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SOME RESULTS BASED ON WEIGHTED DYNAMIC ENTROPIES

Abstract. In this paper, we study some properties of the weighted dynamic measures of entropy introduced in Di Crescenzo and Longobardi (2006). It is shown that the proposed measures could characterize distributions, and characterizations for some distributions in continuous and discrete cases are presented. Further, we define a new stochastic order based on weighted dynamic entropies, and we provide some of its properties. Discrete version results for these classes are explored too.

1. Introduction

Study of information theory is one of the most important aspects in modeling of biological systems. To this aim, the basic measure of uncertainty has been introduced by Shannon (1948) and Wiener (1948). Further, various researchers have studied properties of this information measure for non-negative random variables representing the lifetime of a system or of a unit in dynamic situations; see e.g., Ebrahimi (1996), Ebrahimi and Pellerey (1995), Di Crescenzo and Longobardi (2002) and Nanda and Paul (2006). Moreover, Since the characterizations of distributions have numerous applications in reliability, characterizations in terms of dynamic entropies of various distributions have been proposed in the literature, (Nair and Rajesh (1998) and Ebrahimi (1996)).
In particular, in the context of theoretical neurobiology have been considered some measures of uncertainty based on the notion of the weighted entropy (see, e.g., Johnson and Glantz, 2004, and Belis and Guiasu, 1968).
Among others, Di Crescenzo and Longobardi (2006) defined and studied the notion of weighted entropy for residual and past lifetimes of a component via the notion of weighted residual and past entropies, extending some properties previously given by Belzunce et al. (2004) and Nanda and Paul (2006) to characterize a distribution function via weighted dynamic measures.
In addition, Di Crescenzo and Longobardi (2006) introduced two new classes of distributions based on monotonicity properties of the weighted dynamic entropies.
In this paper, we characterize some distributions of random lifetimes based on weighted dynamic measures in continuous and discrete situations. Also, we explore some properties of the ageing classes defined by monotonicity properties of weighted residual and past entropies.
Stochastic orders based on Weighted residual and past entropies will be considered

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as well. In fact, stochastic comparisons between probability distributions play a fundamental role in statistics and related areas, and various concepts of ordering have been investigated and used as mathematical tools for proving important results in applied probability (see Shaked and Shanthikumar, 2007, for complete list of stochastic orders and their application). Thus, for the same reasons described in Di Crescenzo and Longobardi (2006), stochastic order based on weighted dynamic entropies can be considered. More, we study their presentation properties under some possible transformations.

2. Preliminaries

Let X be a non-negative random variable representing the lifetime of a system or a component having distribution function $F(x)$, survival function $F(x) = P(X > x) = 1 - F(x)$ such that $F(0) = 1$ and probability density function $f(x)$.
It is known that some measures such as the hazard rate $\lambda(t) = f(t)/F(t)$, the reversed hazard rate $\lambda(t) = f(t)/F(t)$, the mean residual life $\mu(t) = E[X - t|X \geq t]$ and the mean past life $\mu(t) = E[t - X|X < t]$ have been applied to characterize the distribution of a system, in the sense that they uniquely characterize the distribution.
It is shown that there are other measures that could be defined and used to characterize distributions, like the functions weighted hazard and reversed hazards, respectively.

$$M(t) = E[X \mid X > t] = \int_t^\infty x \frac{f(x)}{F(t)}\, dx = \mu(t) + t, \tag{1}$$

and

$$M(t) = E[X \mid X \leq t] = \int_0^t x \frac{f(x)}{F(t)}\, dx = t - \mu(t), \tag{2}$$

based on the notion of entropy, Ebrahimi (1996) defined the uncertainty of residual lifetime distributions $H(X,t)$ by

$$H(X,t) = -\int_t^\infty \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)}\, dx, \tag{3}$$

Later Di Crescenzo and Longobardi (2002) have explored the past entropy by truncation the distribution up to some point t of a component via the function

$$H(X,t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)}\, dx. \tag{4}$$

Similarly as for the definition of M and M above, Di Crescenzo and Longobardi (2006) also introduced the notions of weighted residual and past entropies, respectively, as

$$H^w(X,t) = -\int_t^\infty x \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)}\, dx, \tag{5}$$

and

$$H^w(X,t) = -\int_0^t x\frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}\,dx. \tag{6}$$

It should be observed that these functions are not necessary monotone; the following is an example of a pair of distributions that presents non monotonic weighted past entropy and such that an order between their weighted past entropies can be considered.

