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SMOOTHING PROPERTIES OF INHOMOGENEOUS EQUATIONS VIA CANONICAL TRANSFORMS

Abstract. The paper describes a new approach to global smoothing problems for inhomogeneous dispersive evolution equations based on an idea of canonical transformation. In our previous papers [20, 21], we introduced such a method to show global smoothing estimates for homogeneous dispersive equations. It is remarkable that this method allows us to carry out a global microlocal reduction of equations to some low dimensional model cases. The purpose of this paper is to pursue the same treatment for inhomogeneous equations. Especially, time-global smoothing estimates for the operator $a(D_x)$ with lower order terms are the benefit of our new method.

1. Introduction

This article consists partly of a survey of the arguments developed in author's recent paper [21] (Sections 2 and 3) and partly of obtaining new results via the extension and continuation of these arguments (Sections 4 and 5).

Let us first consider the following Schrödinger equation:

$$\begin{cases} (i\partial_t+\Delta_x)\,u(t,x)=0 & \text{in } \mathbb{R}_t\times\mathbb{R}^n_x,\\ u(0,x)=\varphi(x) & \text{in } \mathbb{R}^n_x. \end{cases}$$

We know that the solution operator $e^{it\Delta_x}$ preserves the L^2-norm for each fixed $t\in\mathbb{R}$. On the other hand, the extra gain of regularity of order $1/2$ in x can be observed if we integrate the solution in t. For example we have the estimate

$$\left\|\langle x\rangle^{-s}|D_x|^{1/2}e^{it\Delta_x}\varphi\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}\qquad (s>1/2)$$

for $u=e^{it\Delta_x}\varphi$ and this estimate was first given by Ben-Artzi and Klainerman [3] ($n\ge 3$). Since the independent pioneering works by Constantin and Saut [10], Sjölin [24] and Vega [27], the local, then the global smoothing estimates for Schorödinger or more general dispersive equations have been intensively investigated. (Smoothing for generalised Korteweg-de Vries equations was already studied by Kato [13].) There has already been a lot of literature on this subject: Ben-Artzi and Devinatz [1, 2], Chihara [9], Hoshiro [11, 12], Kato and Yajima [14], Kenig, Ponce and Vega [4]–[8], Linares and Ponce [18], Simon [23], Sugimoto [25, 26], Walther [28, 29], and many others.

In our previous papers [20, 21], we introduced a new method to show global smoothing estimates for Schorödinger equations, or more generally, those for homoge-

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neous dispersive equations:

$$
(1) \qquad \begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0,x) = \varphi(x) & \text{in } \mathbb{R}^n_x. \end{cases}
$$

where $a(\xi)$ is a real-valued function of $\xi = (\xi_1, \dots, \xi_n)$ with the growth of order m, and $a(D_x)$ is the corresponding operator. The main idea was to change the equation

$$
(i\partial_t + a(D_x))\, u(t,x) = 0 \quad \text{to} \quad (i\partial_t + \sigma(D_x))\, v(t,x) = 0,
$$

where the operators $a(D_x)$ and $\sigma(D_x)$ are related with each other by the relation

$$
a(\xi) = (\sigma \circ \psi)(\xi).
$$

Such an idea can be realised by a canonical transformation T in the following way:

$$
a(D_x) \circ T = T \circ \sigma(D_x).
$$

If now operators T and T^{-1} are bounded in $L^2(\mathbb{R}^n_x)$ and in weighted $L^2(\mathbb{R}^n_x)$ respectively, we can reduce global smoothing estimates for $u = e^{ita(D_x)}\varphi$ to those for $v = e^{it\sigma(D_x)}\varphi$. It is remarkable that the method of canonical transformations described above allows us to carry out a global microlocal reduction of equation (1) to the model cases $a(\xi) = |\xi_n|^m$ (elliptic case) or $a(\xi) = \xi_1|\xi_n|^{m-1}$ (non-elliptic case) under a dispersiveness condition.

The purpose of this paper is to pursue the same treatment for inhomogeneous equations:

$$
\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = f(t,x) & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0,x) = 0 & \text{in } \mathbb{R}^n_x. \end{cases}
$$

We will obtain the corresponding results on the global smoothing for solutions to inhomogeneous problems. There are considerably less results on this topic available in the literature. Mostly the Schrödinger equation was treated (e.g. Linares and Ponce [18], Kenig, Ponce and Vega [8]), or the one dimensional case (Kenig, Ponce and Vega [5, 7] or Laurey [17]). Some more general results on the local smoothing for dispersive operators were obtained by Chihara [9] and Hoshiro [12], and for dispersive differential operators by Koch and Saut [15]. In this paper we will extend these results in two directions: we will establish the global smoothing for rather general dispersive equations of different orders in all dimensions. Especially, these kinds of time-global estimate for the operator $a(D_x)$ with lower order terms are the benefit of our new method. The treatment of the inhomogeneous equations may allow one to treat nonlinear equations with lower order terms and with corresponding nonlinearities, see the author's paper [22] for one example.

We will explain the organisation of this paper. In Section 2, we introduce our main tools established by the authors in [21], which originate in the idea of canonical transformation. In Section 3, we list results of smoothing estimates for homogeneous equations which were partially announced in [20] and will be completely given in [21].

We also explain how general cases can be reduced to the model estimates by using canonical transformation. Sections 4 and 5 are devoted to non-homogeneous problems as a counterpart of Section 3. Model estimates will be given in Section 4, and estimates for general cases will be given in Section 5 by using the idea of canonical transformation. Such argument and related results were partly announced in [19].

Finally we comment on the notation used in this paper. As usual, we will denote $D_{x_j} = -i\partial_{x_j}$ and view operators $a(D_x)$ as Fourier multipliers. Constants denoted by letter C in estimates are always positive and may differ on different occasions, but will still be denoted by the same letter.

2. Canonical transforms

We will first review our main tool to reduce general operators to normal forms discussed in [21].

