real number is assigned to every three elements of the spaces. There are lots of fixed point results in 2-metric spaces in the literature. Some of the fixed point results in 2-metric spaces may be obtained in [20, 21, 23, 27, 29, 30, 31, 36].

In 1942 K. Menger [24] introduced the idea of probabilistic metric spaces as a generalization of metric spaces. In these spaces the distance between two points is probabilistic or statistical. The distribution function plays the role of metric in the spaces. Sehgal and Bharucha-Reid established the first fixed point result in probabilistic metric spaces in [35] in 1972. The contraction proved by Sehgal and Bharucha-Reid is known as Sehgal contraction or B-contraction. Subsequently, fixed point theory in probabilistic metric spaces has developed in an extensive way. A comprehensive survey of this development up to 2001 is described by Hadzic and Pap in [19]. Some more recent references may be seen in [4, 5, 11, 13, 14] and [26].

Probabilistic 2-metric spaces are probabilistic generalization of 2-metric spaces. Recently many authors established the fixed point results on these types of spaces. References [2, 6, 9, 17, 18] and [39] are some of the fixed point results in probabilistic

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2-metric spaces.

In the below we give some definitions which are used to prove our main results.

DEFINITION 1. ***2-metric space [15, 16]***
Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if

(i) given distinct elements $x, y \in X$, there exists an element z of X such that $d(x,y,z) \neq 0$,
(ii) $d(x,y,z) = 0$ when at least two of x,y,z are equal,
(iii) $d(x,y,z) = d(x,z,y) = d(y,z,x)$ for all $x,y,z \in X$ and
(iv) $d(x,y,z) \leq d(x,y,w) + d(x,w,z) + d(w,y,z)$ for all $x,y,z,w \in X$.

When d is a 2-metric on X, the ordered pair (X,d) is called a 2-metric space.

DEFINITION 2. ***[19, 34]*** *A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.*

DEFINITION 3. ***Probabilistic metric space [19, 34]***
A probabilistic metric space (briefly, PM-space) is an ordered pair (X,F), where X is a non empty set and F is a mapping from $X \times X$ into the set of all distribution functions. The function $F_{x,y}$ is assumed to satisfy the following conditions for all $x,y,z \in X$,

(i) $F_{x,y}(0) = 0$,
(ii) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $F_{x,y}(t) = F_{y,x}(t)$ for all $t > 0$,
(iv) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$ then $F_{x,z}(t_1 + t_2) = 1$ for all $t_1, t_2 > 0$.

A particular type of probabilistic metric space is Menger space in which the triangular inequality is proved with the help of a t-norm.

Shi, Ren and Wang [37] introduced the following definition of n-th order t-norm.

DEFINITION 4. ***n-th order t-norm [37]***
A mapping $T : \Pi_{i=1}^{n}[0,1] \to [0,1]$ is called a n-th order t-norm if the following conditions are satisfied :

(i) $T(0,0,....,0) = 0$, $T(a,1,1,...,1) = a$ for all $a \in [0,1]$,
(ii) $T(a_1,a_2,a_3,....,a_n) = T(a_2,a_1,a_3,....,a_n) = T(a_2,a_3,a_1,....,a_n) = ... = T(a_2,a_3,a_4,....,a_n,a_1)$,
(iii) $a_i \geq b_i$, i=1,2,3,....,n implies $T(a_1,a_2,a_3,....,a_n) \geq T(b_1,b_2,b_3,....,b_n)$,

(iv) $T(T(a_1,a_2,a_3,....,a_n),b_2,b_3,...b_n)$
$= T(a_1,T(a_2,a_3,....,a_n,b_2),b_3,...,b_n)$
$= T(a_1,a_2,T(a_3,a_4....,a_n,b_2,b_3),b_4,...,b_n)$
$=$..............................
$= T(a_1,a_2,...,a_{n-1},T(a_n,b_2,b_3,...,b_n))$.

When $n = 2$, we have a binary t-norm, which is commonly known as t-norm.

In our main results we use the 3rd-order minimum t-norm which first appeared in the work of C. Shih-sen and Huang Nan-Jing [38]. D. Mihet [25] showed that every 3-rd order t-norm is actually of the form T^2, where T is a t-norm and $T^2(x,y,z) = T(T(x,y),z)$, hence, the definition of a 2-Menger space coincides with that in Golet [17]. Shi, Ren and Wang [37] extended the 3-rd order t-norm to n-th order t-norm.

DEFINITION 5. ***Hadzic type t-norm [19]***
A t-norm Δ is said to be Hadzic type t-norm if the family $\{\Delta^p\}_{p\in N}$ of its iterates, defined for each $s \in (0,1)$ as

$$\Delta^0(s) = 1,\ \Delta^{p+1}(s) = \Delta(\Delta^p(s),s) \text{ for all integer } p \geq 0,$$

is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exits $\eta(\lambda) \in (0,1)$ such that

$$1 \geq s > \eta(\lambda) \Rightarrow \Delta^p(s) \geq 1-\lambda \text{ for all integer } p \geq 0.$$

DEFINITION 6. ***Menger space [19, 34]***
A Menger space is a triplet (X,F,Δ), where X is a non empty set, F is a function defined on $X \times X$ to the set of all distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ *for all* $x,y \in X$,

(ii) $F_{x,y}(s) = 1$ *for all* $s > 0$ *if and only if* $x = y$,

(iii) $F_{x,y}(s) = F_{y,x}(s)$ *for all* $x,y \in X,\ s > 0$ *and*

(iv) $F_{x,y}(u+v) \geq \Delta(F_{x,z}(u),F_{z,y}(v))$ *for all* $u,v \geq 0$ *and* $x,y,z \in X$.

Wen-Zhi Zeng [41] introduced the concept of probabilistic 2-metric spaces.