EXAMPLE 1. Let X and Y be two random variables with densities $f_X(t) = 2\,t$ and $f_Y(t) = 3\,t^2$, $0 < t < 1$ respectively. Then, for $t \in (0,1)$,

$$H^w(X,t) = \frac{2}{3}t\log(2) - \frac{2}{9}t - \frac{2}{3}t\log(t),$$

and

$$H^w(Y,t) = \frac{3}{4}t\log(3) - \frac{3}{8}t - \frac{3}{4}t\log(t),$$

Numerically it can be observed that

$$\begin{aligned} H^w(X,0.2) &\cong 0.13\log(2) + 0.17, \\ H^w(X,0.5) &\cong 0.33\log(2) + 0.12, \\ H^w(X,0.9) &\cong 0.6\log(2) - 0.14, \end{aligned}$$

and that,

$$\begin{aligned} H^w(Y,0.2) &\cong 0.15\log(3) - 0.17, \\ H^w(Y,0.5) &\cong 0.375\log(3) + 0.72, \\ H^w(Y,0.9) &\cong 0.675\log(3) - 0.27, \end{aligned}$$

thus the both the two weighted past entropies are first increasing and then decreasing, hence they are not monotonic. It can also be observed that their difference always has the same sign, and for all $0 < t < 1$ we can write,

$$H^w(Y,t) - H^w(X,t) = \frac{3}{4}t\log(3) - \frac{2}{3}t\log(2) - \frac{11}{72}t - \frac{1}{12}t\log(t) > 0,$$

which means that the weighted past entropy for the random variable Y is bigger than the weighted past entropy for the random variable X. Thus, a stochastic order based on comparison among weighted past entropy can be considered.

Two well-known partial orders which are based on the notion of residual and past entropies, have been proposed already by Ebrahimi and Pellerey (1995) and by Nanda and Paul (2006); their definitions are recalled here.

DEFINITION 1. Let X and Y be two non-negative random variables. Then X is said to be smaller than Y in residual entropy order (denoted as $X \leq^{RE} Y$) if $H(X,t) \leq H(Y,t)$ for all $t \geq 0$.

DEFINITION 2. Let X and Y be two non-negative random variables. Then X is said to be smaller than Y in past entropy order (denoted as $X \leq^{PE} Y$) if $\bar{H}(X,t) \leq \bar{H}(Y,t)$ for all $t \geq 0$.

The following stochastic orders based on weighted dynamic measures could be defined similarly.

DEFINITION 3. Let X and Y be two non-negative random variables. Then, X is said to be smaller than Y in weighted residual entropy order (denoted as $X \leq^{WRE} Y$) if $H^w(X,t) \leq H^w(Y,t)$ for all $t \geq 0$.

DEFINITION 4. Let X and Y be two non-negative random variables. Then, X is said to be smaller than Y in weighted past entropy order (denoted as $X \leq^{WPE} Y$) if $\bar{H}^w(X,t) \leq \bar{H}^w(Y,t)$ for all $t \geq 0$.

Ebrahimi (1996) also introduced two new non-parametric classes of life distributions via monotonicity properties of the uncertainly residual life function: according to his definition a random variable X is said to be decreasing (increasing) in uncertainty of residual life (denoted as $X \in DURL$ $(IURL)$) if $H(X,t)$ is decreasing (increasing). Similarly, a random variable X is said to be decreasing (increasing) in uncertainty of past life (denoted as $X \in DUPL$ $(IUPL)$) if $\bar{H}(X,t)$ is decreasing (increasing) with respect to t. Later, Di Crescenzo and Longobardi (2006) extended these classes to new ones based on weighted residual and past entropies as in the following definitions:

DEFINITION 5. A non-negative random variable X is said to have decreasing (increasing) weighted uncertainty residual life (denoted as $X \in DWURL$ $(IWURL)$) if $H^w(X,t)$ is decreasing (increasing) in $t \geq 0$.