Let $\psi : \Gamma \to \widetilde{\Gamma}$ be a C^∞-diffeomorphism between open sets $\Gamma \subset \mathbb{R}^n$ and $\widetilde{\Gamma} \subset \mathbb{R}^n$. We always assume that

$$C^{-1} \le |\det \partial\psi(\xi)| \le C \quad (\xi \in \Gamma), \tag{2}$$

for some $C > 0$. We set formally

$$\begin{aligned}
I_\psi u(x) &= \mathcal{F}^{-1}\left[\mathcal{F}u(\psi(\xi))\right](x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\psi(\xi))}u(y)\,dyd\xi,\\
I_\psi^{-1} u(x) &= \mathcal{F}^{-1}\left[\mathcal{F}u\big(\psi^{-1}(\xi)\big)\right](x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\psi^{-1}(\xi))}u(y)\,dyd\xi.
\end{aligned}$$

The operators I_ψ and I_ψ^{-1} can be justified by using cut-off functions $\gamma \in C^\infty(\Gamma)$ and $\widetilde{\gamma} = \gamma \circ \psi^{-1} \in C^\infty(\widetilde{\Gamma})$ which satisfy $\operatorname{supp}\gamma \subset \Gamma$, $\operatorname{supp}\widetilde{\gamma} \subset \widetilde{\Gamma}$. We set

$$\begin{aligned}
I_{\psi,\gamma} u(x) &= \mathcal{F}^{-1}\left[\gamma(\xi)\mathcal{F}u(\psi(\xi))\right](x)\\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\Gamma} e^{i(x\cdot\xi - y\cdot\psi(\xi))}\gamma(\xi)u(y)dyd\xi,\\
I_{\psi,\gamma}^{-1} u(x) &= \mathcal{F}^{-1}\left[\widetilde{\gamma}(\xi)\mathcal{F}u\big(\psi^{-1}(\xi)\big)\right](x)\\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\widetilde{\Gamma}} e^{i(x\cdot\xi - y\cdot\psi^{-1}(\xi))}\widetilde{\gamma}(\xi)u(y)dyd\xi.
\end{aligned} \tag{3}$$

In the case that Γ, $\widetilde{\Gamma} \subset \mathbb{R}^n \setminus 0$ are open cones, we may consider the homogeneous ψ and γ which satisfy $\operatorname{supp}\gamma \cap \mathbb{S}^{n-1} \subset \Gamma \cap \mathbb{S}^{n-1}$ and $\operatorname{supp}\widetilde{\gamma} \cap \mathbb{S}^{n-1} \subset \widetilde{\Gamma} \cap \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$. Then we have the expressions for compositions

$$I_{\psi,\gamma} = \gamma(D_x)\cdot I_\psi = I_\psi \cdot \widetilde{\gamma}(D_x), \quad I_{\psi,\gamma}^{-1} = \widetilde{\gamma}(D_x)\cdot I_\psi^{-1} = I_\psi^{-1}\cdot\gamma(D_x), \tag{4}$$

and the identities

$$I_{\psi,\gamma}\cdot I_{\psi,\gamma}^{-1} = \gamma(D_x)^2, \quad I_{\psi,\gamma}^{-1}\cdot I_{\psi,\gamma} = \widetilde{\gamma}(D_x)^2. \tag{5}$$

We have also the formulae

$$I_{\psi,\gamma}\cdot\sigma(D_x)=(\sigma\circ\psi)(D_x)\cdot I_{\psi,\gamma},\quad I^{-1}_{\psi,\gamma}\cdot(\sigma\circ\psi)(D_x)=\sigma(D_x)\cdot I^{-1}_{\psi,\gamma}. \tag{6}$$

We also introduce the weighted L^2-spaces. For the weight function $w(x)$, let $L^2_w(\mathbb{R}^n;w)$ be the set of measurable functions $f:\mathbb{R}^n\to\mathbb{C}$ such that the norm

$$\|f\|_{L^2(\mathbb{R}^n;w)}=\left(\int_{\mathbb{R}^n}|w(x)f(x)|^2\,dx\right)^{1/2}$$

is finite. Then, from the relations (4), (5), and (6), we obtain the following fundamental theorem ([21, Theorem 4.1]):

THEOREM 1. *Assume that the operator $I_{\psi,\gamma}$ defined by* (3) *is $L^2(\mathbb{R}^n;w)$–bounded. Suppose that we have the estimate*

$$\left\|w(x)\rho(D_x)e^{it\sigma(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \tag{7}$$

for all φ such that $\operatorname{supp}\widehat{\varphi}\subset\operatorname{supp}\widetilde{\gamma}$, *where $\widetilde{\gamma}=\gamma\circ\psi^{-1}$. Assume also that the function*

$$q(\xi)=\frac{\gamma\cdot\zeta}{\rho\circ\psi}(\xi) \tag{8}$$

is bounded. Then we have

$$\left\|w(x)\zeta(D_x)e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \tag{9}$$

for all φ such that $\operatorname{supp}\widehat{\varphi}\subset\operatorname{supp}\gamma$, *where $a(\xi)=(\sigma\circ\psi)(\xi)$.*

Note that $e^{ita(D_x)}\varphi(x)$ and $e^{it\sigma(D_x)}\varphi(x)$ are solutions to

$$\begin{cases}(i\partial_t+a(D_x))\,u(t,x)=0,\\ \qquad u(0,x)=\varphi(x),\end{cases}\quad\text{and}\quad\begin{cases}(i\partial_t+\sigma(D_x))\,v(t,x)=0,\\ \qquad v(0,x)=g(x),\end{cases}$$

respectively. Theorem 1 means that smoothing estimates for equation with $\sigma(D_x)$ implies those with $a(D_x)$ if the canonical transformations which relate them are bounded on weighted L^2-spaces. The same thing is true for inhomogeneous equations

$$\begin{cases}(i\partial_t+a(D_x))\,u(t,x)=f(t,x),\\ \qquad u(0,x)=0,\end{cases}\quad\text{and}\quad\begin{cases}(i\partial_t+\sigma(D_x))\,v(t,x)=f(t,x),\\ \qquad v(0,x)=0,\end{cases}$$

whose solutions are $-i\int_0^t e^{i(t-\tau)a(D_x)}f(\tau,x)\,d\tau$ and $-i\int_0^t e^{i(t-\sigma)a(D_x)}f(\tau,x)\,d\tau$, respectively. The only difference is that we need the weighted L^2–boundedness of the operator $I^{-1}_{\psi,q}$ instead of just the L^2–boundedness of it induced by the boundedness of $q(\xi)$:

