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MATHEMATICAL STUDY OF THE SMALL OSCILLATIONS OF A FLOATING BODY IN A BOUNDED TANK CONTAINING AN ALMOST-HOMOGENEOUS LIQUID

Abstract. The authors study the small oscillations of a floating body in a bounded tank containing an almost-homogeneous incompressible inviscid liquid (i.e. a liquid whose density in equilibrium is practically a linear function of the height, which differs very little from a constant). From suitable variational equation and using functional analysis, they obtain two operatorial equations from which they can study the spectrum of the problem. They prove that is composed by a discrete part and an essential part, which fills an interval. Finally, using the weak formulation, the authors prove the existence and uniqueness theorem of the evolution problem.

Notations

In equilibrium position:

Ω: domain occupied by the liquid

S: wall of the tank wetted by the liquid

Γ: horizontal free line

Σ, Σ_0: parts of the boundary of the body that are into contact with the liquid and the air respectively

τ: submerged volume of the body

σ: horizontal line limiting with Σ the domain τ

$\vec{n}_S, \vec{n}_\Gamma, \vec{n}_\Sigma$: unit vectors normal to S, Γ, Σ , directed to the exterior of Ω

$\vec{n}_{\Sigma_0}$: unit vector normal to Σ_0, directed to the interior of the body

G_0: centre of inertia of the body

G_0x, G_0y: horizontal, vertical upwards fixed axes (unit vectors $\vec{x}$,$\vec{y}$)

h: ordinate of Γ and σ

g: acceleration of the gravity

m_0: mass of the body

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p_0: constant pressure above the free line.

At the instant t: the centre of inertia of the body lies in G and the axes G_0xy become the axes GXY (unit vectors $\vec{X},\vec{Y}$). We set: $\overrightarrow{G_0G} = \xi\vec{x}+\eta\vec{y}$; $\widehat{G_0x,GX} = \theta$; ξ, η, θ and their derivates are considered as small

$y = h+\zeta(x,y,t)$: the equation of the moving free line Γ_t, where ζ and its derivatives are small

$\hat{\Sigma}$: position of Σ at the instant t

Σ_t: submerged part of the boundary of the body; its differs from $\hat{\Sigma}$, in two small arcs Σ_t', Σ_t''(on the figure, $\Sigma_t = \hat{\Sigma} - \Sigma_t' - \Sigma_t''$)

Σ_{0t}: part of the boundary of the body into contact with the air

$\hat{\sigma}$ (resp. $\hat{\tau}$): position of σ(resp.τ) at the instant t.

1. Introduction

Studying small oscillations of floating body in a reservoir of limited dimensions, containing a homogeneous incompressible inviscid liquid, is a subject of a great interest in engineering. For example, N. N. Moiseyev [6], [7], in his pioneering work, described the rocking of a vessel in a canal look through methods of the functional analysis and proved that it is a classical vibration problem. This is not the case of the boundless fluid, where damped oscillatory motions appear.

The case of a heterogeneous liquid, that occurs, for example, in the case of the lake water, where the density can increase with the depth, has not been studied yet.

The aim of this work is to study the problem in the particular case, introduced by Capodanno [1], of an almost-homogeneous liquid whose density in equilibrium position is practically a linear function of the height, differing a little bit from a constant.

This hypothesis modifies significantly the spectrum of the problem.

From a variational equation of the problem, the authors obtain two operator equations in a suitable Hilbert space, and they prove that the spectrum is comprised of a countable set of positive real eigenvalues with an accumulation point of infinity, and an essential spectrum which fills an interval and correspond physically to a domain of resonance.

Finally, using the weak formulation, the authors prove the existence and uniqueness theorem of the evolution problem.

2. Position of the problem

We are going to study the small oscillations of a floating body in a bounded reservoir containing a heterogeneous incompressible inviscid liquid, restricting ourselves for simplicity to the planar problem.

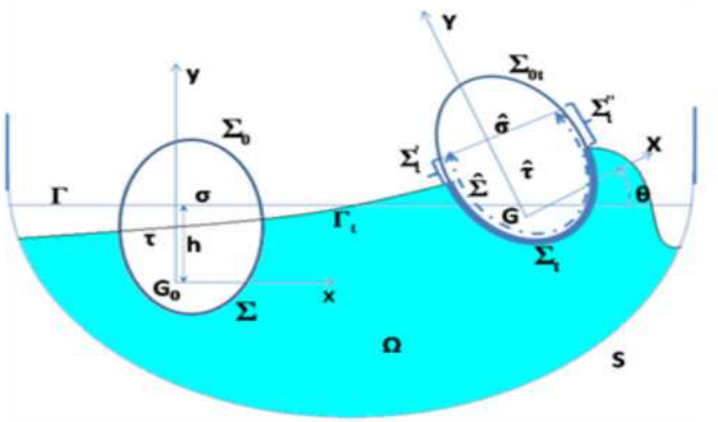

Figure 1: Model of the System.

For the rigour, since a position of the body which is deduced from the equilibrium position by a translation parallel to G_0x, is also an equilibrium position, we supposed that the body is submitted also to a force $-k^2\xi\vec{x}$ acting in G (k^2 positive constant).

We are going to study the small oscillations of the system body-liquid about its equilibrium position, obviously in linear theory.

As usual, we are considering that the linearized velocities and accelerations are "true" velocities and accelerations, in order to avoid writing needless formulas in the following calculations.

3. Study of the equilibrium of the system

3.1- If ρ_0 and P_{st} are the density and the pressure of the liquid in the equilibrium position, we have

$$\overrightarrow{grad}P_{st} = -\rho_0 g\vec{y} \ ,$$

so that P_{st} and ρ_0 are functions of y, with

$$\frac{dP_{st}(y)}{dy} = -\rho_0(y)g$$

Setting

(1) $$R(y) = -\int_h^y \rho_0(w)gdw \ ,$$

we have

(2) $$P_{st} = R(y) + p_0$$

$$R(h) = 0 \; ; \; R'(y) = -\rho_0(y)g \; .$$

3.2- It is easy to verify that the equilibrium conditions for the body are

$$m = m_0$$

where

$$m = \int_\tau \rho_0(y)\, d\tau$$

is the mass of the displaced liquid, and

$$x_{G_\tau} = 0 \; ,$$

where G_τ is the centre of buoyancy, according to the Archimede's theorem.

4. Equation of the small oscillatiuons of the system

If P is the pressure of the liquid, we introduce the dynamic pressure p by

$$P = P_{st} + p = R(y) + p_0 + p \tag{3}$$

4.1- At first, applying the theorem of momentum to the body, we have

$$m_0 \left(\ddot{\xi}\vec{x} + \ddot{\eta}\vec{y} \right) = -m_0 g\vec{y} - k^2 \xi \vec{x} + \int_{\Sigma_t} P \vec{n}_{\Sigma_t} d\Sigma_t + \int_{\Sigma_{0t}} p_0 \vec{n}_{\Sigma_{0t}} d\Sigma_{0t}$$

and, consequently

$$m_0 \left(\ddot{\xi}\vec{x} + \ddot{\eta}\vec{y} \right) = -m_0 g\vec{y} - k^2 \xi \vec{x} + \int_{\hat{\Sigma}} p \vec{n}_{\hat{\Sigma}} d\hat{\Sigma} + \int_{\hat{\Sigma} - \Sigma'_t - \Sigma_t"} R(y) \vec{n}_{\Sigma_t} d\Sigma_t$$

Remark:
p has sense on Σ_t, not on $\hat{\Sigma}$. But, Σ'_t, Σ''_t being of the first order, we may consider that p is defined on $\hat{\Sigma}$.(See the reference [8] p 33, for an analogous remark) $\Box$.