DEFINITION 6. A non-negative random variable X is said to have decreasing (increasing) weighted uncertainty past life (denoted as $X \in DWUPL$ $(IWUPL)$) if $\bar{H}^w(X,t)$ is decreasing (increasing) in $t \geq 0$.

3. Characterizations for continuous distributions

In this section, by using the same ideas as in Di Crescenzo and Longobardi (2006), we show that weighted dynamic entropies can determine the distribution functions. Moreover, we show that we can characterize probability distributions through the relationship between their weighted dynamic entropies and expected inactivity times.

THEOREM 1. *If X has survival distribution $\bar{F}(t)$ and increasing weighted residual entropy $H^w(X,t)$, then $H^w(X,t)$ uniquely determines $\bar{F}$.*

Proof. From the definition of $H^w(X,t)$, we have,

$$\begin{aligned} H^w(X,t) &= -\int_t^{\infty} x\frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}dx \\ &= \frac{1}{F(t)}\Big[-\int_t^{\infty} xf(x)\log f(x)dx + \log F(t)\int_t^{\infty} xf(x)dx\Big]. \end{aligned}$$

By differentiating both sides, we obtain that

$$tf(t)\log f(t) - \frac{f(t)}{F(t)}\int_t^{\infty} xf(x)dx - tf(t)\log F(t) = -f(t)H^w(X,t) + F(t)\frac{d}{dt}H^w(X,t).$$

Equivalently, since $\lambda(t) = \frac{f(t)}{F(t)}$ and $\int_t^{\infty} x\frac{f(x)}{F(t)}dx = \mu(t)+t$, it holds that

$$t\lambda(t)\log\lambda(t) - \lambda(t)[\mu(t)+t] = -\lambda(t)H^w(X,t) - \frac{d}{dt}H^w(X,t). \tag{7}$$

Thus, for any fixed $t > 0$, $\lambda(t)$ is a positive solution of the equation

$$g(x) = x[t\log x - \mu(t) - t + H^w(X,t)] - \frac{d}{dt}H^w(X,t) = 0. \tag{8}$$

Note that $\lim_{x\to+\infty} g(x) = +\infty$, $\lim_{x\to 0} g(x) = -\frac{d}{dt}H^w(X,t) \leq 0$ and

$$g'(x) = t\log x - \mu(t) + H^w(X,t).$$

Hence, $g(x)$ first decreases and then increases in x and has minimum at $x = \exp\{\frac{1}{t}(\mu(t) - H^w(X,t))\}$.
Thus, equation (4) has just one solution for all t, and $H^w(X,t)$ uniquely characterizes $\lambda(t)$ which, in turns, characterizes F. □ □

The next theorem shows that the weighted past entropy uniquely can characterize $F(t)$. The proof, omitted, is the same as for the one of the previous theorem.

THEOREM 2. *If X has distribution $F(t)$ and decreasing weighted past entropy $H^w(X,t)$, then $H^w(X,t)$ uniquely determines F.*

The following theorem shows that the relation between the functions $H^w(X,t)$ and $M(t)$ can determine exponential distribution.