DEFINITION 7. ***probabilistic 2-metric space [41]***
A probabilistic 2-metric space is an order pair (X,F) where X is an arbitrary set and F is a mapping from $X \times X \times X$ into the set of all distribution functions such that the following conditions are satisfied:

(i) $F_{x,y,z}(t) = 0$ *for* $t \leq 0$ *and for all* $x,y,z \in X$,

(ii) $F_{x,y,z}(t) = 1$ *for all* $t > 0$ *iff at least two of* x,y,z *are equal*,

(iii) for distinct points $x,y \in X$ there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ for $t > 0$,

(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$ for all $x,y,z \in X$ and $t > 0$,

(v) $F_{x,y,w}(t_1) = 1$, $F_{x,w,z}(t_2) = 1$ and $F_{w,y,z}(t_3) = 1$ then $F_{x,y,z}(t_1+t_2+t_3) = 1$, for all $x,y,z,w \in X$ and $t_1,t_2,t_3 > 0$.

A special case of the above definition is the following.

DEFINITION 8. ***2-Menger space [38]***
Let X be a nonempty set. A triplet (X,F,Δ) is said to be a 2-Menger space if F is a mapping from $X \times X \times X$ into the set of all distribution functions satisfying the following conditions:

(i) $F_{x,y,z}(0) = 0$,

(ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ if and only if at least two of $x,y,z \in X$ are equal,

(iii) for distinct points $x,y \in X$ there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ for $t > 0$,

(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$, for all $x,y,z \in X$ and $t > 0$,

(v) $F_{x,y,z}(t) \geq \Delta(F_{x,y,w}(t_1), F_{x,w,z}(t_2), F_{w,y,z}(t_3))$

where $t_1,t_2,t_3 > 0$, $t_1+t_2+t_3 = t$, $x,y,z,w \in X$ and Δ is the 3rd order t norm.

DEFINITION 9. ***[18]*** *A sequence $\{x_n\}$ in a 2-Menger space (X,F,Δ) is said to be converge to a limit x if given $\varepsilon > 0, 0 < \lambda < 1$ there exists a positive integer $N_{\varepsilon,\lambda}$ such that*

$$F_{x_n,x,a}(\varepsilon) \geq 1-\lambda \tag{1.1}$$

for all $n > N_{\varepsilon,\lambda}$ and for every $a \in X$.

DEFINITION 10. ***[18]*** *A sequence $\{x_n\}$ in a 2-Menger space (X,F,Δ) is said to be a Cauchy sequence in X if given $\varepsilon > 0, 0 < \lambda < 1$ there exists a positive integer $N_{\varepsilon,\lambda}$ such that*

$$F_{x_n,x_m,a}(\varepsilon) \geq 1-\lambda \tag{1.2}$$

for all $m,n > N_{\varepsilon,\lambda}$ and for every $a \in X$.

DEFINITION 11. ***[18]*** *A 2-Menger space (X,F,Δ) is said to be complete if every Cauchy sequence is convergent in X.*

Recently Choudhury and Das extended the concept of “altering distance function” in the context of Menger spaces in [4]. They have introduced the following Φ-function for this purpose. The definition of Φ-function is as follows:

DEFINITION 12. ***Φ-function [4]***
A function $\phi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ *if and only if* $t = 0$,

(ii) $\phi(t)$ *is strictly monotone increasing and* $\phi(t) \to \infty$ *as* $t \to \infty$,

(iii) ϕ *is left continuous in* $(0, \infty)$,

(iv) ϕ *is continuous at* 0.

With the help of above Φ-function Choudhury and Das [4] introduced a new type of contraction in Menger spaces which is known as ϕ-contraction. The idea of control function has opened new possibilities of proving more fixed point results in Menger spaces. This concept has also applied to a coincidence point problems. Some recent results using Φ-function are noted in [5, 7, 8, 10, 11, 14] and [26].

We will make use of the following function in our theorems.

DEFINITION 13. ***Ψ-function***
A function $\psi : [0,1] \times [0,1] \times [0,1] \to [0,1]$ *is said to be a* Ψ*-function if*

(i) ψ*-is monotone increasing in each variable and continuous,*

(ii) $\psi(x,x,x) > x$ *for all* $0 < x < 1$,

(iii) $\psi(1,1,1) = 1, \psi(0,0,0) = 0$.

An example of ψ-function is given below:

$$\psi(x,y,z) = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}.$$

In this paper we actually establish two Ciric type fixed point results in 2-Menger spaces. We are motivated by the recent result of Ciric [12]. These results generalized some existing results in literature. Our results are also supported by examples.

2. Main Results

THEOREM 1. *Let* (X, F, Δ) *be a complete 2-Menger space with a Hadzic type t-norm* Δ *such that whenever* $x_n \to x$ *and* $y_n \to y$, *for all* $a \in X$ *and* $F_{x_n,y_n,a}(t) \to F_{x,y,a}(t)$. *Let* $S, T : X \to X$ *be two self mappings on X which satisfy the following inequality:*

$$F_{Sx,Ty,a}(t) + q(1 - \max\{F_{x,Ty,a}(t), F_{y,Sx,a}(t)\}) \geq \psi(F_{x,y,a}(\tfrac{t}{k}), F_{x,Sx,a}(\tfrac{t}{k}), F_{y,Ty,a}(\tfrac{t}{k})) \quad (2.1)$$

for all $x, y, a \in X$, $t > 0$, *where* $0 < k < 1$, $q \geq 0$ *and* ψ *is a* Ψ*-function. Then S and T have a common fixed point in X. The fixed point is unique if q=0.*

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}_{n=0}^{\infty}$ in X as follows:

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1} \text{ for all } n \geq 0. \quad (2.2)$$

Putting $x = x_{2n}, y = x_{2n+1}$ in (2.1), for all $a \in X$ and $t > 0$, we have

$$F_{Sx_{2n},Tx_{2n+1},a}(t) + q(1 - \max\{F_{x_{2n},Tx_{2n+1},a}(t), F_{x_{2n+1},Sx_{2n},a}(t)\}) \geq \psi(F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n},Sx_{2n},a}(\tfrac{t}{k}), F_{x_{2n+1},Tx_{2n+1},a}(\tfrac{t}{k})),$$

that is,

$$F_{x_{2n+1},x_{2n+2},a}(t) + q(1 - \max\{F_{x_{2n},x_{2n+2},a}(t), F_{x_{2n+1},x_{2n+1},a}(t)\}) \geq \psi(F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n+1},x_{2n+2},a}(\tfrac{t}{k})).$$