Since $R(h) = 0$, the contribution of Σ'_t and Σ''_t is of the second order; so that we must calculate, writing Σ instead $\hat{\Sigma}$ for simplicity

$$\int_\Sigma R(y) \vec{n}_\Sigma d\Sigma$$

We have, in linear theory

$$R(y) = R(\eta + \theta X + Y) = R(Y) + (\eta + \theta X) R'(Y) + \cdots$$

and then

$$\int_\Sigma R(y) \vec{n}_\Sigma d\Sigma = \int_\Sigma R(Y) \vec{n}_\Sigma d\Sigma - g \int_\Sigma \rho_0(Y)(\eta + \theta X) \vec{n}_\Sigma d\Sigma$$

But, writing σ and τ instead of $\hat{\sigma}$ and $\hat{\tau}$ and using the relation $R(h) = 0$ and Green's formula, we have

$$\int_{\Sigma} R(Y)\vec{n}_{\Sigma} d\Sigma = \int_{\Sigma+\sigma} R(Y)\vec{n} d\omega = \int_{\tau} \rho_0(Y) g d\tau \cdot \vec{Y} = mg\vec{Y}$$

On the other hand, we can write

$$\begin{cases} -g\displaystyle\int_{\Sigma} \rho_0(Y)(\eta + \theta X)\vec{n}_{\Sigma} d\Sigma \\ = -g\displaystyle\int_{\Sigma+\sigma} \rho_0(Y)(\eta + \theta X)\vec{n} d\omega + \rho_0(h) g \int_{\sigma} (\eta + \theta X) dX \cdot \vec{Y} \end{cases}$$

Calculating the first integral in the right-hand side by means of the Green's formula and setting

$$\int_{\sigma} dX = \sigma \,; \qquad \int_{\sigma} X dX = \sigma x_0 \,,$$

we find

$$\begin{cases} -g\displaystyle\int_{\Sigma} \rho_0(Y)(\eta + \theta X)\vec{n}_{\Sigma} d\Sigma \\ = mg\theta\vec{X} + g\left[\displaystyle\int_{\tau} \rho_0'(Y)(\eta + \theta X) d\tau - \rho_0(h)\sigma(\eta + \theta x_0)\right] \cdot \vec{Y} \end{cases}$$

The difference between the dynamic pressure $p_{|\hat{\Sigma}}$ at a point of $\hat{\Sigma}$ at the instant t and the dynamic pressure $p_{|\Sigma}$ at a point of Σ at the same instant t is of the second order (since the dynamic pressure is of the first order).

Therefore, we can write, at the first order

$$\int_{\hat{\Sigma}} p_{|\hat{\Sigma}} \vec{n}_{\hat{\Sigma}} d\hat{\Sigma} = \int_{\Sigma} p_{|\Sigma} \vec{n}_{\Sigma} d\Sigma \,,$$

Analogous remark can be made for the integrals on $\hat{\tau}$ and $\hat{\sigma}$, so that we obtain the linearized equations:

$$m_0 \ddot{\xi} = -k^2 \xi + \int_{\Sigma} p n_{\Sigma x} d\Sigma \tag{4}$$

$$m_0 \ddot{\eta} = g\left[\int_{\tau} \rho_0'(y)(\eta + \theta x) d\tau - \rho_0(h)\sigma(\eta + \theta x_0)\right] + \int_{\Sigma} p n_{\Sigma y} d\Sigma \tag{5}$$

4.2- The theorem of moment of momentum gives, if I_G is the moment of inertia of the body about G and M a point of the boundary of the body:

$$I_G \ddot{\theta} \vec{x} \times \vec{y} = \int_{\Sigma_t} \overrightarrow{GM} \times P\vec{n}_{\Sigma_t} d\Sigma_t + \int_{\Sigma_{0t}} \overrightarrow{GM} \times p_0 \vec{n}_{\Sigma_{0t}} d\Sigma_{0t} \,,$$

and consequently

$$I_G\ddot{\theta} = \int_\Sigma p(Xn_{\Sigma Y} - Yn_{\Sigma X})d\Sigma + \int_\Sigma R(\eta + \theta X + Y)(Xn_{\Sigma Y} - Yn_{\Sigma X})d\Sigma$$

We can write for the second integral of the right-hand side

$$\begin{cases} \int_\Sigma R(\eta + \theta X + Y)(Xn_{\Sigma Y} - Yn_{\Sigma X})d\Sigma \\ = \int_\Sigma R(Y)(Xn_{\Sigma Y} - Yn_{\Sigma X})d\Sigma - g\int_\Sigma \rho_0(Y)(\eta + \theta X)(Xn_{\Sigma Y} - Yn_{\Sigma X})d\Sigma \end{cases}$$

At first, we have

$$\int_\Sigma R(Y)(Xn_{\Sigma Y} - Yn_{\Sigma X})d\Sigma = \int_{\Sigma+\sigma} R(Y)(Xn_{\Sigma Y} - Yn_{\Sigma X})d\Sigma = \int_\tau \rho_0(Y)gX d\tau = 0$$

On the other hand, we write

$$\begin{cases} \int_\Sigma \rho_0(Y)(\eta + \theta X)(Xn_{\Sigma Y} - Yn_{\Sigma X})d\Sigma \\ = \int_{\Sigma+\sigma} \rho_0(Y)(\eta + \theta X)(Xn_{\Sigma Y} - Yn_{\Sigma X})d\omega - \rho_0(h)\int_\sigma (\eta + \theta X)X d\sigma \end{cases}$$

and we transform the first integral by the Green's formula.

Finally, we obtain

$$\begin{cases} \int_\Sigma R(\eta + \theta X + Y)(Xn_{\Sigma Y} - Yn_{\Sigma X})d\Sigma \\ = g\int_\tau \rho_0'(Y)X(\eta + \theta X))d\tau - mgy_{G_t} - \rho_0(h)g\sigma(\eta x_0 + \theta J_\sigma) \end{cases}$$

where y_{G_t} is the ordinate of the centre of buoyancy and we have set

$$\int_\sigma X^2 d\sigma = \sigma J_\sigma$$

Consequently, we have the linearized equation

$$(6) \qquad \begin{cases} I_G\ddot{\theta} = -mgy_{G_t}\theta - \rho_0(h)g\sigma(\eta x_0 + \theta J_\sigma) + g\int_\tau \rho_0'(y)x(\eta + \theta x))d\tau \\ \qquad + \int_\Sigma p(xn_{\Sigma y} - yn_{\Sigma x})d\Sigma \end{cases}$$

4.3- If $\rho^*(x,y,t)$ is the density of the liquid and $\vec{u}(x,y,t)$ is the small displacement of a particle of the liquid with respect to its equilibrium position, we have

(7) $\rho^* \ddot{\vec{u}} = -\overrightarrow{grad} P - \rho^* g \vec{y}$ $\quad$ *in* Ω (*Euler's equation*)

(8) $div\, \dot{\vec{u}} = 0$ $\quad$ *in* Ω (*incompressibility*)

(9) $\frac{\partial \rho^*}{\partial t}\Big|_{x,y(fixed)} + div\, (\rho^* \dot{\vec{u}}) = 0$ $\quad$ *in* Ω (*continuity equation*),

and the boudary conditions

(10) $$\dot{\vec{u}} \cdot \vec{n}_{|S} = 0$$

(11) $$\dot{\vec{u}} \cdot \vec{n}_{|\Gamma} = \dot{\zeta}$$

(12) $$\dot{\vec{u}} \cdot \vec{n}_{|\Sigma} = \left[(\dot{\xi} - \dot{\theta} Y) \vec{X} + (\dot{\eta} + \dot{\theta} X) \vec{Y} \right] \cdot \vec{n}_{|\Sigma}$$

Since we have
$$\overrightarrow{grad} P = -\rho_0(y) g \vec{y} + \overrightarrow{grad}\, p$$

the Euler's equation takes the form

$$\rho^* \ddot{\vec{u}} = -\overrightarrow{grad}\, p - [\rho^* - \rho_0(y)]\, g \vec{y}$$

We set
$$\rho^*(x,y,t) = \rho_0(y) + \tilde{\rho}(x,y,t) + \cdots$$

$\tilde{\rho}$ is the first order with respect to the amplitude of the oscillations and the dots indicate terms of higher order.