THEOREM 3. *Let X be a non-negative continuous random variable with finite mean. Then X has exponential distribution with mean $\frac{1}{\theta}$ iff*

$$H^w(X,t) = A\,M(t) + B, \tag{9}$$

where, for some $\theta > 0$,

$$A = 1 - \log\theta \ \ \textit{and} \ \ B = \frac{1}{\theta}.$$

Proof. Fix $t \geq 0$ according to (1), for the exponential distribution with mean $\frac{1}{\theta}$ we have,

$$H^w(X,t) = (t+\frac{2}{\theta}) - (t+\frac{1}{\theta})\log\theta = M(t)[1-\log\theta] + \frac{1}{\theta} = A\,M(t) + B.$$

Thus, it remains to prove the necessary condition. Suppose that (5) holds. Then, via (2), we have,

$$\frac{d}{dt}H^w(X,t) - \lambda(t)H^w(X,t) = t\lambda(t)\log\lambda(t) - \lambda(t)M(t).$$

Differentiating both sides of (5) with respect to t implies

$$\frac{d}{dt}H^w(X,t) = A\frac{d}{dt}M(t).$$

Also, we can get $\frac{d}{dt}M(t) = \lambda(t)M(t) - t\lambda(t)$, therefore, $\frac{d}{dt}H^w(X,t) = A\ \lambda(t)\ M(t) - At\lambda(t)$. Hence,

$$A\lambda(t)\ M(t) + B\lambda(t) + t\lambda(t)\log\lambda(t) - \lambda(t)M(t) = A\lambda(t)M(t) - At\lambda(t),$$

and

$$B + t\log\lambda(t) - M(t) = At.$$

With a simple calculation,

$$\begin{aligned}\lambda(t) &= \exp\{\frac{M(t)-B+At}{t}\}\\ &= \exp\{\frac{M(t)-B}{t} + A\}.\end{aligned}$$

Also, since $\lambda(t)$ and $M(t)$ are unique for every distribution, the relation between them is unique too. Hence, the proof is completed. □ □

The Pareto distribution is characterized below in terms of weighted residual entropy.

THEOREM 4. *The Pareto distribution with parameters* α *and* β *can be characterized by*

$$H^w(X,t) = M(t)\log M(t) + KM(t), \quad for\ t > \beta, \tag{10}$$

where

$$K = \frac{\alpha+1}{\alpha-1} + \log\frac{\alpha-1}{\alpha^2}.$$

Proof. Fix $t > \beta$. For the Pareto distribution with parameters α and β, by definition of $H^w(X,t)$ it holds

$$\begin{aligned}H^w(X,t) &= \frac{\alpha t}{\alpha-1}\log\frac{t}{\alpha} + \frac{\alpha(\alpha+1)}{(\alpha-1)^2}t\\ &= \frac{\alpha t}{\alpha-1}\log\frac{\alpha t}{\alpha-1} + \frac{\alpha t}{\alpha-1}[\frac{\alpha+1}{\alpha-1} + \log\frac{\alpha-1}{\alpha^2}],\end{aligned}$$

and (10) is satisfied.
To prove the converse, let us assume that (10) holds. Then, we get,

$$\frac{d}{dt}H^w(X,t) = \frac{d}{dt}M(t)\log M(t) + \frac{d}{dt}M(t) + K\frac{d}{dt}M(t).$$

Also, it holds

$$\begin{aligned}\frac{d}{dt}H^w(X,t) - \lambda(t)H^w(X,t) &= \frac{d}{dt}M(t)\log M(t) + \frac{d}{dt}M(t)\\ &+ K\frac{d}{dt}M(t) - \lambda(t)M(t)\log M(t) - K\lambda(t)M(t).\end{aligned}$$

By using the fact that $\frac{d}{dt}M(t) = \lambda(t)M(t) - t\lambda(t)$, we can write

$$\begin{aligned}\frac{d}{dt}H^w(X,t) - \lambda(t)H^w(X,t) &= \lambda(t)M(t)\log M(t) - t\lambda(t)\log M(t) + \lambda(t)M(t) - t\lambda(t)\\ &+ K\lambda(t)M(t) - tK\lambda(t) - \lambda(t)M(t)\log M(t) - K\lambda(t)M(t)\\ &= \lambda(t)M(t) - t\lambda(t)\log M(t) - t\lambda(t) - tK\lambda(t).\end{aligned}$$