Now, for all $a \in X$, $t > 0$ and $n \geq 0$,

$$\max\{F_{x_{2n},x_{2n+2},a}(t), F_{x_{2n+1},x_{2n+1},a}(t)\} = \max\{F_{x_{2n},x_{2n+2},a}(t), 1\} = 1.$$

Therefore, for all $a \in X$, $t > 0$ and $n \geq 0$, we have

$$F_{x_{2n+1},x_{2n+2},a}(t) \geq \psi(F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n+1},x_{2n+2},a}(\tfrac{t}{k})). \quad (2.3)$$

We now claim that for all $a \in X$, $t > 0$ and $n \geq 0$,

$$F_{x_{2n+1},x_{2n+2},a}(\tfrac{t}{k}) \geq F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}). \quad (2.4)$$

If possible, let for some $a \in X$, $s > 0$ and $n \geq 0$,

$$F_{x_{2n+1},x_{2n+2},a}(\tfrac{s}{k}) < F_{x_{2n},x_{2n+1},a}(\tfrac{s}{k}).$$

Then, from (2.3), using the properties of ψ, we have

$$\begin{aligned} F_{x_{2n+1},x_{2n+2},a}(s) &\geq \psi(F_{x_{2n},x_{2n+1},a}(\tfrac{s}{k}), F_{x_{2n},x_{2n+1},a}(\tfrac{s}{k}), F_{x_{2n+1},x_{2n+2},a}(\tfrac{s}{k})) \\ &\geq \psi(F_{x_{2n+1},x_{2n+2},a}(\tfrac{s}{k}), F_{x_{2n+1},x_{2n+2},a}(\tfrac{s}{k}), F_{x_{2n+1},x_{2n+2},a}(\tfrac{s}{k})) \\ &> F_{x_{2n+1},x_{2n+2},a}(\tfrac{s}{k}) \\ &\geq F_{x_{2n+1},x_{2n+2},a}(s), \end{aligned}$$

which is a contradiction.
Therefore (2.4) holds for all $a \in X$, $t > 0$ and $n \geq 0$.
Using (2.4) in (2.3), and by the properties of ψ, for all $a \in X$, $t > 0$ and $n \geq 0$, we have

$$\begin{aligned} F_{x_{2n+1},x_{2n+2},a}(t) &\geq \psi(F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n+1},x_{2n+2},a}(\tfrac{t}{k})) \\ &\geq \psi(F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k})). \end{aligned}$$

We now claim that $0 < F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}) < 1$ for all $t > 0$.
If not, then $F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}) = 1$ for all $t > 0$.
In that case we have

$$F_{x_{2n+1},x_{2n+2},a}(t) \geq \psi(1, 1, 1) = 1,$$

that is,

$$F_{x_{2n+1},x_{2n+2},a}(t) = 1 \text{ for all } n \geq 0.$$

Similarly, for all $a \in X$, $t > 0$ and $n \geq 0$, we can prove that

$$F_{x_{2n},x_{2n+1},a}(t) \geq \psi(1, 1, 1) = 1,$$

that is,

$$F_{x_{2n},x_{2n+1},a}(t) = 1 \text{ for all } n \geq 0.$$

Combining the above two cases, for all $a \in X$, $t > 0$ and $n \geq 0$, we have

$$F_{x_n,x_{n+1},a}(t) = 1 \text{ for all } n \geq 0.$$

If $0 < F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}) < 1$ for all $t > 0$, then

$$\begin{aligned} F_{x_{2n+1},x_{2n+2},a}(t) &\geq \psi(F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n+1},x_{2n+2},a}(\tfrac{t}{k})) \\ &\geq \psi(F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}), F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k})) \\ &> F_{x_{2n},x_{2n+1},a}(\tfrac{t}{k}). \end{aligned} \quad (2.5)$$

Similarly, for all $a \in X$, $t > 0$ and $n \geq 1$, we can prove that

$$F_{x_{2n},x_{2n+1},a}(t) > F_{x_{2n-1},x_{2n},a}(\tfrac{t}{k}). \quad (2.6)$$

Combining (2.5) and (2.6), for all $a \in X$, $n \geq 1$ and $t > 0$, we get

$$F_{x_n,x_{n+1},a}(t) > F_{x_{n-1},x_n,a}(\tfrac{t}{k}). \quad (2.7)$$

By repeated applications of this inequality, for all $a \in X$, $t > 0$ and $n \geq 1$, we obtain

$$F_{x_n,x_{n+1},a}(t) > F_{x_0,x_1,a}(\tfrac{t}{k^n}). \tag{2.8}$$

Taking limit as $n \to \infty$ on both sides, for all $a \in X$ and $t > 0$, we have

$$\lim_{n\to\infty} F_{x_{n+1},x_n,a}(t) = 1. \tag{2.9}$$

Again, by repeated applications of (2.7), it follows that for all $a \in X$, $t > 0$ and $n \geq 0$ and each $i \geq 1$,

$$F_{x_{n+i},x_{n+i+1},a}(t) > F_{x_n,x_{n+1},a}(\tfrac{t}{k^i}). \tag{2.10}$$

We next prove that $\{x_n\}$ is a Cauchy sequence (Definition 10), that is, we prove that for arbitrary $\varepsilon > 0$ and $0 < \lambda < 1$, there exists $N(\varepsilon,\lambda)$ such that for all $a \in X$,

$$F_{x_n,x_m,a}(\varepsilon) \geq 1-\lambda \text{ for all } n,m \geq N(\varepsilon,\lambda).$$

Without loss of generality we can assume that $m > n$.