The continuity equation is, at the first order

$$\frac{\partial \tilde{\rho}}{\partial t} + div\, \left(\rho_0(y) \dot{\vec{u}} \right) = 0$$

or taking into account of (8):

$$\frac{\partial \tilde{\rho}}{\partial t} + \overrightarrow{grad} \rho_0(y) \cdot \dot{\vec{u}} = 0$$

Integrating from the instant of the equilibrium and the instant t, we obtain the linearized continuity equation

(13) $$\tilde{\rho} = -\rho_0'(y) u_y$$

and consequently the linearized Euler's equation

(14) $$\rho_0(y) \ddot{\vec{u}} = -\overrightarrow{grad}\, p + \rho_0'(y) g u_y \vec{y}$$

5. The case of the almost-homogeneous liquid

5.1- Let be h_0 the maximal depth of the reservoir. in Ω , we have

$$|y-h| < h_0$$

We suppose that the density of the liquid in equilibrium position has the form

$$\rho_0(y) = \rho\,[1-\beta(y-h)] + o(\beta h_0) \ , \tag{15}$$

where ρ and β are positive constants, β being sufficiently small so that $(\beta h_0)^2$, $(\beta h_0)^3 \cdots$, are negligible with respect to βh_0.
In this case, the liquid is called ***almost-homogeneous in*** Ω .

5.2- In the following we restrict ourselves to this case. Then, like in the Boussinesq theory of the convective motions of a fluid [2]. we replace in the equation of motion

$$\rho_0(y) \text{ by } \rho \ , \qquad \rho_0'(y) \text{ by } -\rho\beta \ .$$

The linearized Euler's equation (14) becomes

$$\ddot{\vec{u}} = -\frac{1}{\rho}\overrightarrow{grad}\,p - \beta g u_y \vec{y} \tag{16}$$

and it is easy to see that the equations (5) and (6) become

$$m_0\ddot{\eta} = -(m_0\beta+\rho\sigma)g\eta - \rho g x_0\sigma\theta + \int_\Sigma p n_{\Sigma y} d\Sigma \tag{17}$$

$$I_G\ddot{\theta} = -\rho g\sigma x_0\eta - K^2\theta + \int_\Sigma p(x n_{\Sigma y} - y n_{\Sigma x}) d\Sigma, \tag{18}$$

where we have set

$$K^2 = m_0 g y_{G_\tau} + \rho g\sigma J_\sigma + \rho\beta g J_\tau \ ; \qquad J_\tau = \int_\tau X^2 d\tau \ .$$

The positivity of the first coefficient will be justtified in the following.
In the almost-homogeneous case, the equations of motions are (4), (17), (18), (16).

The equation (8) and the bundary conditions give, after integration

$$div\,\vec{u} = 0 \qquad in\ \Omega \tag{19}$$

$$\vec{u}\cdot\vec{n}_{|S} = 0 \tag{20}$$

$$\vec{u}\cdot\vec{n}_{|\Gamma} = \zeta \tag{21}$$

$$\vec{u}\cdot\vec{n}_{|\Sigma} = \xi n_{\Sigma x} + \eta n_{\Sigma y} + \theta(x n_{\Sigma y} - y n_{\Sigma x}) \tag{22}$$

6. A variational equation of the problem

6.1- We introduce the space of kinematically displacements by means of arbitrary complex numbers $\tilde{\xi}, \tilde{\eta}, \tilde{\theta}$ and a smooth vectorial function $\vec{\tilde{u}}$ defined in Ω, so that

$$(23) \qquad div\ \vec{\tilde{u}} = 0 \ \ in\ \Omega\ ; \ \ \tilde{u}_{n|S} = 0; \ \ \tilde{u}_{n|\Sigma} = \tilde{\xi} n_{\Sigma x} + \tilde{\eta} n_{\Sigma y} + \tilde{\theta}(x n_{\Sigma y} - y n_{\Sigma x})$$

where we have set

$$\tilde{u}_{n|S} = \vec{\tilde{u}} \cdot \vec{n}_{|S}\,, \ \ \tilde{u}_{n|\Sigma} = \vec{\tilde{u}} \cdot \vec{n}_{|\Sigma}\,.$$

We have, by virtue of (16)

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \vec{\tilde{u}} d\Omega = -\int_\Omega \overrightarrow{grad} p \cdot \vec{\tilde{u}}\, d\Omega - \rho\beta g \int_\Omega u_y \tilde{u}_y\, d\Omega$$

Using Green's formula and (19), we have

$$\int_\Omega \overrightarrow{grad} p \cdot \vec{\tilde{u}}\, d\Omega = \int_\Gamma p_{|\Gamma}\, \tilde{u}_{n|\Gamma}\, d\Gamma + \int_\Sigma p_{|\Sigma}\, \tilde{u}_{n|\Sigma}\, d\Sigma$$

On Γ_t, we must have $P = p_0$ and consequently

$$R(h + u_{n|\Gamma}) + p_0 + p = p_0,$$

from wich we deduce

$$p_{|\Gamma} = \rho g u_{n|\Gamma}$$

Therefore, we obtain

$$\begin{cases} \displaystyle\int_\Omega \rho \ddot{\vec{u}} \cdot \vec{\tilde{u}}\, d\Omega + \int_\Gamma \rho g u_{n|\Gamma}\, \tilde{u}_{n|\Gamma}\ d\Gamma + \int_\Sigma p \left[\tilde{\xi} n_{\Sigma x} + \tilde{\eta} n_{\Sigma y} + \tilde{\theta}(x n_{\Sigma y} - y n_{\Sigma x}) \right] d\Sigma \\ \displaystyle + \rho\beta g \int_\Omega u_y \tilde{u}_y\, d\Omega = 0 \end{cases}$$

Now, we multiply (4), (17), (18) by $\tilde{\xi}$, $\tilde{\eta}$, $\tilde{\theta}$ repectively and we add to the precedent equation. We obtain a formal variational equation of the problem

$$(24) \quad \begin{cases} \displaystyle\int_\Omega \rho \ddot{\vec{u}} \cdot \vec{\tilde{u}}\, d\Omega + m_0 \left(\ddot{\xi}\tilde{\xi} + \ddot{\eta}\tilde{\eta} \right) + I_G \ddot{\theta}\tilde{\theta} + \int_\Gamma \rho g u_{n|\Gamma}\, \tilde{u}_{n|\Gamma}\ d\Gamma + \rho\beta g \int_\Omega u_y \tilde{u}_y\, d\Omega \\ + a_0(\xi, \eta, \theta; \tilde{\xi}, \tilde{\eta}, \tilde{\theta}) = 0 \end{cases}$$

for all "admissible" $\vec{\tilde{u}}, \tilde{\xi}, \tilde{\eta}, \tilde{\theta}$ and with

$$a_0(\xi, \eta, \theta; \tilde{\xi}, \tilde{\eta}, \tilde{\theta}) = k^2 \xi\tilde{\xi} + (m\beta + \rho\sigma) g \eta\tilde{\eta} + K^2 \theta\tilde{\theta} + \rho g \sigma x_0 \left(\eta\tilde{\theta} + \theta\tilde{\eta} \right)$$

Reciprocally, we are proving that, from the equation (24), we can deduce the equations of motion.