But we know that

$$\frac{d}{dt}H^w(X,t) - \lambda(t)H^w(X,t) = t\lambda(t)\log\lambda(t) - \lambda(t)M(t),$$

therefore

$$t\lambda(t)\log[\lambda(t)M(t)] + t\lambda(t)[K+1] = 2\lambda(t)M(t).$$

On the other hand, one can show that, for a Pareto distribution,

$$\lambda(t)M(t) = \exp\{\frac{2M(t)}{t} - (K+1)\},$$

which implies that a unique solution $\lambda(t) = \frac{1}{M(t)}\exp\{\frac{2M(t)}{t} - (K+1)\}$ is obtained and the theorem is proved. □ □

In the following theorem we present a distribution having finite support that can be characterized from the relationship between the weighted past entropy and the mean inactivity time.

THEOREM 5. *A non-negative random variable X having probability density function cx^{c-1}, for $0 < x < 1$ is characterized by*

$$H^w(X,t) = M(t)\log M(t) + KM(t), \tag{11}$$

where

$$K = \frac{c-1}{c+1} + \log\frac{c+1}{c^2}.$$

Proof. The proof is the same as for Theorem 1 . □

The following proposition describes the relationship between $H(X,t)$ and $H^w(x,t)$ for symmetric random variables.

PROPOSITION 1. Let X be a random variable with support in $[0,c]$, and symmetric with respect to $\frac{c}{2}$, i.e. such that $F(x) = F(c-x)$ for $0 \le x \le c$. Then

$$H^w(X,t) = c\, H(X,c-t) - H^w(X,c-t) \quad \forall t \in [0,c]$$

Proof. It is known that $f(x) = f(c-x)$ and hence,

$$H^w(X,t) = -\int_0^t x \, \frac{f(c-x)}{F(c-t)} \log \frac{f(c-x)}{F(c-t)} \, dx.$$

Letting $u = c-x$, we get

$$\begin{aligned} H^w(X,t) &= -\int_{c-t}^{c} (c-u)\, \frac{f(u)}{F(c-t)} \log \frac{f(u)}{F(c-t)} \, du \\ &= -c \int_{c-t}^{c} \frac{f(u)}{F(c-t)} \log \frac{f(u)}{F(c-t)} \, du + \int_{c-t}^{c} u \, \frac{f(u)}{F(c-t)} \log \frac{f(u)}{F(c-t)} \, du, \end{aligned}$$

which provides the required result. □ □

The next example is a straightforward consequence of above proposition.

EXAMPLE 2. Let X be a non-negative random variable with uniform distribution on (a,b). Then $H^w(X,E(X)) = H^w(X,E(X))$.

Proof. Via $H^w(X,t) = H^w(X,t)$, we can get $(b-t)\log(b-t) = (t-a)\log(t-a)$ and obviously leads to $t = \frac{a+b}{2} = E(X)$. □

PROPOSITION 2. Let X be a non-negative random variable,
(i) If $H^w(X,t)$ is increasing (decreasing) in t then,

$$\exp\{H^w(X,t)\} \ge (\le) \lambda(t)^{-t} e^{\mu(t)+t}.$$

(ii) If $H^w(X,t)$ is increasing (decreasing) in t then,

$$\exp\{H^w(X,t)\} \le (\ge) \lambda(t)^{-t} e^{t-\mu(t)}.$$

Proof. The results could be obtained directly from the following statements

$$\frac{d}{dt} H^w(X,t) = \lambda(t)[H^w(X,t) + t\log\lambda(t) - \mu(t) - t],$$

and

$$\frac{d}{dt} H^w(X,t) = \lambda(t)[t - \mu - t\log\lambda(t) - H^w(X,t)].$$

□

4. Characterizations for discrete distributions

In most aspects of reliability theory, time is assumed to be continuous, but there exist some problems involving discrete situations. The following theorems show that weighted residual and past entropies uniquely determine distribution functions even in the discrete case.
Assume X to be a discrete random variable with discrete density $p(t_k) = P(X = t_k)$, survival function $F(t_k) = P(X > t_k)$ and discrete hazard rate $\lambda(t_k) = \frac{p(t_k)}{F(t_k)}$. Similarly as far the continue case characterization result holds.

