CONTENTS

K. Benmeriem and C. Bouzar, An algebra of generalized Roumieu ultradistributions ..................................... 101

C. Bouzar - M. T. Khalladi, Linear differential equations in the algebra of almost periodic generalized functions ........................................ 111

R. Estrada - J. Vindas, On distributional point values and boundary values of analytic functions .................................................. 121

G. R. Franssens, The convolution and multiplication of one-dimensional associated homogeneous distributions ........................................ 127

A. Kamiński - S. Sorek, Remarks on proofs of diagonal theorem and its applications in the theory of distributions ........................................ 139

L. Radzhabova, Cauchy type problems for a class of integral equations of Volterra type .......................................................... 151

M. Ruzhansky - M. Sugimoto, Smoothing properties of inhomogeneous equations via canonical transforms ........................................... 165

J. Schmeelk, An impulsive differential equation in an infinite dimensional Fock space .............................................................. 183

V. Valmorin, A global construction of algebras of generalized functions ......................................................... 197

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Khaled BENMERIEM and Chikh BOUZAR

AN ALGEBRA OF GENERALIZED ROUMIEU ULTRADISTRIBUTIONS

Abstract. We introduce an algebra of generalized functions containing Roumieu ultradistributions.

1. Introduction

Schwartz distributions [13] have natural extensions Roumieu ultradistributions [12], see [8] for a detailed study and [9] for applications. Gevrey ultradistributions are particular, but important, case of Roumieu ultradistributions.

Colombeau generalized functions [3] and [4], introduced in connection with the problem of multiplication of Schwartz distributions, have been developed and applied in linear and nonlinear problems, see [7].

The problem of multiplication of ultradistributions is still posed. So, it is natural to search for algebras of generalized functions containing spaces of ultradistributions, to study and to apply them. Generalized Gevrey ultradistributions have been introduced and studied in [1] and [2].

The aim of this paper is to introduce an algebra of generalized functions containing Roumieu ultradistributions.

2. Roumieu ultradistributions

Let $\langle M_p \rangle_{p \in \mathbb{Z}_+}$ be a sequence of positive numbers, recall the following properties:

(H1) Logarithmic convexity:

$$M_p^2 \leq M_{p-1}M_{p+1}, \forall p \geq 1.$$  

(H2) Stability under ultradifferentiation:

$$\exists A > 0, \exists H > 0, M_{p+q} \leq AH^{p+q}M_pM_q, \forall p \geq 0, \forall q \geq 0.$$  

(H2)' Stability under differentiation:

$$\exists A > 0, \exists H > 0, M_{p+1} \leq AH^pM_p, \forall p \geq 0.$$  

(H3)' Non-quasi-analyticity:

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty.$$
The associated function of the sequence \((M_p)_{p \in \mathbb{Z}_+}\) is the function defined by

\[ M(t) = \sup_p \frac{t^p}{M_p}, \quad t \in \mathbb{R}_+^* \]

**Example 1.** The well-known Gevrey sequence \((M_p)_{p \in \mathbb{Z}_+} = (p!^\alpha)_{p \in \mathbb{Z}_+}, \alpha > 0\), has an associated function equivalent to the function \(M_\sigma(t) = t^{\frac{\alpha}{2}}\).

An important result on the associated function is given in the following proposition, see [8].

**Proposition 1.** Let the sequence \((M_p)_{p \in \mathbb{Z}_+}\) satisfy condition \((H1)\), then it satisfies \((H2)\) if and only if \(\exists A > 0, \exists H > 0, \forall t > 0,\)

\[ 2M(t) \leq M(Ht) + \ln(AM_0). \]

The class of ultradifferentiable functions of class \(M\), denoted \(E^M(\Omega)\), is the space of all \(f \in C^\infty(\Omega)\) satisfying for every compact subset \(K\) of \(\Omega, \exists c > 0, \forall \alpha \in \mathbb{Z}_+^n,\)

\[ \sup_{x \in K} |\partial^\alpha f(x)| \leq c|\alpha| + 1M|\alpha|. \]

**Example 2.** If \((M_p)_{p \in \mathbb{Z}_+} = (p!^\alpha)_{p \in \mathbb{Z}_+}\), we obtain \(E^\alpha(\Omega)\) the Gevrey space of order \(\sigma\), and \(\mathcal{A}(\Omega) := E^1(\Omega)\) is the space of real analytic functions on the open set \(\Omega\).