We take

$$\tilde{\xi}, \tilde{\eta}, \tilde{\theta} \quad \text{arbitrary in } \mathbb{C}$$

$\vec{\tilde{u}}$ smooth in Ω, such that $\tilde{u}_{n|S} = 0$, $\quad \tilde{u}_{n|\Sigma} = \tilde{\xi} n_{\Sigma x} + \tilde{\eta} n_{\Sigma y} + \tilde{\theta}(x n_{\Sigma y} - y n_{\Sigma x})$, but not verifing $div\ \vec{\tilde{u}} = 0$.

Then, introducing a multiplier λ associated to the constraint $div\ \vec{\tilde{u}} = 0$, we replace the equation (24) by

$$\begin{cases} \int_\Omega \rho \ddot{\vec{u}} \cdot \vec{\tilde{u}}\, d\Omega + m_0 \left(\ddot{\xi}\tilde{\xi} + \ddot{\eta}\tilde{\eta} \right) + I_G \ddot{\theta}\tilde{\theta} + \int_\Gamma \rho g u_{n|\Gamma}\, \tilde{u}_{n|\Gamma}\ d\Gamma + \rho\beta g \int_\Omega u_y \tilde{u}_y\, d\Omega \\ + a_0(\xi, \eta, \theta; \tilde{\xi}, \tilde{\eta}, \tilde{\theta}) + \int_\Omega \lambda div\ \vec{\tilde{u}}\, d\Omega = 0 \end{cases}$$

We have

$$\int_\Omega \lambda div\ \vec{\tilde{u}}\, d\Omega = \int_\Omega \left[div\ \left(\lambda \vec{\tilde{u}} \right) - \overrightarrow{grad}\lambda \cdot \vec{\tilde{u}} \right]\ d\Omega$$

or using Grren's formula

$$\begin{cases} \int_\Omega \lambda div\ \vec{\tilde{u}}\, d\Omega = \int_\Gamma \lambda_{|\Gamma}\, \tilde{u}_{n|\Gamma}\, d\Gamma + \int_\Sigma \lambda_{|\Sigma} \left[\tilde{\xi} n_{\Sigma x} + \tilde{\eta} n_{\Sigma y} + \tilde{\theta}(x n_{\Sigma y} - y n_{\Sigma x}) \right] d\Sigma \\ \qquad - \int_\Omega \overrightarrow{grad}\lambda \cdot \vec{\tilde{u}}\, d\Omega \end{cases}$$

Carrying in the variational equation, we obtain

$$\begin{cases} \int_\Omega \left(\rho \ddot{\vec{u}} - \overrightarrow{grad}\lambda + \rho\beta g u_y \vec{y} \right) \cdot \vec{\tilde{u}}\, d\Omega + \left[m_0 \ddot{\xi} + k^2 \xi + \int_\Sigma \lambda_{|\Sigma}\, n_{\Sigma x} d\Sigma \right] \tilde{\xi} \\ + \left(m_0 \ddot{\eta} + (m\beta + \rho\sigma) g\eta + \rho g x_0 \sigma\theta + \int_\Sigma \lambda_{|\Sigma}\, n_{\Sigma y} d\Sigma \right) \tilde{\eta} \\ + \left[I_G \ddot{\theta} + K^2 \theta + \rho g x_0 \sigma\eta + \int_\Sigma \lambda_{|\Sigma}\, (x n_{\Sigma y} - y n_{\Sigma x}) d\Sigma \right] \tilde{\theta} \\ + \int_\Gamma \left(\rho g u_{n|\Gamma} + \lambda_{|\Gamma} \right) \tilde{u}_{n|\Gamma}\, d\Gamma = 0 \end{cases}$$

We take $\vec{\tilde{u}} \in \left[D(\Omega) \right]^2$ (so that $\tilde{u}_{n|S} = 0,\ \tilde{u}_{n|\Sigma} = 0$), $\tilde{\xi}, \tilde{\eta}, \tilde{\theta}$ equal to zero(according to $\tilde{u}_{n|\Sigma} = 0$), and we have

$$\int_\Omega \left(\rho \ddot{\vec{u}} - \overrightarrow{grad}\lambda + \rho\beta g u_y \vec{y} \right) \cdot \vec{\tilde{u}}\, d\Omega = 0 \qquad \forall \vec{\tilde{u}} \in \left[D(\Omega) \right]^2$$

and consequently

$$\rho \ddot{\vec{u}} - \overrightarrow{grad}\lambda + \rho\beta g u_y \vec{y} = 0 \qquad in \left[D'(\Omega) \right]^2.$$

Turning to a smooth $\vec{\tilde{u}}$ and using the precedent result, since $\tilde{u}_{n|\Gamma}, \tilde{\xi}, \tilde{\eta}, \tilde{\theta}$ are arbitrary, we see that the square brackets are equal to zero and that

$$\rho g u_{n|\Gamma} + \lambda_{|\Gamma} = 0.$$

Setting $\lambda = -p$ [11], [7], we find the equations of motion.

6.2- We remark that the linear stability of the equilibrium of the system can take place only if the quadratic form associated with $a_0(\xi,\eta,\theta;\tilde{\xi},\tilde{\eta},\tilde{\theta})$, i.e.

$$k^2\xi^2 + (m\beta+\rho\sigma)g\eta^2 + 2\rho g\sigma x_0\eta\theta + K^2\theta^2$$

is positive definite, then if

$$K^2(m\beta+\rho\sigma)g > \rho^2 g^2\sigma^2 x_0^2,$$

In particular; this inequality is satisfied if

$$K^2 > \rho g\sigma x_0^2$$

or

$$m_0 g y_{G_\tau} + \rho g\sigma(J_\sigma - \sigma x_0^2) + \rho\beta g J_\tau > 0$$

Since $(J_\sigma - \sigma x_0^2)$ is positive by virtue of the Schwarz inequality, this conditions is satisfied if G_τ is sufficiently close to G.

7. Transformation of the variational equation of the problem

7.1- We introduce the spaces [4]:

$$J_0(\Omega) = \left\{\vec{u} \in \left[L^2(\Omega)\right]^2;\ div\vec{u} = 0;\ u_{n|\partial\Omega} = 0\right\}$$
$$J_{0,S}(\Omega) = \left\{\vec{u} \in \left[L^2(\Omega)\right]^2;\ div\vec{u} = 0;\ u_{n|S} = 0\right\}$$
$$G(\Omega) = \left\{\vec{u} = \overrightarrow{grad}\,p\ ;\ p \in H^1(\Omega);\ \int_{\partial\Omega} p\, d\omega = 0\right\}$$
$$G_{h,S}(\Omega) = \left\{\vec{u} = \overrightarrow{grad}\,p;\ p \in H^1(\Omega);\ \Delta p = 0;\quad \frac{\partial p}{\partial n}|_S = 0;\ \int_{\Gamma+\Sigma} p\, d\omega = 0\right\}$$
$$G_{0,\Gamma+\Sigma}(\Omega) = \left\{\vec{u} = \overrightarrow{grad}\,p;\ p \in H^1(\Omega)\ ;\ p_{|\Gamma+\Sigma} = 0\right\}$$

and the orthogonal decompositions in $\left[L^2(\Omega)\right]^2$

$$\left[L^2(\Omega)\right]^2 = J_0(\Omega) \oplus G(\Omega) = J_0(\Omega) \oplus G_{h,S}(\Omega) \oplus G_{0,\Gamma+\Sigma}(\Omega)$$
$$J_{0,S}(\Omega) = J_0(\Omega) \oplus G_{h,S}(\Omega)$$

7.2- We seek $\vec{u}$ in $J_{0,S}(\Omega)$ and p in $G(\Omega)$.