THEOREM 6. *If X has a survival function $F(t)$ with support T such that $T = \{t_j, j \in N, t_j < t_{j+1}\}$ and an increasing weighted residual entropy $H^w(X,t)$, then $H^w(X,t)$ uniquely determines F.*

Proof. Fixed $t_j \in T$, note that in the discrete case equation(1) can be rewritten as

$$\begin{aligned} F(t_j)H^w(X,t_j) &= -\sum_{k=j}^{\infty} t_k p(t_k)\log\frac{p(t_k)}{F(t_j)} \\ &= -\sum_{k=j}^{\infty} t_k p(t_k)\log p(t_k) + \log F(t_j)\sum_{k=j}^{\infty} t_k p(t_k). \end{aligned}$$

Equivalently,

$$\sum_{k=j}^{\infty} t_k p(t_k)\log p(t_k) = \log F(t_j)\sum_{k=j}^{\infty} t_k p(t_k) - F(t_j)H^w(X,t_j).$$

and similarly, for t_{j+1}, we can write

$$\sum_{k=j+1}^{\infty} t_k p(t_k)\log p(t_k) = \log F(t_{j+1})\sum_{k=j+1}^{\infty} t_k p(t_k) - F(t_{j+1})H^w(X,t_{j+1}),$$

so that

$$\begin{aligned} t_j p(t_j)\log p(t_j) &= \log F(t_j)\sum_{k=j}^{\infty} t_k p(t_k) - \log F(t_{j+1})\sum_{k=j+1}^{\infty} t_k p(t_k) \\ &- F(t_j)H^w(X,t_j) + F(t_{j+1})H^w(X,t_{j+1}) \\ &= \log F(t_j)[t_j p(t_j)] + [\log\frac{F(t_j)}{F(t_{j+1})}]\sum_{k=j+1}^{\infty} t_k p(t_k) \\ &- F(t_j)H^w(X,t_j) + F(t_{j+1})H^w(X,t_{j+1}). \end{aligned}$$

Letting $p(t_j) = F(t_j) - F(t_{j+1})$, this leads to

$$\begin{aligned} &t_j[F(t_j) - F(t_{j+1})]\log F(t_j) - F(t_{j+1}) = t_j[F(t_j) - F(t_{j+1})]\log F(t_j) \\ &+\log\frac{F(t_j)}{F(t_{j+1})}]\sum_{k=j+1}^{\infty} t_k p(t_k) - F(t_j)H^w(X,t_j) + F(t_{j+1})H^w(X,t_{j+1}). \end{aligned}$$

Considering $rj = \frac{F(t_{j+1})}{F(t_j)}$, then

$$t_jF(t_j)(1-r_j)\log(1-r_j) = -\log r_j \sum_{k=j+1}^{\infty} t_k p(t_k) - F(t_j)H^w(X,t_j) + r_jF(t_j)H^w(X,t_{j+1}),$$

which is equivalent to

$$t_j(1-r_j)\log(1-r_j) = -r_j\log r_j \sum_{k=j+1}^{\infty} t_k \frac{p(t_k)}{F(t_{j+1})} - H^w(X,t_j) + r_jH^w(X,t_{j+1}).$$