Now,

$$\varepsilon = \varepsilon\tfrac{1-k}{1-k} > \varepsilon(1-k)(1+k+k^2+.........+k^{m-n-1}).$$

Then, by the monotone increasing property of F, and for all $a \in X$, we have

$$F_{x_n,x_m,a}(\varepsilon) \geq F_{x_n,x_m,a}(\varepsilon(1-k)(1+k+k^2+.........+k^{m-n-1})),$$

that is,

$$F_{x_n,x_m,a}(\varepsilon) \geq \Delta(F_{x_n,x_{n+1},a}(\varepsilon(1-k)),\Delta(F_{x_{n+1},x_{n+2},a}(\varepsilon k(1-k)),\Delta(.........,$$
$$\Delta(F_{x_{m-2},x_{m-1},a}(\varepsilon k^{m-n-2}(1-k)),F_{x_{m-1},x_m,a}(\varepsilon k^{m-n-1}(1-k)))......))). \tag{2.11}$$

Putting $t = (1-k)\varepsilon k^i$ in (2.10), for all $a \in X$, we get

$$F_{x_{n+i},x_{n+i+1},a}((1-k)\varepsilon k^i) > F_{x_n,x_{n+1},a}((1-k)\varepsilon).$$

Then, by (2.11), for all $a \in X$, we have

$$F_{x_n,x_m,a}(\varepsilon) \geq \Delta(F_{x_n,x_{n+1},a}(\varepsilon(1-k)),\Delta(F_{x_n,x_{n+1},a}(\varepsilon(1-k)),\Delta(.........,$$
$$\Delta(F_{x_n,x_{n+1},a}(\varepsilon(1-k)),F_{x_n,x_{n+1},a}(\varepsilon(1-k)))......))),$$

that is,

$$F_{x_n,x_m,a}(\varepsilon) \geq \Delta^{(m-n)}F_{x_n,x_{n+1},a}(\varepsilon(1-k)). \tag{2.12}$$

Since the t-norm Δ is a Hadzic type t-norm, the family $\{\Delta^p\}$ of its iterates is equicontinuous at the point $s = 1$, that is, there exists $\eta(\lambda) \in (0,1)$ such that for all $m > n$,

$$\Delta^{(m-n)}(s) \geq 1-\lambda \text{ whenever } \eta(\lambda) < s \leq 1. \tag{2.13}$$

Since, $F_{x_0,x_1,a}(t) \to 1$ as $t \to \infty$ and $0 < k < 1$, there exists an positive integer $N(\varepsilon,\lambda)$ such that for all $a \in X$,

$$F_{x_0,x_1,a}(\tfrac{(1-k)\varepsilon}{k^n}) > \eta(\lambda) \text{ for all } n \geq N(\varepsilon,\lambda). \tag{2.14}$$

From (2.14) and (2.10), with $n = 0$, $i = n$ and $t = (1-k)\varepsilon$, for all $a \in X$, we get

$$F_{x_n,x_{n+1},a}(\varepsilon(1-k)) > F_{x_0,x_1,a}(\tfrac{(1-k)\varepsilon}{k^n}) > \eta(\lambda) \text{ for all } n \geq N(\varepsilon,\lambda).$$

Then, from (2.13) with $s = F_{x_n,x_{n+1},a}(\varepsilon(1-k))$, we have

$$\Delta^{(m-n)}(F_{x_n,x_{n+1},a}(\varepsilon(1-k))) \geq 1-\lambda.$$

It then follows from (2.12) that for all $a \in X$,

$$F_{x_n,x_m,a}(\varepsilon) \geq 1-\lambda \text{ for all } m,n \geq N(\varepsilon,\lambda).$$

Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there is some $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Then,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = z \text{ and } \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = z. \tag{2.15}$$

Now, we prove that $Tz = z$.

Putting $x = x_{2n}$, $y = z$ in the inequality (2.1), for all $a \in X$ and $t > 0$, we have

$$F_{Sx_{2n},Tz,a}(t) + q(1 - \max\{F_{x_{2n},Tz,a}(t), F_{z,Sx_{2n},a}(t)\}) \geq \psi(F_{x_{2n},z,a}(\tfrac{t}{k}), F_{x_{2n},Sx_{2n},a}(\tfrac{t}{k}), F_{z,Tz,a}(\tfrac{t}{k})). \tag{2.16}$$

Taking limit as $n \to \infty$ in (2.16) for all $a \in X$ and $t > 0$, we have

$$F_{z,Tz,a}(t) + q(1 - \max\{F_{z,Tz,a}(t), F_{z,z,a}(t)\}) \geq \psi(F_{z,z,a}(\tfrac{t}{k}), F_{z,z,a}(\tfrac{t}{k}), F_{z,Tz,a}(\tfrac{t}{k})), \tag{2.17}$$

(since by our assumption for all $a \in X$, $x_n \to x, y_n \to y$ implies $F_{x_n,y_n,a} \to F_{x,y,a}$)
that is,

$$F_{z,Tz,a}(t) \geq \psi(1, 1, F_{z,Tz,a}(\tfrac{t}{k})).$$

We claim that $F_{z,Tz,a}(\frac{t}{k}) = 1$, if not, then we have

$$F_{z,Tz,a}(t) \geq \psi(1, 1, F_{z,Tz,a}(\tfrac{t}{k})) > F_{z,Tz,a}(\tfrac{t}{k}). \tag{2.18}$$

(by the properties of ψ)

By repeated applications of (2.18), for all $a \in X$ and $t > 0$, we obtain

$$F_{z,Tz,a}(t) > F_{z,Tz,a}(\tfrac{t}{k^n}).$$

Taking limit as $n \to \infty$ on both sides, for all $t > 0$,

$$F_{z,Tz,a}(t) \geq \lim_{n\to\infty} F_{z,Tz,a}(\frac{t}{k^n}) = 1,$$

which implies

$$F_{z,Tz,a}(t) = 1.$$

Thus $z = Tz$.
Similarly we can prove that $Sz = z$.
Now, we prove the uniqueness of the fixed point for the case where $q = 0$. Let z and w be two distinct common fixed points of S and T. Then, we have $0 < F_{z,w,a}(t) < 1$ for some $a \in X$ and $t > 0$.
Then, by the inequality (2.1), for all $a \in X$ and $t > 0$, we get

$$F_{Sz,Tw,a}(t) \geq \psi(F_{z,w,a}(\tfrac{t}{k}), F_{z,Sz,a}(\tfrac{t}{k}), F_{w,Tw,a}(\tfrac{t}{k})),$$

that is,

$$\begin{aligned} F_{z,w,a}(t) &\geq \psi(F_{z,w,a}(\tfrac{t}{k}), F_{z,z,a}(\tfrac{t}{k}), F_{w,w,a}(\tfrac{t}{k})) \\ &= \psi(F_{z,w,a}(\tfrac{t}{k}), 1, 1) \\ &> F_{z,w,a}(\tfrac{t}{k}) \text{ (by the properties of } \psi) \\ &\geq F_{z,w,a}(t), \text{ which is a contradiction.} \end{aligned}$$

Hence $z = w$.