THEOREM 2. *Assume that the operator $I_{\psi,\gamma}$ defined by* (3) *is $L^2(\mathbb{R}^n;w)$–bounded. Suppose that we have the estimate*

$$\left\| w(x)\rho(D_x)\int_0^t e^{i(t-\tau)\sigma(D_x)} f(\tau,x)\,d\tau \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|v(x)f(t,x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}$$

for all f such that $\operatorname{supp}\mathcal{F}_x f(t,\cdot)\subset\operatorname{supp}\widetilde{\gamma}$*, where $\widetilde{\gamma}=\gamma\circ\psi^{-1}$. Also assume that the operator $I^{-1}_{\psi,q}$ defined by* (3) *with $q(\xi)=(\gamma\cdot\zeta)/(\rho\circ\psi)(\xi)$ is $L^2(\mathbb{R}^n;v)$–bounded. Then we have*

$$\left\| w(x)\zeta(D_x)\int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\,d\tau \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|v(x)f(t,x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}$$

for all f such that $\operatorname{supp}\mathcal{F}_x f(t,\cdot)\subset\operatorname{supp}\gamma$*, where $a(\xi)=(\sigma\circ\psi)(\xi)$.*

The proof of Theorem 1 is given in [21], and that of Theorem 2 is just a slight modification of it, hence here we omit them.

As for the $L^2(\mathbb{R}^n;w)$–boundedness of the operator $I_{\psi,\gamma}$, we have criteria for some special weight functions. For $\kappa\in\mathbb{R}$, let $L^2_\kappa(\mathbb{R}^n)$, $\dot{L}^2_\kappa(\mathbb{R}^n)$ be the set of measurable functions f such that the norm

$$\|f\|_{L^2_\kappa(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\langle x\rangle^\kappa f(x)|^2\,dx\right)^{1/2}, \qquad \|f\|_{\dot{L}^2_\kappa(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} ||x|^\kappa f(x)|^2\,dx\right)^{1/2}$$

is finite, respectively. Then we have the following mapping properties ([21, Theorems 4.2 and 4.3]).

THEOREM 3. *Suppose $\kappa\in\mathbb{R}$. Assume that all the derivatives of entries of the $n\times n$ matrix $\partial\psi$ and those of γ are bounded. Then the operators $I_{\psi,\gamma}$ and $I^{-1}_{\psi,\gamma}$ defined by* (3) *are $L^2_\kappa(\mathbb{R}^n)$–bounded.*

THEOREM 4. *Let Γ, $\widetilde{\Gamma}\subset\mathbb{R}^n\setminus 0$ be open cones. Suppose $|\kappa|<n/2$. Assume $\psi(\lambda\xi)=\lambda\psi(\xi)$, $\gamma(\lambda\xi)=\gamma(\xi)$ for all $\lambda>0$ and $\xi\in\Gamma$. Then the operators $I_{\psi,\gamma}$ and $I^{-1}_{\psi,\gamma}$ defined by* (3) *are $L^2_\kappa(\mathbb{R}^n)$–bounded and $\dot{L}^2_\kappa(\mathbb{R}^n)$–bounded.*

We remark that the following result due to Kurtz and Wheeden [16, Theorem 3] is essentially used to prove Theorem 4:

LEMMA 1. *Suppose $|\kappa|<n/2$. Assume that $m(\xi)\in C^n(\mathbb{R}^n\setminus 0)$ and all the derivative of $m(\xi)$ satisfies $|\partial^\gamma m(\xi)|\le C_\gamma|\xi|^{-|\gamma|}$ for all $\xi\ne 0$ and $|\gamma|\le n$. Then $m(D_x)$ is $L^2_\kappa(\mathbb{R}^n)$ and $\dot{L}^2_\kappa(\mathbb{R}^n)$–bounded.*

3. Smoothing estimates for homogeneous dispersive equations

In author's paper [21], it is explained how to derive smoothing estimates for general homogeneous dispersive equations from model estimates as an application of the canonical transformations described in Section 2. We will repeat it here to help readers to understand the later part of this paper concerning estimates for inhomogeneous equations.

Let us consider the solution

$$u(t,x) = e^{ita(D_x)}\varphi(x)$$

to the homogeneous equation

$$\begin{cases} (i\partial_t + a(D_x))\,u(t,x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0,x) = \varphi(x) & \text{in } \mathbb{R}^n_x, \end{cases}$$

where we always assume that function $a(\xi)$ is real-valued. Let $a_m(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$, the *principal* part of $a(\xi)$, be a positively homogeneous function of order m, that is, satisfy $a_m(\lambda\xi) = \lambda^m a_m(\xi)$ for all $\lambda > 0$ and $\xi \neq 0$.

First we consider the case that $a(\xi)$ has no lower order terms, and assume that $a(\xi)$ is *dispersive*:

$$\textbf{(H)} \qquad a(\xi) = a_m(\xi), \qquad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0),$$

where $\nabla = (\partial_1, \dots, \partial_n)$ and $\partial_j = \partial_{\xi_j}$. A typical example is $a(\xi) = a_m(\xi) = |\xi|^m$. Especially, $a(\xi) = a_2(\xi) = |\xi|^2$ is the case of the Schrödinger equation.

The following result ([21, Theorem 5.1]) is a generalisation of the one given by Ben-Artzi and Klainerman [3] which treated the case $a(\xi) = |\xi|^2$ and $n \geq 3$:

THEOREM 5. *Assume* (H). *Suppose* $n \geq 1$, $m > 0$, *and* $s > 1/2$. *Then we have*

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{10}$$

We review how to prove Theorem 5. The main idea is reducing them to the special cases $a(D_n) = |D_n|^m$, $D_1 |D_n|^{m-1}$, where $D_x = (D_1, \dots, D_n)$, by using Theorem 1. The following estimates ([21, Theorem 3.1, Corollary 3.3]) for them act as model ones:

PROPOSITION 1. *Suppose* $n = 1$ *and* $m > 0$. *Then we have*

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x)}$$

for all $x \in \mathbb{R}$. Suppose $n = 2$ and $m > 0$. Then we have

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x,y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \le C\|\varphi\|_{L^2\left(\mathbb{R}^2_{x,y}\right)}$$

for all $x \in \mathbb{R}$.