By using $M(t_{j+1}) = \sum_{k=j+1}^{\infty} t_k \frac{p(t_k)}{F(t_{j+1})}$, the previous equation can be restated as

$$g(x) = t_j(1-x)\log(1-x) + M(t_{j+1})x\log x + H^w(X,t_j) - xH^w(X,t_{j+1}) = 0, \quad (12)$$

where $0 < x < 1$. Observe that

$$g(1) = H^w(X,t_j) - H^w(X,t_{j+1}) \leq 0 \quad \textit{and} \quad g(0) = H^w(X,t_j) \geq 0.$$

and

$$g'(x) = M(t_{j+1})\log x - t_j\log(1-x) + M(t_{j+1}) - H^w(X,t_{j+1}) - t_j,$$

$$g''(x) = \frac{M(t_{j+1})}{x} + \frac{t_j}{1-x} = 0.$$

It is clear that $x = \frac{-M(t_{j+1})}{t_j - M(t_{j+1})}$ can be the only solution of above equation. But, if $t_j < M(t_{j+1})$ then $x > 1$, and if $t_j > M(t_{j+1})$ then $x < 0$, thus there exist a solution for $g''(x) = 0$.
Hence, $g'(x) = 0$ has almost one solution, which implies that equation (6) has a unique positive solution in $(0,1)$ for all j. Since $\lambda_j = 1 - r_j$ and since λ_j uniquely characterize F, we have the required result. □ □

Using similar arguments, the following statement can be proved.

THEOREM 7. *If X has distribution $F(t)$ with support T such that $T = \{t_j, j \in N, t_j < t_{j+1}\}$ and a decreasing weighted past entropy, then $H^w(X,t)$ uniquely characterizes $F(t)$.*

5. A new ordering of weighted dynamic entropies

Some results dealing with the ordering based on dynamic entropy ordering are presented in this section. Results for the role of transformation in ordering of weighted dynamic entropies are discussed as well.
The following example shows in case for random variables with exponential distributions both weighted residual entropy and residual entropy orders can be satisfied.

EXAMPLE 3. Let X_1 and X_2 be two random variables exponentially distributed with parameters θ_1 and θ_2, where $0 < \theta_1 < \theta_2 < 1$. Then it is simple to claim that $X_1 >^{WRE} X_2$ because

$$H^w(X_i,t) = (t+\frac{2}{\theta_i}) + (t+\frac{1}{\theta_i})\log\frac{1}{\theta_i}, \quad i = 1,2$$

Also, $H(X_i,t) = 1-\log\theta_i$, that is independent of t and decreasing in θ_i, which gives $X >^{RE} Y$.

The following theorems state the relation between PE, RE, WRE and WPE orders under monotonic convex transformation.

THEOREM 8. *Let X and Y be two non-negative random variables such that $X >^{RE} Y$ and let ϕ be a strictly increasing convex function such that $\phi(0) = 0$ and $\phi'(0) > 1$, then $\phi(X) >^{WRE} \phi(Y)$.*

Proof. Observe that if ϕ is increasing then

$$H^w(\phi(X),t) = H^{w,\,\phi}(X,\phi^{-1}(t)) + E[\phi(X)\log\phi'(X) \mid X > \phi^{-1}(t)], \tag{13}$$

where $H^{w,\,\phi}(X,t) = -\int_t^\infty \phi(x)\frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}dx$.
By (7), we get,

$$\begin{aligned} H^w(\phi(X),t) - H^w(\phi(Y),t) &= H^{w,\,\phi}(X,\phi^{-1}(t)) - H^{w,\,\phi}(Y,\phi^{-1}(t)) \\ &+ E[\phi(X)\log\phi'(X) \mid X > \phi^{-1}(t)] \\ &- E[\phi(Y)\log\phi'(Y) \mid Y > \phi^{-1}(t)]. \end{aligned}$$

Note that

$$H^{w,\,\phi}(X,\phi^{-1}(t)) - H^{w,\,\phi}(Y,\phi^{-1}(t))$$

$$\begin{aligned} &= \int_{\phi^{-1}(t)}^\infty \phi(x)\,[\frac{g(x)}{G(\phi^{-1}(t))}\log\frac{g(x)}{G(\phi^{-1}(t))} - \frac{f(x)}{F(\phi^{-1}(t))}\log\frac{f(x)}{F(\phi^{-1}(t))}\,]\,dx \\ &> t\,[\,H(X,\phi^{-1}(t)) - H(Y,\phi^{-1}(t))\,]\,, \end{aligned}$$