Taking $S = T$ in the Theorem 1 we get the following Corollary.

COROLLARY 1. *Let (X, F, Δ) be a complete 2-Menger space with a Hadzic type t-norm Δ such that whenever $x_n \to x$ and $y_n \to y$, for all $a \in X$ and $F_{x_n,y_n,a}(t) \to F_{x,y,a}(t)$. Let $T : X \to X$ be a self mapping on X which satisfies the following inequality:*

$$F_{Tx,Ty,a}(t) + q(1 - \max\{F_{x,Ty,a}(t), F_{y,Tx,a}(t)\}) \geq \psi(F_{x,y,a}(\tfrac{t}{k}), F_{x,Tx,a}(\tfrac{t}{k}), F_{y,Ty,a}(\tfrac{t}{k}))$$

for all $x, y, a \in X$, $t > 0$, where $0 < k < 1$, $q \geq 0$ and ψ is a Ψ-function. Then T has a fixed point in X. The fixed point is unique if q=0.

Now we give the following example to support the above Corollary 1.

EXAMPLE 1. Let $X = \{\alpha, \beta, \gamma, \delta\}$, the t-norm Δ is a 3rd order minimum t-norm and F be defined as

$$F_{\alpha,\beta,\gamma}(t) = F_{\alpha,\beta,\delta}(t) = \begin{cases} 0, & \text{if } \ t \leq 0, \\ 0.40, & \text{if } \ 0 < t < 7, \\ 1, & \text{if } \ t \geq 7, \end{cases}$$

$$F_{\alpha,\gamma,\delta}(t) = F_{\beta,\gamma,\delta}(t) = \begin{cases} 0, & \text{if } \ t \leq 0, \\ 0.95, & \text{if } \ 0 < t < 1, \\ 1, & \text{if } \ t \geq 1, \end{cases}$$

Then (X, F, Δ) is a complete 2-Menger space. If we define $T : X \to X$ as follows: $T\alpha = \gamma, T\beta = \delta, T\gamma = \gamma, T\delta = \gamma$ then the mappings T satisfies all the conditions of the Corollary 1 where $\psi(x,y,z) = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}$ and γ is the unique fixed point of T in X for $q = 0$.

In our next theorem we use the control function ϕ (Definition 12) in the inequality (2.1) with $q = 0$. Here we also use the minimum t-norm. We prove our next theorem by different arguments from the first theorem.

THEOREM 2. *Let (X, F, Δ) be a complete 2-Menger space with the 3rd order minimum t-norm Δ. Let $S, T : X \to X$ be two self mappings on X which satisfy the following inequality:*

$$F_{Sx,Ty,a}(\phi(t)) \geq \psi(F_{x,y,a}(\phi(\tfrac{t}{c})), F_{x,Sx,a}(\phi(\tfrac{t}{c})), F_{y,Ty,a}(\phi(\tfrac{t}{c}))) \qquad (2.19)$$

for all $x, y, a \in X$, $t > 0$ where $0 < c < 1$, ϕ is a Φ-function and ψ is a Ψ-function. Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}_{n=0}^{\infty}$ in X as follows:

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1} \text{ for all } n \geq 0. \qquad (2.20)$$

Putting $x = x_{2n}, y = x_{2n+1}$ in (2.19), for all $t > 0$, $n \geq 0$ and for all $a \in X$, we have

$$F_{Sx_{2n},Tx_{2n+1},a}(\phi(t)) \geq \psi(F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c})), F_{x_{2n},Sx_{2n},a}(\phi(\tfrac{t}{c})), F_{x_{2n+1},Tx_{2n+1},a}(\phi(\tfrac{t}{c}))),$$

that is,

$$F_{x_{2n+1},x_{2n+2},a}(\phi(t)) \geq \psi(F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c})), F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c})), F_{x_{2n+1},x_{2n+2},a}(\phi(\tfrac{t}{c}))). \qquad (2.21)$$

We now claim that for all $t > 0, n \geq 0$ and $a \in X$,

$$F_{x_{2n+1},x_{2n+2},a}(\phi(\tfrac{t}{c})) \geq F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c})). \quad (2.22)$$

If possible, let for some $s > 0, n \geq 0$ and $a \in X$,

$$F_{x_{2n+1},x_{2n+2},a}(\phi(\tfrac{s}{c})) < F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{s}{c})).$$

Then, from (2.21), using the properties of ψ, we have for $s > 0, n \geq 0$ and $a \in X$,

$$\begin{aligned} F_{x_{2n+1},x_{2n+2},a}(\phi(s)) &\geq \psi(F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{s}{c})), F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{s}{c})), F_{x_{2n+1},x_{2n+2},a}(\phi(\tfrac{s}{c}))) \\ &\geq \psi(F_{x_{2n+1},x_{2n+2},a}(\phi(\tfrac{s}{c})), F_{x_{2n+1},x_{2n+2},a}(\phi(\tfrac{s}{c})), F_{x_{2n+1},x_{2n+2},a}(\phi(\tfrac{s}{c}))) \\ &> F_{x_{2n+1},x_{2n+2},a}(\phi(\tfrac{s}{c})) \\ &\geq F_{x_{2n+1},x_{2n+2},a}(\phi(s)), \end{aligned}$$

which is a contradiction.

Therefore (2.22) holds for all $t > 0, n \geq 0$ and $a \in X$.