A differential operator of infinite order \(P(D) = \sum_{\gamma \in \mathbb{Z}_+^n} a_\gamma D^\gamma\) is called an ultradifferential operator of class \(\{M_p\}_{p \in \mathbb{Z}_+}\), if for every \(h > 0\) there exist \(c > 0\) such that \(\forall \gamma \in \mathbb{Z}_+^n,\)

\[ |a_\gamma| \leq c\frac{h^{\gamma}}{M_{|\gamma|}}. \]

The basic properties of the space \(E^M(\Omega)\) are summarized in the following proposition, for the proof see [10] and [8].

**Proposition 2.** Let the sequence \((M_p)_{p \in \mathbb{Z}_+}\) satisfy condition \((H1)\), then the space \(E^M(\Omega)\) is an algebra moreover, if \((M_p)_{p \in \mathbb{Z}_+}\) satisfies \((H2)\)', then \(E^M(\Omega)\) is stable by differential operators of finite order with coefficients in \(E^M(\Omega)\), and if \((M_p)_{p \in \mathbb{Z}_+}\) satisfies \((H2)\) then any ultradifferential operator of class \(M\) operates also as a sheaf homomorphism.

The space \(\mathcal{D}^M(\Omega) = E^M(\Omega) \cap \mathcal{D}(\Omega)\) is not trivial if and only if the sequence \((M_p)_{p \in \mathbb{Z}_+}\) satisfies \((H3)\)'.

**Remark 1.** The sequence \((p!^\alpha)_{p \in \mathbb{Z}_+}\) satisfies \((H3)\) if and only if \(\sigma > 1\).
3. Generalized Roumieu ultradistributions

To define the algebra of generalized Roumieu ultradistributions, we first introduce the algebra of moderate elements and its ideal of null elements.

Let $\Omega$ be a non void open set of $\mathbb{R}^n$ and $I = [0, 1]$.

**Remark 2.** In the sequel, we will always suppose that the sequence $(M_p)_{p \in \mathbb{Z}_+}$ satisfies the conditions $(H1), (H2), (H3)$ and $M_0 = 1$.

**Definition 2.** The space of moderate elements, denoted $\mathcal{L}_m^{(M)} (\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty (\Omega)$ satisfying for every compact $K$ of $\Omega$, $\forall \alpha \in \mathbb{Z}^n_+, \exists k > 0, \exists c > 0, \exists \epsilon_0 \in I, \forall \epsilon \leq \epsilon_0$,

\[
\sup_{x \in K} |\partial^\alpha f_\varepsilon (x)| \leq c \exp \left( -M \left( \frac{k}{\epsilon} \right) \right).
\]

The space of null elements, denoted $\mathcal{N}^{(M)} (\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty (\Omega)$ satisfying for every compact $K$ of $\Omega$, $\forall \alpha \in \mathbb{Z}^n_+, \exists k > 0, \exists c > 0, \exists \epsilon_0 \in I, \forall \epsilon \leq \epsilon_0$,

\[
\sup_{x \in K} |\partial^\alpha f_\varepsilon (x)| \leq c \exp \left( M \left( \frac{k}{\epsilon} \right) \right).
\]

The main properties of the spaces $\mathcal{L}_m^{(M)} (\Omega)$ and $\mathcal{N}^{(M)} (\Omega)$ are given in the following proposition.

**Proposition 3.** 1) The space of moderate elements $\mathcal{L}_m^{(M)} (\Omega)$ is an algebra stable by derivation.

2) The space $\mathcal{N}^{(M)} (\Omega)$ is an ideal of $\mathcal{L}_m^{(M)} (\Omega)$.