COROLLARY 1. *Suppose $n \ge 1$, $m > 0$, and $s > 1/2$. Then we have*

$$\left\| \langle x_n \rangle^{-s} |D_n|^{(m-1)/2} e^{it|D_n|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{11}$$

Suppose $n \ge 2$, $m > 0$, and $s > 1/2$. Then we have

$$\left\| \langle x_1 \rangle^{-s} |D_n|^{(m-1)/2} e^{itD_1|D_n|^{m-1}} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{12}$$

We assume (H). Let $\Gamma \subset \mathbb{R}^n \setminus 0$ be a sufficiently small conic neighbourhood of $e_n = (0,\dots 0,1)$, and take a cut-off function $\gamma(\xi) \in C^\infty(\Gamma)$ which is positively homogeneous of order 0 and satisfies $\operatorname{supp}\gamma \cap \mathbb{S}^{n-1} \subset \Gamma \cap \mathbb{S}^{n-1}$. By the microlocalisation and the rotation of the initial data φ, we may assume $\operatorname{supp}\widehat{\varphi} \subset \operatorname{supp}\gamma$. The dispersive assumption $\nabla a_m(e_n) \ne 0$ in this direction implies the following two possibilities:

(i) $\partial_n a_m(e_n) \ne 0$. Then, by Euler's identity $a_m(\xi) = (1/m)\nabla a_m(\xi)\cdot\xi$, we have $a_m(e_n) \ne 0$. Hence, in this case, we may assume that $a(\xi) (> 0)$ and $\partial_n a(\xi)$ are bounded away from 0 for $\xi \in \Gamma$.

(ii) $\partial_n a_m(e_n) = 0$. Then there exits $j \ne n$ such that $\partial_j a_m(e_n) \ne 0$, say $\partial_1 a_m(e_n) \ne 0$. Hence, in this case, we may assume $\partial_1 a(\xi)$ is bounded away from 0 for $\xi \in \Gamma$. We remark $a(e_n) = 0$ by Euler's identity.

The estimate with the case $n = 1$ is given by estimate (11) in Corollary 1. In fact, we have $a(\xi) = a(1)|\xi|^m$ for $\xi > 0$ in this case. Hence we may assume $n \ge 2$. We remark that it is sufficient to show theorem with $1/2 < s < n/2$ because the case $s \ge n/2$ is easily reduced to this case. We will use the notation $\xi = (\xi_1,\dots,\xi_n)$, $\eta = (\eta_1,\dots,\eta_n)$.

In the case (i), we take

$$\sigma(\eta) = |\eta_n|^m, \quad \psi(\xi) = (\xi_1,\dots,\xi_{n-1}, a(\xi)^{1/m}). \tag{13}$$

Then we have $a(\xi) = (\sigma\circ\psi)(\xi)$ and

$$\det \partial\psi(\xi) = \begin{vmatrix} E_{n-1} & 0 \\ * & (1/m)a(\xi)^{1/m-1}\partial_n a(\xi) \end{vmatrix}, \tag{14}$$

where E_{n-1} is the identity matrix of order $n-1$. We remark that (2) is satisfied since $\det\partial\psi(e_n) = (1/m)a(e_n)^{1/m-1}\partial_n a(e_n) \ne 0$. By estimate (11) in Corollary 1, we have estimate (7) in Theorem 1 with $\sigma(D_x) = |D_n|^m$, $w(x) = \langle x\rangle^{-s}$, and $\rho(\xi) = |\xi_n|^{(m-1)/2}$. Note here the trivial inequality $\langle x\rangle^{-s} \le \langle x_n\rangle^{-s}$. If we take $\zeta(\xi) = |\xi|^{(m-1)/2}$, then $q(\xi) = \gamma(\xi)\big(|\xi|/a(\xi)^{1/m}\big)^{(m-1)/2}$ defined by (8) is a bounded function. On the other hand, $I_{\psi,\gamma}$ is L^2_{-s}–bounded for $1/2 < s < n/2$ by Theorem 4. Hence, by Theorem 1, we have estimate (9), that is, estimate (10).

In the case (ii), we take

$$\sigma(\eta) = \eta_1|\eta_n|^{m-1}, \quad \psi(\xi) = \big(a(\xi)|\xi_n|^{1-m}, \xi_2, \dots, \xi_n\big)$$

Then we have $a(\xi) = (\sigma\circ\psi)(\xi)$ and

$$\det\partial\psi(\xi) = \begin{vmatrix} \partial_1 a(\xi)|\xi_n|^{1-m} & * \\ 0 & E_{n-1} \end{vmatrix}.$$

Since $\det\partial\psi(e_n) = \partial_1 a(e_n) \neq 0$, (2) is satisfied. Similarly to the case (i), the estimate for $\sigma(D_x) = D_1|D_n|^{m-1}$ is given by estimate (12) in Corollary 1, which implies estimate (10) again by Theorem 1.

As another advantage of the method explained here, we can also consider the case that $a(\xi)$ has lower order terms, and assume that $a(\xi)$ is dispersive in the following sense:

$$(\mathbf{L}) \quad \begin{aligned} &a(\xi) \in C^\infty(\mathbb{R}^n), \qquad \nabla a(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0), \\ &|\partial^\alpha(a(\xi) - a_m(\xi))| \le C_\alpha|\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \ge 1. \end{aligned}$$

or equivalently

$$(\mathbf{L}) \quad \begin{aligned} &a(\xi) \in C^\infty(\mathbb{R}^n), \qquad |\nabla a(\xi)| \ge C\langle\xi\rangle^{m-1} \quad (\xi \in \mathbb{R}^n) \quad \text{for some } C > 0, \\ &|\partial^\alpha(a(\xi) - a_m(\xi))| \le C_\alpha|\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \ge 1. \end{aligned}$$

The last lines of these assumptions simply amount to saying that the principal part a_m of a is positively homogeneous of order m for $|\xi| \ge 1$.