since for $X > \phi^{-1}(t)$, by monotonicity of ϕ, it holds $\phi(X) > t$. Also, $X >^{RE} Y$ leads to $H(X,\phi^{-1}(t)) - H(Y,\phi^{-1}(t)) > 0$, and consequently

$$H^{w,\,\phi}(X,\phi^{-1}(t)) - H^{w,\,\phi}(Y,\phi^{-1}(t)) > 0, \tag{14}$$

and

$$E[\phi(X)\log\phi'(X) \mid X > \phi^{-1}(t)] - E[\phi(Y)\log\phi'(Y) \mid Y > \phi^{-1}(t)]$$

$$= \int_{\phi^{-1}(t)}^\infty \phi(x)\log\phi'(x)\,[\,\frac{f(x)}{F(\phi^{-1}(t))} - \frac{g(x)}{G(\phi^{-1}(t))}\,]\,dx.$$

If ϕ is convex, then for $x > \phi^{-1}(t)$ we have $\log\phi'(x) > \log\phi'(\phi^{-1}(t))$, and therefore

$$E[\phi(X)\log\phi'(X) \mid X > \phi^{-1}(t)] - E[\phi(Y)\log\phi'(Y) \mid Y > \phi^{-1}(t)]$$

$$> t\log\phi'(\phi^{-1}(t))\int_{\phi^{-1}(t)}^{\infty} [\ \frac{f(x)}{F(\phi^{-1}(t))} - \frac{g(x)}{G(\phi^{-1}(t))}\ ]\ dx = 0. \tag{15}$$

Using (8) and (15) the required result is obtained. □ □

With similar arguments we can prove the following statement:

THEOREM 9. *(i) If $X >^{PE} Y$ and ϕ is decreasing and convex function such that $\phi'(0) > 1$, then $\phi(X) >^{WRE} \phi(Y)$.*
(ii) If $X <^{PE} Y$ and ϕ is increasing and convex function such that $\phi'(0) > 1$, then $\phi(X) <^{WPE} \phi(Y)$.
(iii) If $X <^{RE} Y$ and ϕ is decreasing and convex function such that $\phi'(0) > 1$, then $\phi(X) <^{WPE} \phi(Y)$.

Some results for classes of weighted dynamic measures are achievable, the following remark shows that the classes of dynamic measures and weighted dynamic measures could be implied of each other.

REMARK 1. According to Di Crescenzo and Longobardi (2006) we have

$$\frac{d}{dt}H^w(X,t) = t\frac{d}{dt}H(X,t),$$

and

$$\frac{d}{dt}H^w(X,t) = t\frac{d}{dt}H(X,t).$$

Thus, we can conclude that

$$X \in DURL(IURL) \Leftrightarrow X \in DWURL(IWURL),$$

$$X \in DUPL(IUPL) \Leftrightarrow X \in DWUPL(IWUPL).$$

Also, we can extend the properties that have been noticed in Belzunce et al.(2004):

$$IFR(DFR) \Rightarrow DMRL(IMRL) \Rightarrow DURL(IURL) \Leftrightarrow DWURL(IWURL).$$

Below is an example which shows that uniform distribution has not monotone $H^w(X,t)$ and $H^w(X,t)$.

EXAMPLE 4. Suppose that a non-negative random variable X has uniform distribution with support (a,b). Then,

$$H^w(X,t) = \frac{b-t}{2}\log(b-t)\ \ ;\ \ b > t > a.$$

Thus, $H^w(X,t)$ is increasing in t if $b-e^{-1}<t<b$ and it is decreasing in t if $a<t<b-e^{-1}$ i.e., it is neither $IWURL$ nor $DWURL$.
Further,

$$H^w(X,t)=\frac{t-a}{2}\log(t-a)\ \ ;\ \ b>t>a,$$

therefore $H^w(X,t)$ is increasing in t if $a+e^{-1}<t<b$ and it is decreasing in t if $a<t<a+e^{-1}$, so it is neither $IWUPL$ nor $DWUPL$.

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