Using (2.22) in (2.21) and by the properties of ψ, for all $t > 0$, $n \geq 0$ and $a \in X$, we have

$$\begin{aligned} F_{x_{2n+1},x_{2n+2},a}(\phi(t)) &\geq \psi(F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c})), F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c})), F_{x_{2n+1},x_{2n+2},a}(\phi(\tfrac{t}{c}))) \\ &\geq \psi(F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c})), F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c})), F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c}))) \\ &> F_{x_{2n},x_{2n+1},a}(\phi(\tfrac{t}{c})). \end{aligned} \quad (2.23)$$

Similarly, for all $t > 0, n > 0$ and $a \in X$, we can prove that

$$F_{x_{2n},x_{2n+1},a}(\phi(t)) > F_{x_{2n-1},x_{2n},a}(\phi(\tfrac{t}{c})). \quad (2.24)$$

Combining (2.23) and (2.24), for all $n \geq 1, t > 0$ and $a \in X$, we get

$$F_{x_n,x_{n+1},a}(\phi(t)) > F_{x_{n-1},x_n,a}(\phi(\tfrac{t}{c})).$$

By repeated applications of this inequality, for all $t > 0, n \geq 1$ and $a \in X$, we have

$$F_{x_n,x_{n+1},a}(\phi(t)) > F_{x_0,x_1,a}(\phi(\tfrac{t}{c^n})). \quad (2.25)$$

Taking limit as $n \to \infty$ on both sides of (2.25), for all $t > 0$ and $a \in X$, we obtain

$$\lim_{n\to\infty} F_{x_{n+1},x_n,a}(\phi(t)) = 1. \quad (2.26)$$

Again, by virtue of a property of ϕ, given $s > 0$ we can find $t > 0$ such that $s > \phi(t)$. Thus the above limit implies that for all $s > 0$ and $a \in X$,

$$\lim_{n\to\infty} F_{x_n,x_{n+1},a}(s) = 1. \quad (2.27)$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and $0 < \lambda < 1$ for which we can find some $a \in X$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\varepsilon) < 1 - \lambda. \quad (2.28)$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (2.28) so that

$$F_{x_{m(k)},x_{n(k)-1},a}(\varepsilon) \geq 1 - \lambda. \quad (2.29)$$

If $\varepsilon_1 < \varepsilon$, then we have

$$F_{x_{m(k)},x_{n(k)},a}(\varepsilon_1) \leq F_{x_{m(k)},x_{n(k)},a}(\varepsilon).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $n(k) > m(k) > k$ and satisfying (2.28), (2.29) whenever ε is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\varepsilon_2 > 0$ such that $\phi(\varepsilon_2) < \varepsilon$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\phi(\varepsilon_2)) < 1 - \lambda \quad (2.30)$$

and

$$F_{x_{m(k)},x_{n(k)-1},a}(\phi(\varepsilon_2)) \geq 1-\lambda. \tag{2.31}$$

Now, we get the following possible cases:

Case-I: $m(k)$ is odd and $n(k)$ is even for an infinite number of values of k. Then, there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ with $n(l) > m(l) > l$ such that

$$F_{x_{m(l)},x_{n(l)},a}(\phi(\varepsilon_2)) < 1-\lambda \tag{2.32}$$

and

$$F_{x_{m(l)},x_{n(l)-1},a}(\phi(\varepsilon_2)) \geq 1-\lambda. \tag{2.33}$$

Then,

$$x_{m(l)} = Sx_{m(l)-1} \text{ and } x_{n(l)} = Tx_{n(l)-1}.$$

By (2.32), we have

$$\begin{aligned} 1-\lambda &> F_{x_{m(l)},x_{n(l)},a}(\phi(\varepsilon_2)) \\ &= F_{Sx_{m(l)-1},Tx_{n(l)-1},a}(\phi(\varepsilon_2)) \\ &\geq \psi(F_{x_{m(l)-1},x_{n(l)-1},a}(\phi(\tfrac{\varepsilon_2}{c})), F_{x_{m(l)-1},Sx_{m(l)-1},a}(\phi(\tfrac{\varepsilon_2}{c})), F_{x_{n(l)-1},Tx_{n(l)-1},a}(\phi(\tfrac{\varepsilon_2}{c}))), \end{aligned}$$

that is,

$$1-\lambda > \psi(F_{x_{m(l)-1},x_{n(l)-1},a}(\phi(\tfrac{\varepsilon_2}{c})), F_{x_{m(l)-1},x_{m(l)},a}(\phi(\tfrac{\varepsilon_2}{c})), F_{x_{n(l)-1},x_{n(l)},a}(\phi(\tfrac{\varepsilon_2}{c}))). \tag{2.34}$$

Since ϕ is strictly increasing and $0 < c < 1$, we can choose $\eta_1, \eta_2 > 0$ such that $\phi(\frac{\varepsilon_2}{c}) = \phi(\varepsilon_2) + \eta_1 + \eta_2$.

Therefore,

$$\begin{aligned} F_{x_{m(l)-1},x_{n(l)-1},a}(\phi(\tfrac{\varepsilon_2}{c})) \geq \Delta(&F_{x_{m(l)-1},x_{n(l)-1},x_{m(l)}}(\eta_1), F_{x_{m(l)-1},x_{m(l)},a}(\eta_2), \\ &F_{x_{m(l)},x_{n(l)-1},a}(\phi(\varepsilon_2))). \end{aligned} \tag{2.35}$$

Again, by (2.27) we have for sufficiently large l and by the property of ϕ,

$$F_{x_{m(l)-1},x_{n(l)-1},x_{m(l)}}(\eta_1) \geq 1-\lambda \tag{2.36}$$

and

$$F_{x_{m(l)-1},x_{m(l)},a}(\eta_2) \geq 1-\lambda. \tag{2.37}$$

Using (2.33), (2.36) and (2.37) in (2.35), we have

$$\begin{aligned} F_{x_{m(l)-1},x_{n(l)-1},a}(\phi(\tfrac{\varepsilon_2}{c})) &\geq \Delta(F_{x_{m(l)-1},x_{n(l)-1},x_{m(l)}}(\eta_1), F_{x_{m(l)-1},x_{m(l)},a}(\eta_2), \\ &\qquad F_{x_{m(l)},x_{n(l)-1},a}(\phi(\varepsilon_2))) \\ &\geq \Delta(1-\lambda, 1-\lambda, 1-\lambda) \\ &= 1-\lambda. \end{aligned} \tag{2.38}$$

Again, by (2.27) we have for sufficiently large l,

$$F_{x_{m(l)-1},x_{m(l)},a}(\phi(\tfrac{\varepsilon_2}{c})) \geq 1-\lambda \tag{2.39}$$

and

$$F_{x_{n(l)-1},x_{n(l)},a}(\phi(\tfrac{\varepsilon_2}{c})) \geq 1-\lambda. \tag{2.40}$$

Using (2.38), (2.39) and (2.40) in (2.34) for $\varepsilon_2 > 0$, $0 < c < 1$, for some $a \in X$ and by the property of ψ, we have

$$1-\lambda > \psi(F_{x_{m(l)-1},x_{n(l)-1},a}(\phi(\tfrac{\varepsilon_2}{c})), F_{x_{m(l)-1},x_{m(l)},a}(\phi(\tfrac{\varepsilon_2}{c})), F_{x_{n(l)-1},x_{n(l)},a}(\phi(\tfrac{\varepsilon_2}{c})))$$

$$\geq \psi(1-\lambda, 1-\lambda, 1-\lambda) > 1-\lambda,$$

which is a contradiction.