**Proof.** 1) Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{L}_m^{(M)} (\Omega)$ and $K$ be a compact of $\Omega$, then $\forall \beta \in \mathbb{Z}^n_+, \exists k_1 = k_1 (\beta) > 0, \exists c_1 = c_1 (\beta) > 0, \exists \epsilon_1 \in I, \forall \varepsilon \leq \epsilon_1$,

\[
\sup_{x \in K} |\partial^\beta f_\varepsilon (x)| \leq c_1 \exp M \left( \frac{k_1}{\epsilon} \right),
\]

$\forall \beta \in \mathbb{Z}^n_+, \exists k_2 = k_2 (\beta) > 0, \exists c_2 = c_2 (\beta) > 0, \exists \epsilon_2 \in I, \forall \varepsilon \leq \epsilon_2$,

\[
\sup_{x \in K} |\partial^\beta g_\varepsilon (x)| \leq c_2 \exp M \left( \frac{k_2}{\epsilon} \right).
\]
Let $\alpha \in \mathbb{Z}^m_+$, then

$$|\partial^n (f_{\varepsilon} g_{\varepsilon}) (x) | \leq \sum_{\beta=0}^n \left( \frac{\alpha}{\beta} \right) |\partial^{n-\beta} f_{\varepsilon} (x) | |\partial^\beta g_{\varepsilon} (x) |$$

from proposition 1, we have $\exists \forall A > 0, \exists H > 0, \forall \iota > 0,$

$$2M (i) \leq M (Ht) + \ln (AM_0).$$

For $k = H (\max \{k_1 (\beta), k_2 (\beta) : \beta \leq \alpha \}), \iota \leq \min \{\varepsilon_{1\beta}, \varepsilon_{2\beta} ; |\beta| \leq |\alpha| \}$ and $x \in K$, we have for $t = \frac{k}{\varepsilon}$

$$\exp \left( -M \left( \frac{k}{\varepsilon} \right) \right) |\partial^n (f_{\varepsilon} g_{\varepsilon}) (x) | \leq \exp(\ln (AM_0)) \sum_{\beta=0}^n \left( \frac{\alpha}{\beta} \right) \exp \left( -M \left( \frac{k_1}{\varepsilon} \right) \right) \times |\partial^{n-\beta} f_{\varepsilon} (x) | \exp \left( -M \left( \frac{k_2}{\varepsilon} \right) \right) |\partial^\beta g_{\varepsilon} (x) | \leq A \sum_{\beta=0}^n \left( \frac{\alpha}{\beta} \right) c_1 (\alpha - \beta) c_2 (\beta) = c (\alpha),$$

i.e. $(f_{\varepsilon} g_{\varepsilon})_x \in \mathcal{L}^{(M)}_\varepsilon (\Omega)$. It is clear, from (5) that for every compact $K$ of $\Omega$, $\forall \beta \in \mathbb{Z}^m_+$, $\exists k_1 = k_1 (\beta) > 0, \exists c_1 = c_1 (\beta + 1) > 0, \exists \varepsilon_{1\beta} \in I$ such that $\forall x \in K, \forall \varepsilon \leq \varepsilon_{1\beta},$

$$|\partial^\beta \partial^n (f_{\varepsilon} g_{\varepsilon}) (x) | \leq c_1 \exp \left( M \left( \frac{k_1}{\varepsilon} \right) \right),$$

i.e. $(\partial^\beta f_{\varepsilon})_x \in \mathcal{L}^{(M)}_\varepsilon (\Omega)$.

2) If $(g_{\varepsilon})_x \in \mathcal{N}^{(M)} (\Omega)$, for every $K$ compact of $\Omega$, $\forall \beta \in \mathbb{Z}^m_+, \forall k_2 > 0, \exists c_2 = c_2 (\beta, k_2) > 0, \exists \varepsilon_{2\beta} \in I$,

$$|\partial^n g_{\varepsilon} (x) | \leq c_2 \exp \left( -M \left( \frac{k_2}{\varepsilon} \right) \right) \times \forall x \in K, \forall \varepsilon \leq \varepsilon_{2\beta}$$