The following result ([21, Theorem 5.4]) is also derived from Corollary 1:

THEOREM 6. *Assume* (L). *Suppose $n \ge 1$, $m > 0$, and $s > 1/2$. Then we have*

$$(15) \qquad \Big\|\langle x\rangle^{-s}\langle D_x\rangle^{(m-1)/2}e^{ita(D_x)}\varphi(x)\Big\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}.$$

We review how to prove Theorem 6. We sometimes decompose the initial data φ into the sum of the *low frequency* part φ_l and the *high frequency* part φ_h, where $\operatorname{supp}\widehat{\varphi_l} \subset \{\xi : |\xi| < 2R\}$ and $\operatorname{supp}\widehat{\varphi_h} \subset \{\xi : |\xi| > R\}$ with sufficiently large $R > 0$. Each part can be realised by multiplying $\chi(D_x)$ or $(1-\chi)(D_x)$ to $\varphi(x)$, hence to $u(t,x)$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ is an appropriate cut-off function. For high frequency part, the same argument as in the proof of Theorem 5 is valid. (Furthermore, we can use Theorem 3 instead of Theorem 4 to assure the boundedness of $I_{\psi,\gamma}$, hence we need not assume $n \geq 2$.) We show how to get the estimates for low frequency part. Because of the compactness of it, we may assume $\partial_j a(\xi) \neq 0$ with some j, say $j = n$, on a bounded set $\Gamma \subset \mathbb{R}^n$ and $\operatorname{supp}\widehat{\varphi} \subset \Gamma$. Since we have $a(\xi)+c > 0$ on Γ with some constant $c > 0$ and

$$\left\| \langle x\rangle^{-s}\langle D_x\rangle^{(m-1)/2} e^{ita(D_x)}\varphi \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} = \left\| \langle x\rangle^{-s}\langle D_x\rangle^{(m-1)/2} e^{it(a(D_x)+2c)}\varphi \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},$$

we may assume $a(\xi) \geq c > 0$ on Γ without loss of generality. We take a cut-off function $\gamma(\xi) \in C_0^\infty(\Gamma)$, and choose $\psi(\xi)$ and $\sigma(\eta)$ in the same way as (13). Assumption (2) is also verified if we notice (14). By estimate (11) in Corollary 1, we have estimate (7) in Theorem 1 with $\sigma(D_x) = |D_n|^m$, $w(x) = \langle x\rangle^{-s}$ $(s > 1/2)$, and $\rho(\xi) = |\xi_n|^{(m-1)/2}$ as in the proof of Theorem 5. If we take $\zeta(\xi) = \langle\xi\rangle^{(m-1)/2}$, then $q(\xi) = \gamma(\xi)\big(\langle\xi\rangle/a(\xi)^{1/m}\big)^{(m-1)/2}$ defined by (8) is a bounded function. On the other hand, $I_{\psi,\gamma}$ is L^2_{-s}–bounded for all $s > 1/2$ by Theorem 3. Hence, by Theorem 1, we have estimate (9), that is, estimate (15).

Finally, we introduce an intermediate assumption between (H) and (L), and discuss what happens if we do not have the condition $\nabla a(\xi) \neq 0$:

$$\textbf{(HL)}\qquad \begin{aligned} &a(\xi) = a_m(\xi) + r(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0), \quad r(\xi) \in C^\infty(\mathbb{R}^n) \\ &|\partial^\alpha r(\xi)| \leq C\langle\xi\rangle^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha. \end{aligned}$$

In view of the proof of Theorem 6, we see that Theorem 5 remains valid if we replace assumption (H) by (HL) and functions $\varphi(x)$ in the estimates by its (sufficiently large) high frequency part $\varphi_h(x)$. However we cannot control the low frequency part $\varphi_l(x)$, and so have only the time local estimates on the whole. We just put such a result ([21, Theorem 5.6]) below without its proof:

THEOREM 7. *Assume* (HL). *Suppose $n \geq 1$, $m > 0$, $s > 1/2$, and $T > 0$. Then we have*

$$\int_0^T \left\| \langle x\rangle^{-s}\langle D_x\rangle^{(m-1)/2} e^{ia(D_x)} \right\|^2_{L^2(\mathbb{R}^n_x)} dt \leq C\|\varphi\|^2_{L^2(\mathbb{R}^n)},$$

where $C > 0$ is a constant depending on $T > 0$.

We remark that Theorem 6 is the time global version (that is, the estimate with $T = \infty$) of Theorem 7, and the extra assumption $\nabla a(\xi) \neq 0$ is needed for that. Since the assumption $\nabla a(\xi) \neq 0$ for large ξ is automatically satisfied by assumption (HL), Theorem 6 means that the condition $\nabla a(\xi) \neq 0$ for small ξ assures the time global estimate. In this sense, the low frequency part have a responsibility for the time global smoothing.

4. Model estimates for inhomogeneous equations

We now turn to deal with inhomogeneous equations, for which we also have similar smoothing estimates. Such estimates are necessary for nonlinear applications, and they can be obtained by further developments of the presented methods. Let us consider the solution

$$u(t,x) = -i\int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\,d\tau$$

to the equation

$$\begin{cases} (i\partial_t + a(D_x))\,u(t,x) = f(t,x) & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0,x) = 0 & \text{in } \mathbb{R}^n_x. \end{cases}$$

We will give model estimates for it below, where we write $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $D_x = (D_1, D_2 \dots, D_n)$. We also write $x = x_1$, $D_x = D_1$ in the case $n = 1$, and $(x,y) = (x_1,x_2)$, $(D_x,D_y) = (D_1,D_2)$ in the case $n = 2$.

PROPOSITION 2. *Suppose $n = 1$ and $m > 0$. Let $a(\xi) \in C^\infty(\mathbb{R}\setminus 0)$ be a real-valued function which satisfies $a(\lambda\xi) = \lambda^m a(\xi)$ for all $\lambda > 0$ and $\xi \neq 0$. Then we have*

$$\left\| a'(D_x)\int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\,d\tau \right\|_{L^2(\mathbb{R}_t)} \le C\int_{\mathbb{R}} \|f(t,x)\|_{L^2(\mathbb{R}_t)}\,dx \tag{16}$$

for all $x \in \mathbb{R}$. Suppose $n = 2$ and $m > 0$. Then we have

$$\begin{aligned} \left\| |D_x|^{m-1}\int_0^t e^{i(t-\tau)|D_x|^{m-1}D_y} f(\tau,x,y)\,d\tau \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x)} \\ \le C\int_{\mathbb{R}} \|f(t,x,y)\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x)}\,dy \end{aligned} \tag{17}$$

for all $y \in \mathbb{R}$.