Case-II: $m(k)$ is even and $n(k)$ is odd for an infinite number of values of k. Then there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ such that (2.32) and (2.33) hold. This case is similar to Case-I and we can get a contradiction.

Case-III: $m(k)$ and $n(k)$ both are even for an infinite number of values of k. Then there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ with $n(l) > m(l) > l$ such that (2.32) and (2.33) hold.
As $0 < c < 1$ we can choose $\varepsilon_3 < \varepsilon_2$ such that $\frac{\varepsilon_3}{c} \geq \varepsilon_2$. Therefore, by the property of ϕ we can take $\phi(\frac{\varepsilon_3}{c}) \geq \phi(\varepsilon_2)$. Also by the property of ϕ we can choose $s_1, s_2 > 0$ such that $\phi(\varepsilon_2) = \phi(\varepsilon_3) + s_1 + s_2$.
Now, by (2.32), we have

$$\begin{aligned} 1-\lambda &> F_{x_{m(l)},x_{n(l)},a}(\phi(\varepsilon_2)) \\ &\geq \Delta(F_{x_{m(l)},x_{n(l)},x_{m(l)+1}}(s_1), F_{x_{m(l)},x_{m(l)+1},a}(s_2), F_{x_{m(l)+1},x_{n(l)},a}(\phi(\varepsilon_3))). \end{aligned} \tag{2.41}$$

Now, by the inequality (2.19), we have

$$\begin{aligned} F_{x_{m(l)+1},x_{n(l)},a}(\phi(\varepsilon_3)) &\geq \psi(F_{x_{m(l)},x_{n(l)-1},a}(\phi(\tfrac{\varepsilon_3}{c})), F_{x_{m(l)},x_{m(l)+1},a}(\phi(\tfrac{\varepsilon_3}{c})), F_{x_{n(l)-1},x_{n(l)},a}(\phi(\tfrac{\varepsilon_3}{c}))). \\ &\geq \psi(F_{x_{m(l)},x_{n(l)-1},a}(\phi(\varepsilon_2)), F_{x_{m(l)},x_{m(l)+1},a}(\phi(\tfrac{\varepsilon_3}{c})), \\ &\qquad F_{x_{n(l)-1},x_{n(l)},a}(\phi(\tfrac{\varepsilon_3}{c}))). \end{aligned} \tag{2.42}$$

By (2.27), we have for sufficiently large l,

$$F_{x_{m(l)},x_{m(l)+1},a}(\phi(\tfrac{\varepsilon_3}{c})) \geq 1-\lambda, \tag{2.43}$$

$$F_{x_{n(l)-1},x_{n(l)},a}(\phi(\tfrac{\varepsilon_3}{c})) \geq 1-\lambda. \tag{2.44}$$

Using (2.33), (2.43), (2.44) in (2.42) for $a \in X$, we have

$$\begin{aligned} F_{x_{m(l)+1},x_{n(l)},a}(\phi(\varepsilon_3)) &\geq \psi(F_{x_{m(l)},x_{n(l)-1},a}(\phi(\varepsilon_2)), F_{x_{m(l)},x_{m(l)+1},a}(\phi(\tfrac{\varepsilon_3}{c})), \\ &\qquad F_{x_{n(l)-1},x_{n(l)},a}(\phi(\tfrac{\varepsilon_3}{c}))) \\ &\geq \psi(1-\lambda, 1-\lambda, 1-\lambda) > 1-\lambda. \end{aligned} \tag{2.45}$$

Again, by (2.27), we have for sufficiently large l,

$$F_{x_{m(l)},x_{n(l)},x_{m(l)+1}}(s_1) \geq 1-\lambda \tag{2.46}$$

and

$$F_{x_{m(l)},x_{m(l)+1},a}(s_2) \geq 1-\lambda. \tag{2.47}$$

Now, using (2.45), (2.46), (2.47) in (2.41), we have

$$\begin{aligned} 1-\lambda &> F_{x_{m(l)},x_{n(l)},a}(\phi(\varepsilon_2)) \\ &\geq \Delta(1-\lambda, 1-\lambda, 1-\lambda) = 1-\lambda, \end{aligned}$$

which is a contradiction.

Case-IV: $m(k)$ and $n(k)$ both are odd for an infinite number of values of k. Then there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ such that (2.32) and (2.33) hold. This case is similar to Case-III and we can get a contradiction.

Combining all the above four cases we conclude that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there is some $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Then,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = z \text{ and } \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = z. \tag{2.48}$$

Now, we claim that $Tz = z$.

Let us choose c_0, c_1 such that $0 < c < c_0 < c_1 < 1$. (2.49)

Now, for all $t > 0$, $a \in X$, we have

$$F_{z,Tz,a}(\phi(t)) \geq \Delta(F_{z,Tz,Sx_{2n}}(\phi(t) - \phi(c_0t) - \phi(c_1t)), F_{z,Sx_{2n},a}(\phi(c_0t)), F_{Sx_{2n},Tz,a}(\phi(c_1t))). \tag{2.50}$$

As Δ is continuous, taking liminf as $n \to \infty$ on both sides of the above inequality, for all $t > 0$, $a \in X$, we have

$$\begin{aligned} F_{z,Tz,a}(\phi(t)) &\geq \Delta(\liminf_{n\to\infty} F_{z,Tz,Sx_{2n}}(\phi(t) - \phi(c_0t) - \phi(c_1t)), \liminf_{n\to\infty} F_{z,Sx_{2n},a}(\phi(c_0t)), \\ &\quad \liminf_{n\to\infty} F_{Sx_{2n},Tz,a}(\phi(c_1t))) \\ &= \Delta(1, 1, \liminf_{n\to\infty} F_{Sx_{2n},Tz,a}(\phi(c_1t))). \end{aligned} \tag{2.51}$$