We set

$$\vec{u} = \vec{v} + \vec{U}; \quad \vec{v} \in J_0(\Omega), \quad \vec{U} \in G_{h,S}(\Omega)$$

The Euler's equation (16) becomes

$$\ddot{\vec{v}} + \ddot{\vec{U}} = -\frac{1}{\rho}\overrightarrow{grad}\, p - \beta g (v_y + U_y)\vec{y}$$

If P_0 is the orthogonal projector from $\left[L^2(\Omega)\right]^2$ in $J_0(\Omega)$, we have

$$\ddot{\vec{v}} = -\beta g P_0 (v_y \vec{y}) - \beta g P_0 (U_y \vec{y}) \tag{25}$$

Setting

$$\tilde{\vec{u}} = \tilde{\vec{v}} + \tilde{\vec{U}},$$

we have, $J_0(\Omega)$ and $G_{h,S}(\Omega)$ being orthogonal

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \tilde{\vec{u}} d\Omega = \int_\Omega \rho \left(\ddot{\vec{v}} \cdot \tilde{\vec{v}} + \ddot{\vec{U}} \cdot \tilde{\vec{U}} \right) d\Omega$$

Since $v_{n|\partial\Omega} = 0$, we have $u_{n|\partial\Omega} = U_{n|\partial\Omega}$ and the variational erquation (24) takes the form

$$\left\{ \begin{aligned} & \int_\Omega \rho \left(\ddot{\vec{v}} \cdot \tilde{\vec{v}} + \ddot{\vec{U}} \cdot \tilde{\vec{U}} \right) d\Omega + m_0 \left(\ddot{\xi}\tilde{\xi} + \ddot{\eta}\tilde{\eta} \right) + I_G \ddot{\theta}\tilde{\theta} + \rho g \int_\Gamma U_{n|\Gamma} \tilde{U}_{n|\Gamma}\ d\Gamma \\ & + \rho\beta g \int_\Omega (v_y + U_y) \left(\tilde{v}_y + \tilde{U}_y \right) d\Omega + a_0(\xi, \eta, \theta; \tilde{\xi}, \tilde{\eta}, \tilde{\theta}) = 0 \end{aligned} \right. \tag{26}$$

But we have

$$\beta g \int_\Omega v_y \tilde{v}_y\ d\Omega = \beta g \int_\Omega P_0 (v_y \vec{y}) \cdot \tilde{\vec{v}}\ d\Omega;$$
$$\beta g \int_\Omega U_y \tilde{v}_y\ d\Omega = \beta g \int_\Omega P_0 (U_y \vec{y}) \cdot \tilde{\vec{v}}\ d\Omega,$$

so that appears in (26)

$$\int_\Omega \left[\ddot{\vec{v}} + \beta g P_0 (v_y \vec{y}) + \beta g P_0 (U_y \vec{y}) \right] \cdot \tilde{\vec{v}}\ d\Omega = 0.$$

Therefore, we obtain the new variational equation

$$\left\{ \begin{aligned} & \int_\Omega \rho \ddot{\vec{U}} \cdot \tilde{\vec{U}}\ d\Omega + m_0 \left(\ddot{\xi}\tilde{\xi} + \ddot{\eta}\tilde{\eta} \right) + I_G \ddot{\theta}\tilde{\theta} + \rho g \int_\Gamma U_{n|\Gamma} \tilde{U}_{n|\Gamma}\ d\Gamma \\ & + \rho\beta g \int_\Omega (v_y + U_y) \tilde{U}_y\ d\Omega + a_0(\xi, \eta, \theta; \tilde{\xi}, \tilde{\eta}, \tilde{\theta}) = 0 \end{aligned} \right. \tag{27}$$

7.3- We introduce the space

$$V=\left\{\begin{array}{l} W=(\vec{U},\xi,\eta,\theta)^t \;\; with \;\; \vec{U}=\overrightarrow{grad}\,\Phi,\;\; \Phi\in\tilde{H}^1(\Omega)=\left\{\Phi\in H^1(\Omega),\;\; \int_{\Gamma+\Sigma}\Phi\,d\omega=0\right\};\\ div\vec{U}=0\;\; in\,\Omega\,,\;\; U_{n|S}=0,\;\; U_{n|\Gamma}\in L^2(\Gamma),\;\;\; U_{n|\Sigma}=\xi n_{\Sigma x}+\eta n_{\Sigma y}+\theta(xn_{\Sigma y}-yn_{\Sigma x});\\ \xi,\eta,\theta\in\mathbb{C} \end{array}\right\}$$

equipped with the Hilbertian norm defined by

$$\|W\|_V^2=\int_\Omega\left|\vec{U}\right|^2 d\Omega+\int_\Gamma\left|U_{n|\Gamma}\right|^2 d\Gamma+|\xi|^2\,\|n_{\Sigma x}\|^2_{L^2(\Sigma)}+|\eta|^2\,\|n_{\Sigma y}\|^2_{L^2(\Sigma)}+|\theta|^2\,\|xn_{\Sigma y}-yn_{\Sigma x}\|^2_{L^2(\Sigma)}$$

and the space χ completion of V for the norm associated with the scalar product

$$(W,\tilde{W})_\chi=\int_\Omega\rho\vec{U}\cdot\vec{\tilde{U}}\;d\Omega+m_0\left(\xi\tilde{\xi}+\eta\tilde{\eta}\right)+I_G\theta\tilde{\theta}.$$

7.4- Now, we introduce a few operators.

We set

$$\beta gP_0\left(v_y\vec{y}\right)=A_{11}\vec{v}\;;\qquad\qquad \beta gP_0\left(U_y\vec{y}\right)=A_{12}W,$$

A_{11} and A_{12} being bounded operators from $J_0(\Omega)$ and χ into $J_0(\Omega)$.

The equation (25) becomes

$$\ddot{\vec{v}}+A_{11}\vec{v}+A_{12}W=0 \tag{28}$$

and we have

$$\beta g\int_\Omega v_y\tilde{v}_y\;d\Omega=(A_{11}\vec{v},\vec{\tilde{v}})_{J_0(\Omega)}\;;\qquad\qquad \beta g\int_\Omega U_y\tilde{v}_y\;d\Omega=(A_{12}W,\vec{\tilde{v}})_{J_0(\Omega)}$$

7.5- The operator A_{11} has a basic role in the problem. It was studied in [1]. It is self-adjoint and its spectrum is identical to its essential spectrum, denoted by $\sigma_{ess}(A_{11})$, and it is the interval $[0,\beta g]$.

We sketch the proof. By a Weyl's theorem [4], it is sufficient to prove that, for every $0<\nu<1$, there exists a sequence $\{\vec{v}_k\}$ such that

$$\frac{\left\|\frac{1}{\beta g}A_{11}\vec{v}_k-\nu\vec{v}_k\right\|}{\|\vec{v}_k\|}\longrightarrow 0\quad when\;\; k\to+\infty$$

We construct a sequence $\{\vec{v}_{nm}\}$ such that $\vec{v}_{nm} = \left(\frac{\partial\Delta_{q_{nm}}}{\partial y}, -\frac{\partial\Delta_{q_{nm}}}{\partial x}\right)^t$ with $q_{nm} = e^{i(nx+my)}q(x,y)$, $q(x,y) \in D(\Omega)$ and equal to 1 in a circle $|x-x_o| \leq r$ contained in Ω.