COROLLARY 2. *Suppose $n \ge 1$, $m > 0$, and $s > 1/2$. Let $a(\xi) \in C^\infty(\mathbb{R}\setminus 0)$ be a real-valued function which satisfies $a(\lambda\xi) = \lambda^m a(\xi)$ for all $\lambda > 0$ and $\xi \neq 0$. Then we have*

$$\left\| \langle x_n\rangle^{-s} a'(D_n)\int_0^t e^{i(t-\tau)a(D_n)} f(\tau,x)\,d\tau \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\langle x_n\rangle^s f(t,x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}.$$

Suppose $n \ge 2$, $m > 0$, and $s > 1/2$. Then we have

$$\left\| \langle x_1\rangle^{-s} |D_n|^{m-1}\int_0^t e^{i(t-\tau)D_1|D_n|^{m-1}} f(\tau,x)\,d\tau \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\langle x_1\rangle^s f(t,x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}.$$

Proposition 2 with the case $n = 1$ is a unification of the results by Kenig, Ponce and Vega who treated the cases $a(\xi) = \xi^2$ ([5, p.258]), $a(\xi) = |\xi|\xi$ ([7, p.160]), and

$a(\xi)=\xi^3$ ([6, p.533]). Corollary 2 is a straightforward result of Proposition 2 and Cauchy–Schwarz's inequality. They act as model estimates for inhomogeneous equations just like Proposition 1 and Corollary 1 do for homogeneous ones. In [21], Corollary 1 is given straightforwardly from the translation invariance of Lebesgue measure, by using a newly introduced method (comparison principle).

Since we unfortunately do not know the comparison principle for inhomogeneous equations, we will give a direct proof to Proposition 2. Note that we have another expression of the solution to inhomogeneous equation

$$\begin{cases} (i\partial_t + a(D_x))\,u(t,x) = f(t,x) & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ \qquad\qquad u(0,x) = 0 & \text{in } \mathbb{R}^n_x, \end{cases}$$

using the weak limit $R(\tau \pm i0)$ of the resolvent $R(\tau \pm i\varepsilon)$ as $\varepsilon \searrow 0$, where $R(\lambda) = (a(D_x)-\lambda)^{-1}$:

$$\begin{aligned} u(t,x) &= -i\int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\,d\tau. \\ &= \mathcal{F}_\tau^{-1} R(\tau - i0)\mathcal{F}_t f^+ + \mathcal{F}_\tau^{-1} R(\tau + i0)\mathcal{F}_t f^- \end{aligned} \tag{18}$$

(see Sugimoto [25] and Chihara [9]). Here $\mathcal{F}_t$ denotes the Fourier Transformation in t and $\mathcal{F}_\tau^{-1}$ its inverse, and $f^\pm(t,x) = f(t,x)Y(\pm t)$ is the characteristic function $Y(t)$ of the set $\{t \in \mathbb{R} : t > 0\}$.

Proof of Estimate (16). Let us use a variant of the argument of Chihara [9, Section 4]. We set $R(\lambda) = (a(D_x)-\lambda)^{-1}$ and show the estimate

$$\left|a'(D_x)R(s \pm i0)g(x)\right| \le C\int_{\mathbb{R}} |g(x)|\,dx,$$

where $C>0$ is a constant independent of $s \in \mathbb{R}, x \in \mathbb{R}$ and $g \in L^1(\mathbb{R})$. Then, on account of the expression (18), Plancherel's theorem, and Minkowski's inequality, we have the desired result. For this purpose, we consider the kernel

$$k(s,x) = \mathcal{F}^{-1}\left[a'(\xi)(a(\xi)-(s \pm i0))^{-1}\right](x)$$

and show its uniform boundedness. By the scaling argument, everything is reduced to show the estimates

$$\sup_{x\in\mathbb{R}} |k(\pm 1,x)| \le C \qquad \text{and} \qquad \sup_{x\in\mathbb{R}} |k(0,x)| \le C.$$

By using an appropriate partition of unity $\widehat{\phi}_1(\xi)+\widehat{\phi}_2(\xi)+\widehat{\phi}_3(\xi)=1$, we split $k(\pm 1,x)$ into the corresponding three parts $k = k_1+k_2+k_3$, where $\widehat{\phi}_1$ has its support near the origin, $\widehat{\phi}_2$ near the point $\xi^m = \pm 1$, and $\widehat{\phi}_3$ away from these points. The estimate for k_1 is trivial. The other estimates are reduced to the boundedness of

$$k_0^\pm(x) = \mathcal{F}^{-1}\left[(\xi \pm i0))^{-1}\right](x) = \mp i\sqrt{2\pi}Y(\pm x). \tag{19}$$

In fact,

$$k_2(\pm 1,x) = \mathcal{F}^{-1}\Big[(\xi-(\alpha\pm i0))^{-1}\widehat{\psi}(\xi)\Big](x) = (e^{i\alpha x}k_0^{\mp}) * \psi(x)$$

where $\alpha \in \mathbb{R}$ is a point which solves $a(\alpha) = \pm 1$, and

$$\widehat{\psi}(\xi) = a'(\xi)\frac{\xi-\alpha}{a(\xi)-(\pm 1)}\widehat{\phi}_2(\xi) \in C_0^\infty(\mathbb{R}).$$

Furthermore, if we notice

$$\frac{a'(\xi)}{a(\xi)-s} = m\left(\frac{s}{(a(\xi)-s)\xi}+\frac{1}{\xi}\right),$$

we have

$$\frac{1}{m}k_3(\pm 1,x) = \pm\mathcal{F}^{-1}\left[\frac{\widehat{\phi}_3(\xi)}{(a(\xi)\mp 1)\xi}\right](x) + k_0^{\pm}(x) - k_0^{\pm} * (\phi_1(x)+\phi_2(x)).$$

It is easy to deduce the estimates for k_2 and k_3. It is also easy to verify

$$\frac{a'(\xi)}{a(\xi)\pm i0} = \frac{m}{\xi\pm i0} + c\delta$$

with a constant c and Dirac's delta function δ, and have the estimate for $k(0,x)$. □

Proof of Estimate (17). We set $R(\lambda) = \big(|D_x|^{m-1}D_y - \lambda\big)^{-1}$ and show the estimate

$$\big\||D_x|^{m-1}R(s\pm i0)g(x,y)\big\|_{L^2(\mathbb{R}_x)} \le C\int \|g(x,y)\|_{L^2(\mathbb{R}_x)}\,dy,$$

where $C>0$ is a constant independent of $s\in\mathbb{R}$, $y\in\mathbb{R}$ and $g\in L^1(\mathbb{R}^2)$. Then, by the expression (18), Plancherel's theorem, and Minkowski's inequality again, we have the desired result.