(by (2.48))

Now, for all $t > 0$, $n \geq 0$, $a \in X$ and using the inequality (2.19), we have

$$F_{Sx_{2n},Tz,a}(\phi(c_1t)) \geq \psi(F_{x_{2n},z,a}(\phi(\tfrac{c_1t}{c})), F_{x_{2n},Sx_{2n},a}(\phi(\tfrac{c_1t}{c})), F_{z,Tz,a}(\phi(\tfrac{c_1t}{c}))). \tag{2.52}$$

Taking liminf as $n \to \infty$ on both sides of (2.52), we have for all $t > 0$, $a \in X$,

$$\liminf_{n\to\infty} F_{Sx_{2n},Tz,a}(\phi(c_1t)) \geq \psi(\liminf_{n\to\infty} F_{x_{2n},z,a}(\phi(\frac{c_1t}{c})), \liminf_{n\to\infty} F_{x_{2n},Sx_{2n},a}(\phi(\frac{c_1t}{c})), F_{z,Tz,a}(\phi(\frac{c_1t}{c}))),$$

that is,

$$\liminf_{n\to\infty} F_{Sx_{2n},Tz,a}(\phi(c_1t)) \geq \psi(1, 1, F_{z,Tz,a}(\phi(\frac{c_1t}{c}))) > F_{z,Tz,a}(\phi(\frac{c_1t}{c})),$$

(by (2.48))

that is,

$$\liminf_{n\to\infty} F_{Sx_{2n},Tz,a}(\phi(c_1t)) > F_{z,Tz,a}(\phi(\frac{t}{(\frac{c}{c_1})})). \tag{2.53}$$

(since ψ is monotone increasing)

We now take $\frac{c}{c_1} = p$. Then, by (2.49), $0 < p < 1$. Hence we get from (2.53) that for all $t > 0$, $a \in X$

$$\liminf_{n\to\infty} F_{Sx_{2n},Tz,a}(\phi(c_1t)) > F_{z,Tz,a}(\phi(\frac{t}{p})). \tag{2.54}$$

Combining (2.51) and (2.54) for all $t > 0$, $a \in X$ we get

$$F_{z,Tz,a}(\phi(t)) > F_{z,Tz,a}(\phi(\tfrac{t}{p})).\ (0 < p < 1)$$

By repeated applications of this inequality, for all $t > 0$, $a \in X$, we obtain

$$F_{z,Tz,a}(\phi(t)) > F_{z,Tz,a}(\phi(\tfrac{t}{p^n})). \tag{2.55}$$

Taking limit as $n \to \infty$ on both sides of (2.55), for all $t > 0$, $a \in X$ we get

$$F_{z,Tz,a}(\phi(t)) \geq \lim_{n\to\infty} F_{z,Tz,a}(\phi(\frac{t}{p^n})) = 1.$$

Therefore by a property of ϕ we get, $z = Tz$.
Similarly, we can prove that $z = Sz$.
Thus z is a common fixed point of S and T.

Now we prove the uniqueness of the common fixed point. Let z and w be two distinct common fixed points of S and T. Then the properties of ϕ imply $0 < F_{z,w,a}(\phi(t)) < 1$ for some $t > 0$. Let $a \in X$ be any element different from z and w. Then, by the inequality (2.19) for $t > 0, a \in X$ we get

$$F_{Sz,Tw,a}(\phi(t)) \geq \psi(F_{z,w,a}(\phi(\tfrac{t}{c})), F_{z,Sz,a}(\phi(\tfrac{t}{c})), F_{w,Tw,a}(\phi(\tfrac{t}{c}))),$$

that is,

$$\begin{aligned} F_{z,w,a}(\phi(t)) &\geq \psi(F_{z,w,a}(\phi(\tfrac{t}{c})), F_{z,z,a}(\phi(\tfrac{t}{c})), F_{w,w,a}(\phi(\tfrac{t}{c}))) \\ &= \psi(F_{z,w,a}(\phi(\tfrac{t}{c})), 1, 1) \\ &\geq \psi(F_{z,w,a}(\phi(\tfrac{t}{c})), F_{z,w,a}(\phi(\tfrac{t}{c})), F_{z,w,a}(\phi(\tfrac{t}{c}))) \text{ [since } 1 > F_{z,w,a}(\phi(\tfrac{t}{c})) > 0] \\ &> F_{z,w,a}(\phi(\tfrac{t}{c})) \\ &\geq F_{z,w,a}(\phi(t)), \text{ which is a contradiction.} \end{aligned}$$

Hence $z = w$ is the unique common fixed point of S and T.

Next we give the following example to validate our result.

EXAMPLE 2. Let $X = \{\alpha, \beta, \gamma, \delta\}$, the t-norm Δ is a 3rd order minimum t-norm and F be defined as

$$F_{\alpha,\beta,\gamma}(t) = F_{\alpha,\beta,\delta}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.40, & \text{if } 0 < t < 4, \\ 1, & \text{if } t \geq 4, \end{cases}$$

$$F_{\alpha,\gamma,\delta}(t) = F_{\beta,\gamma,\delta}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Then (X, F, Δ) is a complete 2-Menger space. If we define $S, T : X \to X$ as follows: $S\alpha = \delta, S\beta = \gamma, S\gamma = \gamma, S\delta = \delta$ and $T\alpha = \delta, T\beta = \gamma, T\gamma = \gamma, T\delta = \gamma$ then the mappings S and T satisfy all the conditions of the Theorem 2 where $\phi(t) = t, \psi(x,y,z) = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}$ and γ is the unique common fixed point of S and T.

This example also satisfies Theorem 1 for $q = 0$.

Remark: In the present paper we establish two Ciric type fixed points results in 2-Menger spaces. The idea of the theorems are borrowed from a recent result of Ciric [12]. In recent times many authors established many fixed point results on Ciric contractions in Menger spaces. Some of the results may be noted as [1, 40]. These results are Ciric type generalized contractions. In both of these results the authors use a single valued function. But in our present paper we use two functions. In our main results we use ψ-function defined as $\psi(x,y,z) = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}$. Our present results also extend some of our results proved in [10].

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