We can prove that

$$\frac{1}{\beta g}A_{11}\vec{v}_{nm} = \left(\frac{\partial^3 q_{nm}}{\partial x^2\partial y}, -\frac{\partial^3 q_{nm}}{\partial x^3}\right)^t$$

and that

$$\frac{1}{\beta g}A_{11}\vec{v}_{nm} - \frac{n^2}{n^2+m^2}\vec{v}_{nm} = O(n^2+m^2),$$

where $\frac{O(n^2+m^2)}{n^2+m^2}$ is uniformly bounded in Ω.

For every $\varepsilon > 0$, it is possible to find a rational number $\frac{\tilde{m}}{\tilde{n}}$ such that

$$\nu < \frac{\tilde{n}^2}{\tilde{n}^2+\tilde{m}^2} < \nu+\varepsilon.$$

Choosing $m = k\tilde{m}$, $n = k\tilde{n}$, we can prove that the sequence $\{\vec{v}_{k\tilde{n},k\tilde{m}}\}$ satisfies the Weyl's theorem.

7.6- On the other hand, we have

$$\left|\int_\Omega \beta g\ v_y\tilde{U}_y\ d\Omega\right| \leq \beta g\ \|v_y\|_{L^2(\Omega)}\left\|\tilde{U}\right\|_{L^2(\Omega)} \leq c\ \|\vec{v}\|_{J_0(\Omega)}\left\|\tilde{W}\right\|_\chi,$$

(c positive constant),

so that we can write

$$\int_\Omega \beta g\ v_y\tilde{U}_y\ d\Omega = (A_{21}\vec{v},\tilde{W})_\chi\ ,$$

A_{21} being a bounded operator from $J_0(\Omega)$ into χ.
It is easy to see that A_{12} and A_{21} are mutually adjoint.

In the same manner, we can write

$$\int_\Omega \beta g\ U_y\tilde{U}_y\ d\Omega = (A_{22}W,\tilde{W})_\chi.$$

whereA_{22} is non negative self-adjoint bounded operator from χ into χ.

7.7- Finally, the variational equation (27) takes the form

(29)

$$(\ddot{W},\tilde{W})_\chi + \int_\Gamma \rho g\, U_{n|\Gamma}\tilde{U}_{n|\Gamma}\ d\Gamma + a_0(\xi,\eta,\theta;\tilde{\xi},\tilde{\eta},\tilde{\theta}) + \rho(A_{21}\vec{v}+A_{22}W,\tilde{W})_\chi = 0, \qquad \forall\tilde{W}\in V$$

8. Operatorial equation of the problem

Let us recall that

$$a_0(\xi,\eta,\theta;\xi,\eta,\theta) \quad \texttt{and} \quad |\xi|^2 \|n_{\Sigma x}\|^2_{L^2(\Sigma)} + |\eta|^2 \|n_{\Sigma y}\|^2_{L^2(\Sigma)} + |\theta|^2 \|x n_{\Sigma y} - y n_{\Sigma x}\|^2_{L^2(\Sigma)}$$

define the squares of norms, which are equivalent to the classical norm of $\mathbb{C}^3$.

8.1- We set

$$a(W,\tilde{W}) = \rho g \int_\Gamma U_{n|\Gamma} \tilde{U}_{n|\Gamma} \; d\Gamma + a_0(\xi,\eta,\theta;\tilde{\xi},\tilde{\eta},\tilde{\theta}) \tag{30}$$

We prove that the hermitian sesquilinear form $a(W,\tilde{W})$ is continuous and coercive on $V \times V$.

It is sufficient to prove that $a(W,\tilde{W})^{1/2}$ defines on V a norm that is equivalent to the norm of V.

Obviously, we have, C_1 being a suitable positive constant

$$a(W,W) \leq C_1 \|W\|^2_V \qquad \forall W \in V.$$

We prove that there exists $C_2 > 0$ such that

$$\|W\|^2_V \leq C_2 a(W,W) \qquad \forall W \in V,$$

or

$$\begin{cases} \left[\int_\Omega \left|\vec{U}\right|^2 d\Omega + \int_\Gamma \left|U_{n|\Gamma}\right|^2 d\Gamma + |\xi|^2 \|n_{\Sigma x}\|^2_{L^2(\Sigma)} + |\eta|^2 \|n_{\Sigma y}\|^2_{L^2(\Sigma)} + |\theta|^2 \|x n_{\Sigma y} - y n_{\Sigma x}\|^2_{L^2(\Sigma)}\right] \\ \leq C_2 \left[\int_\Gamma \rho g \left|U_{n|\Gamma}\right|^2 d\Gamma + a_0(\xi,\eta,\theta;\xi,\eta,\theta)\right] \end{cases}$$

By virtue of the equivalence of the norms mentioned above, it is sufficient to prove that there exists $C_3 > 0$ such that

$$\int_\Omega \left|\vec{U}\right|^2 d\Omega \leq C_3 \left[\int_\Gamma \left|U_{n|\Gamma}\right|^2 d\Gamma + \int_\Sigma |\xi n_{\Sigma x} + \eta n_{\Sigma y} + \theta(x n_{\Sigma y} - y n_{\Sigma x})|^2 \, d\Sigma\right]$$

or

$$\int_\Omega \left|\overrightarrow{grad}\Phi\right|^2 d\Omega \leq C_3 \left[\int_\Gamma \left|\frac{\partial\Phi}{\partial n}|_\Gamma\right|^2 d\Gamma + \int_\Sigma \left|\frac{\partial\Phi}{\partial n}|_\Sigma\right|^2 d\Sigma\right] \tag{31}$$

This inequality can be proved by using a method that is in [9].

We sketch the proof. We consider the Neumann problem

$$\Delta\Phi = 0 \quad in \;\; \Omega\,; \quad \frac{\partial\Phi}{\partial n}|_S = 0\,, \quad \frac{\partial\Phi}{\partial n}|_\Gamma = \delta \in L^2(\Gamma)\,; \quad \frac{\partial\Phi}{\partial n}|_\Sigma = \varepsilon \in L^2(\Sigma).$$

The Green's formula gives its variational formulation

$$(32)\qquad \int_\Omega \overrightarrow{grad}\Phi \cdot \overrightarrow{grad}\Psi \, d\Omega = \int_\Gamma \delta\Psi|_\Gamma \, d\Gamma + \int_\Sigma \varepsilon\Psi|_\Sigma \, d\Sigma \qquad \forall \Psi \in \widetilde{H}^1(\Omega)$$

Choosing $\Psi = \Phi$, using a trace theorem and the equivalence in $\widetilde{H}^1(\Omega)$ of $\|\Phi\|_{\widetilde{H}^1(\Omega)}$ and $\left\|\overrightarrow{grad}\Phi\right\|_{\left[L^2(\Omega)\right]^2}$, we obtain (31).

8.2- Now, we are going to prove that the embedding from V into χ, obviously dense and continuous, is compact.

We still sketch the method indicated in [9].

Let a sequence $\{W^p\} = \left(\vec{U}^p, \xi^p, \eta^p, \theta^p\right)^t \in V$ that converges weakly in V to $W^* = \left(\vec{U}^*, \xi^*, \eta^*, \theta^*\right)^t \in V \subset \chi$.
Obviously, ξ^p, η^p, θ^p converge strongly to ξ^*, η^*, θ^* in $\mathbb{C}$.