First we note, we may assume $\widehat{g}(\xi,\eta)=0$ for $\xi<0$. Then we have

$$\begin{aligned}
&|D_x|^{m-1}R(s\pm i0)g(x,y)\\
=&(2\pi)^{-2}\int_0^\infty\int_{-\infty}^\infty e^{i(x\xi+y\eta)}|\xi|^{m-1}\big(|\xi|^{m-1}\eta-(s\pm i0)\big)^{-1}\widehat{g}(\xi,\eta)\,d\xi d\eta\\
=&(2\pi)^{-2}\int_0^\infty\int_{-\infty}^\infty e^{ix\xi}|\xi|^{m-1}\big(|\xi|^{m-1}\eta-(s\pm i0)\big)^{-1}\widehat{g_y}(\xi,\eta)\,d\xi d\eta\\
=&(2\pi)^{-2}\int_{-\infty}^\infty\int_0^\infty e^{ixb}(a-(s\pm i0))^{-1}\widehat{g_y}(b,ab^{-(m-1)})\,dadb\\
=&(2\pi)^{-2}\int_{-\infty}^\infty\int_0^\infty e^{ixb}\mathcal{F}_a\Big[(a-(s\pm i0))^{-1}\Big]\mathcal{F}_a^{-1}\Big[\widehat{g_y}(b,ab^{-(m-1)})\Big]\,dadb\\
=&(2\pi)^{-1}\int_{-\infty}^\infty\int_0^\infty e^{ixb}e^{-isa}k_0^{\mp}(-a)b^{m-1}\widetilde{g_y}(b,ab^{m-1})\,dadb,
\end{aligned}$$

hence we have

$$\mathcal{F}_x\big[|D_x|^{m-1}R(s\pm i0)g(x,y)\big](b) = \int_{-\infty}^{\infty} e^{-isa}k_0^{\mp}(-a)b^{m-1}\widetilde{g_y}(b,ab^{m-1})\,da$$

for $b \geq 0$, and it vanishes for $b < 0$. Here $g_y(x,\cdot) = g(x,\cdot+y)$, and $\widetilde{g_y}$ denotes its partial Fourier transform with respect to the first variable. We have also used here the change of variables $a = \xi^{m-1}\eta, b = \xi$ and Parseval's formula. Note that $\partial(a,b)/\partial(\xi,\eta) = b^{m-1}$ and $k_0^{\mp}$ is a bounded function defined by (19). Then we have the estimate

$$\begin{aligned}\big|\mathcal{F}_x\big[|D_x|^{m-1}R(s\pm i0)g(x,y)\big](b)\big| &\leq \sqrt{2\pi}\int_{-\infty}^{\infty}\big|b^{m-1}\widetilde{g_y}(b,ab^{m-1})\big|\,da \\ &= \sqrt{2\pi}\int_{-\infty}^{\infty}|\widetilde{g_y}(b,a)|\,da,\end{aligned}$$

and, by Plancherel's theorem and Minkowski's inequality, we have

$$\begin{aligned}\big\||D_x|^{m-1}R(s\pm i0)g(x,y)\big\|_{L^2(\mathbb{R}_x)} &\leq \sqrt{2\pi}\int_{-\infty}^{\infty}\|g_y(x,a)\|_{L^2(\mathbb{R}_x)}\,da \\ &= \sqrt{2\pi}\int_{-\infty}^{\infty}\|g(x,y)\|_{L^2(\mathbb{R}_x)}\,dy,\end{aligned}$$

which is the desired estimate. $\square$

5. Smoothing estimates for dispersive inhomogeneous equations

Let us consider the inhomogeneous equation

$$\begin{cases}(i\partial_t + a(D_x))\,u(t,x) = f(t,x) & \text{in } \mathbb{R}_t\times\mathbb{R}^n_x, \\ u(0,x) = 0 & \text{in } \mathbb{R}^n_x,\end{cases}$$

where we always assume that function $a(\xi)$ is real-valued. Let the principal part $a_m(\xi) \in C^\infty(\mathbb{R}^n\setminus 0)$, be a positively homogeneous function of order m. Recall the dispersive conditions we used in Section 3:

$$\textbf{(H)}\qquad a(\xi) = a_m(\xi), \qquad \nabla a_m(\xi) \neq 0 \quad (\xi\in\mathbb{R}^n\setminus 0),$$

$$\textbf{(L)}\qquad \begin{aligned}&a(\xi)\in C^\infty(\mathbb{R}^n), \qquad \nabla a(\xi)\neq 0 \quad (\xi\in\mathbb{R}^n), \quad \nabla a_m(\xi)\neq 0 \quad (\xi\in\mathbb{R}^n\setminus 0), \\ &|\partial^\alpha(a(\xi)-a_m(\xi))| \leq C_\alpha|\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi|\geq 1.\end{aligned}$$

The following is a counterpart of Theorem 5 which treated homogeneous equations:

THEOREM 8. *Assume* (H). *Suppose* $m > 0$ *and* $s > 1/2$. *Then we have*

$$\left\| \langle x \rangle^{-s} |D_x|^{m-1} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\langle x \rangle^s f(t,x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \tag{20}$$

in the case $n \ge 2$, *and*

$$\left\| \langle x \rangle^{-s} a'(D_x) \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \le C \|\langle x \rangle^s f(t,x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \tag{21}$$

in the case $n = 1$.

Chihara [9] proved Theorem 8 with $m > 1$ under the assumption (H). As was pointed out in [9, p.1958], we cannot replace $a'(D_x)$ by $|D_x|^{m-1}$ in estimate (21) for the case $n = 1$, but there is another explanation for this obstacle. If we decompose $f(t,x) = \chi_+(D_x) f(t,x) + \chi_-(D_x) f(t,x)$, where $\chi_\pm(\xi)$ is a characteristic function of the set $\{\xi \in \mathbb{R} : \pm\xi \ge 0\}$, then we easily obtain

$$\begin{aligned} &\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \\ \le& C \left(\left\| \langle x \rangle^{s} |D_x|^{-(m-1)/2} f_+(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} + \left\| \langle x \rangle^{s} |D_x|^{-(m-1)/2} f_-(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \right) \end{aligned}$$

from Theorem 8. But we cannot justify the estimate

$$\left\| \langle x \rangle^{s} |D_x|^{-(m-1)/2} f_\pm(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \le C \left\| \langle x \rangle^{s} |D_x|^{-(m-1)/2} f(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)}$$

for $s > 1/2$ by Lemma 1 because it requires $s < n/2$ and it is impossible for $n = 1$.