Setting

$$\vec{U}^p = \overrightarrow{grad}\Phi^p, \qquad \vec{U}^* = \overrightarrow{grad}\Phi^*,$$

we have

$$\overrightarrow{grad}(\Phi^p - \Phi^*) \to 0 \qquad weakly\ \ in\ \ \left[L^2(\Omega)\right]^2$$

$$\Phi^p - \Phi^* \to 0 \qquad weakly\ \ in\ \ \widetilde{H}^1(\Omega),$$

so that

$$(\Phi^p - \Phi^*)_{|\Gamma} \to 0 \qquad strongly\ \ in\ \ L^2(\Gamma);$$

$$(\Phi^p - \Phi^*)_{|\Sigma} \to 0 \qquad strongly\ \ in\ \ L^2(\Sigma).$$

On the other hand, with obvious notations, we have $\delta^p - \delta^*$ (resp.$\varepsilon^p - \varepsilon^*$) $\to 0$ weakly in $L^2(\Gamma)$ (resp.$L^2(\Sigma)$).

From (32), we deduce

$$\int_\Omega \left|\overrightarrow{grad}(\Phi^p - \Phi^*)\right|^2 d\Omega = \int_\Gamma (\delta^p - \delta^*)\overline{(\Phi^p - \Phi^*)}\,|_\Gamma \, d\Gamma + \int_\Sigma (\varepsilon^p - \varepsilon^*)\overline{(\Phi^p - \Phi^*)}\,|_\Sigma \, d\Sigma\,,$$

so that

$$\int_\Omega \left|\overrightarrow{grad}(\Phi^p - \Phi^*)\right|^2 d\Omega \to 0$$

and finally

$$\|W^p - W^*\|_\chi^2 \to 0;$$

i.e. the sequence $\{W^p\}$ converge strongly to W^* in χ.

8.3- The variational equation (29) takes the form

$$(\ddot{W},\tilde{W})_\chi + a(W,\tilde{W}) + \rho(A_{21}\vec{v} + A_{22}W,\tilde{W})_\chi = 0, \qquad \forall\tilde{W} \in V \tag{33}$$

Let be A the unbounded operator of χ associated with the form $a(.,.)$ and the pair (V,χ).

The equation (33) is equivalent to the operatorial equation [5]:

$$\ddot{W} + AW + \rho(A_{21}\vec{v} + A_{22}W) = 0, \qquad W \in V \tag{34}$$

Then, we have obtained both operatorial equation (28) and (34) of the problem.

In order to eliminate the unbounded operator$A$, we set classically

$$A^{1/2}W = w \in \chi$$

and we obtain the equations with bounded coefficients

$$\ddot{\vec{v}} + A_{11}\vec{v} + A_{12}A^{-1/2}w = 0 \tag{35}$$

$$A^{-1}\ddot{w} + \rho A^{-1/2}A_{21}\vec{v} + \left[I_\chi + A^{-1/2}A_{22}A^{-1/2}\right]w = 0 \tag{36}$$

$$\vec{v} \in J_0(\Omega); \qquad w \in \chi.$$

9. The spectrum of the problem

We will prove at the end of the paper the existence of the spectrum.

9.1- We seek the solutions that depend on the time according to the law $e^{i\omega t}$; ω real.

We obtain

$$\begin{cases} \omega^2\vec{v} = A_{11}\vec{v} + A_{12}A^{-1/2}w \\ \omega^2 A^{-1}w = \rho A^{-1/2}A_{21}\vec{v} + \left(I_\chi + \rho A^{-1/2}A_{22}A^{-1/2}\right)w \end{cases} \tag{37}$$

or, setting $\mu = \omega^2$

$$\begin{cases} \vec{v} = \mu A_{11}\vec{v} + \mu A_{12}A^{-1/2}w \\ A^{-1}w = \mu\rho A^{-1/2}A_{21}\vec{v} + \mu\left(I_\chi + \rho A^{-1/2}A_{22}A^{-1/2}\right)w \end{cases} \tag{38}$$

9.2- At first, we study the spectrum in the interval $\omega^2 > \beta g$.

Since $\|A_{11}\| = \beta g$, $I - \mu A_{11}$ has a bounded inverse; setting $R(\mu) = (I - \mu A_{11})^{-1}$, we obtain from (38)

$$\vec{v} = \mu R(\mu) A_{12} A^{-1/2} w$$

and then

$$Q(\mu) w = \left[\mu^2 \rho A^{-1/2} A_{21} R(\mu) A_{12} A^{-1/2} + \mu \left(I_\chi + \rho A^{-1/2} A_{22} A^{-1/2}\right) - A^{-1}\right] w = 0$$

$Q(\mu)$ is a self-adjoint operatorial function, that is holomorphic in the domain $|\mu| < (\beta g)^{-1}$.We have
$Q(0) = -A^{-1}$ compact; $Q'(0) = I_\chi + \rho A^{-1/2} A_{22} A^{-1/2}$ strongly positive.

Therefore [4], for every ε, $0 < \varepsilon < (\beta g)^{-1}$,there is in $]0, \varepsilon[$ a countable set of eigenvalues μ_k, which tend to zero when $k \to +\infty$; the eigenelements $\{w_k\}$ form a Riesz basis in a subspace of χ, which has a finite defect.

Consecontly, for the problem, there is a countable set of real, positive eigenvalues ${\omega_k}^2 = \mu_k^{-1}$, which tend to infinity, when $k \to +\infty$.

9.3- Now, we study the spectrum in the interval $0 \leq \omega^2 \leq \beta g$.

From (37), we deduce the equation

$$\left(I_\chi + \rho A^{-1/2} A_{22} A^{-1/2} - \omega^2 A^{-1}\right) w = -\rho A^{-1/2} A_{21} \vec{v}$$

If βg is sufficiently small, the coefficient of w is a strongly positive, self-adjoint, bounded operator; consequently, It has an inverse having the same properties and we can write

$$w = -\rho \left(I_\chi + \rho A^{-1/2} A_{22} A^{-1/2} - \omega^2 A^{-1}\right)^{-1} A^{-1/2} A_{21} \vec{v}\ .$$

Substituting in the first equation (37), we obtain

$$A_{11} \vec{v} - V\left(\omega^2\right) \vec{v} = \omega^2 \vec{v}, \qquad \vec{v} \in J_0(\Omega)\ , \tag{39}$$

with

$$V\left(\omega^2\right) = \rho A_{12} A^{-1/2} \left(I_\chi + \rho A^{-1/2} A_{22} A^{-1/2} - \omega^2 A^{-1}\right)^{-1} A^{-1/2} A_{21}$$

$V\left(\omega^2\right)$ is analytical function in $[0, \beta g]$ and, for each ω^2, $V\left(\omega^2\right)$ is a compact self-adjoint operator, since $A^{-1/2}$ is compact from χ into χ.

We are going to apply the method indicated in [4].
Setting

$$Z\left(\omega^2\right) = A_{11} - V\left(\omega^2\right),$$

we obtain the equation

$$\left(Z\left(\omega^2\right) - \omega^2 I_{J_0(\Omega)}\right) \vec{v} = 0, \qquad \vec{v} \in J_0(\Omega). \tag{40}$$

Let $\omega_1^2 \in \sigma_{ess}(A_{11})$. By a classical Weyl's theorem [4], [9], the operator $Z(\omega_1^2)$ verifies

$$\sigma_{ess}[Z(\omega_1^2)] = \sigma_{ess}(A_{11}) = [0, \beta g]$$

For each $\omega_2^2 \in \sigma_{ess}[Z(\omega_1^2)]$,there exists a "Weyl's sequence" [4], [9], $\{\vec{v}_n\} \in J_0(\Omega)$, that depends on ω_1^2 and ω_2^2, such that

$$\vec{v}_n \to 0 \text{ weakly; } \inf \left\|\vec{v}_n\right\| > 0; \qquad \left(Z\left(\omega_1^2\right) - \omega_2^2 I_{J_0(\Omega)}\right)\vec{v}_n \to 0 \text{ in } J_0(\Omega).$$

Choosing $\omega_2^2 = \omega_1^2$, the corresponding Weyl's sequence $\left\{\vec{v}_n\right\}$ depends on ω_1^2 only and verifies $\left(Z\left(\omega_1^2\right) - \omega_1^2 I_{J_0(\Omega)}\right)\vec{v}_n \to 0$ in $J_0(\Omega)$, so that ω_1^2 belongs to the spectrum of the problem (40). ω_1^2 being arbitrary in $[0, \beta g]$; the spectrum of the problem in this interval coincides with its essential spectrum $[0, \beta g]$

9.4-Conclusion

The spectrum of the problem is composed by an essential part, which fills the closed interval $[0, \beta g]$, and a discrete part that lies outside this interval and is comprised of a countable set of positive real eingenvalues, whose accumulation point is the infinity. Physically, the interval $[0, \beta g]$ is a domain of resonance.

10. Existence and uniqueness theorem

We use the equation (28) and (34) for unknown $\vec{v}, W$.

We introduce the spaces

$$V_0 = J_0(\Omega) \oplus V, \qquad H_0 = J_0(\Omega) \oplus \chi$$

The imbeding $V_0 \subset H_0$ is obviously dense and continous, but it is not compact, because the identical operator $I_{J_0(\Omega)}$ is not compact.

From the equation (28) and (34), we deduce

$$\rho(\ddot{\vec{v}}, \tilde{\vec{v}})_{J_0(\Omega)} + (\ddot{W}, \tilde{W})_\chi + \rho(A_{11}\vec{v} + A_{12}W, \tilde{\vec{v}})_{J_0(\Omega)} + \left((A + \rho A_{22})W + \rho A_{21}\vec{v}, \tilde{W}\right)_\chi = 0$$

Itroducing the operator C from H_0 onto H_0 defined by

$$C = \begin{pmatrix} \rho I_{J_0(\Omega)} & 0 \\ & \\ 0 & I_\chi \end{pmatrix}$$

and setting

$$X = (\vec{v}, W)^t \in V_0,$$

we can write the last equation in the form

$$(C\ddot{X},\tilde{X})_{H_0} + \hat{a}(X,\tilde{X}) = 0 \qquad \forall \hat{X} \in V_0 \tag{41}$$

with

$$\hat{a}(X,\tilde{X}) = \rho\left[(A_{11}\vec{v} + A_{12}W, \tilde{\vec{v}})_{J_0(\Omega)} + (A_{21}\vec{v} + A_{22}W, \tilde{W})_\chi\right] + (W,\tilde{W})_V$$

Since the operators are bounded, $\hat{a}(X,\tilde{X})$ is continuous in $V_0 \times V_0$ and we have, using the definition of the A_{ij}:

$$\hat{a}(X,X) = \rho\beta g \int_\Omega |v_y + U_y|^2\, d\Omega\ + \|W\|_V^2.$$

Let be λ real positive; we have

$$\hat{a}(X,X) + \lambda\|X\|_{H_0}^2 = \rho\beta g \int_\Omega |v_y + U_y|^2\, d\Omega\ + \lambda\left(\|\vec{v}\|_{J_0(\Omega)}^2 + \|W\|_\chi^2\right) + \|W\|_V^2$$

and then

$$\hat{a}(X,X) + \lambda\|X\|_{H_0}^2 \geq \min(1,\lambda)\|X\|_{V_0}^2,$$

so that $\hat{a}(.,.)$ is V_0- coercive with respect to H_0.

Therefore, we can apply a known theorem [3; pp 667-670]:
If we have the initial data

$$X_0 = \left((\vec{v})^0, W^0\right)^t \in V_0; \qquad \dot{X}_0 = \left((\dot{\vec{v}})^0, \dot{W}^0\right)^t \in H_0$$

the problem (41) has one and only one solution $X(.)$ suth that

$$X(t) \in L^2(0,T;V_0); \qquad \dot{X}(t) \in L^2(0,T;V_0)$$

where T is an arbitrary positive constant.

11. On the existence of the spectrum

The equation (35), (36) can take the form:

$$Q\ddot{\hat{X}} + B\hat{X} = 0, \tag{42}$$

$$\hat{X} = (\vec{v}, w)^t \in H_0$$

$$Q = \begin{pmatrix} \rho I_{J_0(\Omega)} & 0 \\ \\ 0 & A^{-1} \end{pmatrix}; \qquad B = \begin{pmatrix} \rho A_{11} & \rho A_{12}A^{-1/2} \\ \\ \rho A^{-1/2}A_{21} & I_\chi + \rho A^{-1/2}A_{22}A^{-1/2} \end{pmatrix}$$

These operators are bounded and self-adjoint. Q is positive definite like A^{-1}.

By direct calculations, we have

$$(B\hat{X},\hat{X})_{H_0} = \int_{\Omega} |v_x + U_y|^2 \, d\Omega \; + \|W\|_V^2.$$

From $(B\hat{X},\hat{X})_{H_0}$, we deduce $v_y + U_y = 0, W = 0$ or $w = 0$.

$W = 0$ is equivalent to $\vec{U} = 0, \xi = 0$, $\eta = 0, \theta = 0$. Then we have $v_y = 0$. Since $div\vec{v} = 0$, we have $\frac{\partial v_x}{\partial x} = 0$ in the sence of distributions; on the other hand, we have $v_{n|S} = 0$.

We denote by Π the part of a parallel to G_0x, which is interior to Ω. it is known [10] that v_x is an absolutely continuous function on Π which has almost everywhere a classical derivate equal to zero. therefore, $v_x = constant$ on Π, from which we deduce, by using the condition on S, $\vec{v} = 0$ a.e in Ω.

Finally $(B\hat{X},\hat{X})_{H_0} = 0$ only for $\hat{X} = 0$ and B is positive definite.
It is easy to see that

$$((Q+B)\hat{X},\hat{X})_{H_0} \geq \rho\|\vec{v}\|^2_{J_0(\Omega)} + \|W\|_V^2 \geq C'\|\hat{X}\|^2_{H_0}$$

where C' is a suitable positive constant, so that $Q+B$ is strongly positive in H_0 and therefore has an inverse.
Seeking the solution of (42) that depend on time according to the law $e^{\nu t}$, we have

$$(\nu^2 Q + B)\hat{X} = 0, \qquad \hat{X} \in H_0. \tag{43}$$

$\nu = 0$ and $\nu = \pm 1$ are not eigenvalues, since B is positive definite and $(Q+B)$ strongly positive.

Writing the equation in the form

$$\left[\nu^2(Q+B) + (1-\nu^2)B\right]\hat{X} = 0$$

and setting

$$(Q+B)^{1/2}\hat{X} = \kappa; \qquad \Lambda = (Q+B)^{-1/2}B(Q+B)^{-1/2},$$

We obtain the equation

$$\Lambda\kappa = \frac{\nu^2}{\nu^2 - 1}\kappa \tag{44}$$

Λ being self-adjoint and positive definite, has a spectrum lying on the semi-axis real positive.

The condition $\frac{\nu^2}{\nu^2-1}$ real posive is equivalent to

$$\nu^2 \text{ real} > 1 \text{ or} < 0.$$

The first case is impossible, because $(\nu^2 Q + B)$ is strongly positive and then, (43) has only the solution $\hat{X}$.

Consequensly, we must have $\nu = i\omega$, ω real according to the calculations in the in the paragraph 9.

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