As a counterpart of Theorem 6, we have

THEOREM 9. *Assume* (L). *Suppose* $n \ge 1$, $m > 0$, *and* $s > 1/2$. *Then we have*

$$\left\| \langle x \rangle^{-s} \langle D_x \rangle^{m-1} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\langle x \rangle^s f(t,x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)}. \tag{22}$$

The following result is a straightforward consequence of Theorem 9 and the L^2_s–boundedness of $|D_x|^{(m-1)/2} \langle D_x \rangle^{-(m-1)/2}$ with $(1/2 <) s < n/2$ and $m \ge 1$ (which is assured by Lemma 1):

COROLLARY 3. *Assume* (L). *Suppose* $n \ge 2$, $m \ge 1$, *and* $s > 1/2$. *Then we have*

$$\left\| \langle x \rangle^{-s} |D_x|^{m-1} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\langle x \rangle^s f(t,x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)}.$$

We remark that the same argument of canonical transformations as used for homogeneous equations in Section 3 works for inhomogeneous ones, as well. That is, the proofs of Theorems 8 and 9 are carried out by reducing them to model estimates in Corollary 2. We omit the details because the argument is essentially the same, but we just remark that we use Theorem 2 instead of Theorem 1.

The following is a counterpart of Theorem 7:

THEOREM 10. *Assume* (HL). *Suppose* $n \geq 1$, $m > 0$, $s > 1/2$, *and* $T > 0$. *Then we have*

$$\int_0^T \left\| \langle x \rangle^{-s} \langle D_x \rangle^{m-1} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\, d\tau \right\|^2_{L^2(\mathbb{R}^n_x)} dt \leq C \int_0^T \|\langle x \rangle^s f(t,x)\|^2_{L^2(\mathbb{R}^n_x)}\, dt,$$

where $C > 0$ *is a constant depending on* $T > 0$.

Proof. By multiplying $\chi(D_x)$ and $(1-\chi)(D_x)$ to $f(t,x)$, we decompose it into the sum of low frequency part and high frequency part, where $\chi(\xi)$ is an appropriate cut-off function. As in the proof of Theorem 6, the estimate for the high frequency part can be reduced to Corollary 2 by using Theorem 2 instead of Theorem 1, together with the boundedness result Theorem 3. Here we note that, for $t \in [0,T]$,

$$\int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\, d\tau = \int_0^t e^{i(t-\tau)a(D_x)} \chi_{[0,T]}(\tau) f(\tau,x)\, d\tau,$$

where $\chi_{[0,T]}$ denotes the characteristic function of the interval $[0,T]$. The estimate for the low frequency part is trivial. In fact, if $\operatorname{supp}_\xi \mathcal{F}_x f(t,\xi) \subset [\xi; |\xi| \leq R]$, we have

$$\begin{aligned} &\int_0^T \left\| \langle x \rangle^{-s} \langle D_x \rangle^{m-1} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\, d\tau \right\|^2_{L^2(\mathbb{R}^n_x)} dt \\ \leq &\int_0^T \left(\int_0^T \left\| \langle D_x \rangle^{m-1} e^{i(t-\tau)a(D_x)} f(\tau,x) \right\|_{L^2(\mathbb{R}^n_x)} d\tau \right)^2 dt \\ \leq &CT^2 \langle R \rangle^{2(m-1)} \int_0^T \|\langle x \rangle^s f(t,x)\|^2_{L^2(\mathbb{R}^n_x)}\, dt. \end{aligned}$$

by Plancherel's theorem. □

If we combine Theorem 8 with Theorem 5, we have a result for the equation

$$\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = f(t,x) & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0,x) = \varphi(x) & \text{in } \mathbb{R}^n_x. \end{cases} \tag{23}$$

COROLLARY 4. *Assume* (H). *Suppose $m > 0$ and $s > 1/2$. Then the solution u to equation* (23) *satisfies*

$$\left\| \langle x\rangle^{-s} |D_x|^{-(m-1)/2} a'(D_x) u(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \left(\|\varphi\|_{L^2(\mathbb{R})} + \left\| \langle x\rangle^{s} |D_x|^{-(m-1)/2} f(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \right)$$

in the case $n = 1$, and

$$\left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} u(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \left(\|\varphi\|_{L^2(\mathbb{R}^n)} + \left\| \langle x\rangle^{s} |D_x|^{-(m-1)/2} f(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \right)$$

in the case $n \geq 2$.

If we combine Theorem 9 with Theorem 6, we have the following:

COROLLARY 5. *Assume* (L). *Suppose $n \geq 1$, $m > 0$, and $s > 1/2$. Then the solution u to equation* (23) *satisfies*

$$\left\| \langle x\rangle^{-s} \langle D_x\rangle^{(m-1)/2} u(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \left(\|\varphi\|_{L^2(\mathbb{R})} + \left\| \langle x\rangle^{s} \langle D_x\rangle^{-(m-1)/2} f(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \right).$$

If we combine Theorem 10 with Theorem 7, we have the following:

COROLLARY 6. *Assume* (HL). *Suppose $n \geq 1$, $m > 0$, $s > 1/2$, and $T > 0$. Then the solution u to equation* (23) *satisfies*

$$\int_0^T \left\| \langle x\rangle^{-s} \langle D_x\rangle^{(m-1)/2} u(t,x) \right\|^2_{L^2(\mathbb{R}^n_x)} dt \leq C \left(\|\varphi\|^2_{L^2(\mathbb{R}^n)} + \int_0^T \left\| \langle x\rangle^{s} \langle D_x\rangle^{-(m-1)/2} f(t,x) \right\|^2_{L^2(\mathbb{R}^n_x)} dt \right),$$

where $C > 0$ is a constant depending on $T > 0$.

Corollary 6 is an extension of the result by Hoshiro [12], which treated the case that $a(\xi)$ is a polynomial. The proof relied on Mourre's method, which is known in spectral and scattering theories. Here we use the argument of canonical transformations, extending the result and simplifying the proof.

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