Let $\alpha \in \mathbb{Z}^m_+$ and $k > 0$, then

$$\exp \left( M \left( \frac{k}{\varepsilon} \right) \right) |\partial^n (f_{\varepsilon} g_{\varepsilon}) (x) | \leq \exp \left( M \left( \frac{k}{\varepsilon} \right) \right) \sum_{\beta=0}^n \left( \frac{\alpha}{\beta} \right) |\partial^{n-\beta} f_{\varepsilon} (x) | \times \left| \partial^\beta g_{\varepsilon} (x) \right|.$$
for \( t = \frac{k}{\varepsilon} \) in (7)
\[
\exp \left( M \left( \frac{k}{\varepsilon} \right) \right) |\partial^\alpha (f \varepsilon g \varepsilon) (x)| \leq A \sum_{\beta=0}^{\alpha} \left( \frac{\alpha}{\beta} \right) \left| \exp \left( -M \left( \frac{k_1}{\varepsilon} \right) \right) |\partial^{\alpha-\beta} f \varepsilon (x)| \right| \times \exp \left( M \left( \frac{k_2}{\varepsilon} \right) \right) |\partial^\beta g \varepsilon (x)|,
\]
which shows that \((f \varepsilon g \varepsilon) \varepsilon \in \mathcal{N}^{(M)}(\Omega)\).

**Definition 3.** The algebra of generalized Roumieu ultradistributions of class \( \{M_p\}_{p \in \mathbb{Z}^+} \), denoted \( G^{(M)}(\Omega) \), is the quotient algebra
\[
G^{(M)}(\Omega) = \frac{E^{(M)}(\Omega)}{N^{(M)}(\Omega)}.
\]

**Example 3.** If \((M_p)_{p \in \mathbb{Z}^+} = (p^\alpha)_{p \in \mathbb{Z}^+}\) we obtain \( G^a(\Omega) \) the algebra of generalized Gevrey ultradistributions of [1].

### 4. Embedding of Roumieu ultradistributions with compact support

Let \( N = (N_p)_{p \in \mathbb{Z}^+} \) be a sequence satisfying the conditions \((H1), (H2), (H3)' \) and \( N_0 = 1 \), the space \( S^{(N)} (\mathbb{R}^n) \) is the space of functions \( \psi \in C^\infty (\mathbb{R}^n) \) such that \( \forall b > 0 \), we have
\[
\|\psi\|_{b,N} = \sup_{\alpha, \beta \in \mathbb{Z}^n} \int b^{(\alpha+\beta)N_{(\alpha+\beta)}} |\partial^\alpha \psi (x)| \, dx < \infty.
\]
(8)

Define \( \Sigma^{(N)} \) as the set of functions \( \phi \in S^{(N)} (\mathbb{R}^n) \) satisfying
\[
\int \phi (x) \, dx = 1 \text{ and } \int x^{\alpha} \phi (x) \, dx = 0, \forall \alpha \in \mathbb{Z}^n \setminus \{0\}.
\]

**Lemma 1.** There exist functions in \( \Sigma^{(N)} \).

**Definition 4.** The net \( \phi_\varepsilon = e^{-m \phi (\cdot / \varepsilon)} \), \( \varepsilon \in I \), where \( \phi \) satisfies the conditions of lemma 4, is called a \( N \)-mollifier net. The space \( E^{(M)}(\Omega) \) is embedded into \( G^{(M)}(\Omega) \) by the standard canonical injection
\[
I: E^{(M)}(\Omega) \rightarrow G^{(M)}(\Omega), \quad f \mapsto [f] = cI (f_\varepsilon),
\]
(9)
where \( f_\epsilon = f, \forall \epsilon \in I \).

The following proposition gives the embedding of Roumieu ultradistributions into \( G^{(M)}(\Omega) \). Let \( M \) and \( N \) two sequences satisfying \((H1),(H2)\) and \((H3)^\prime\) with \( M_0 = N_0 = 1 \) and \( \phi \in \Sigma^{(N)} \).

**THEOREM 1.** The map

\[
J_0 : \mathcal{E}^{(MN)}_J(\Omega) \rightarrow G^{(M)}(\Omega)
\]

\[
T \mapsto [T] = cl \left( (T \ast \phi_\epsilon) / \Omega \right)_x
\]

is an embedding.

**Proof.** Let \( T \in \mathcal{E}^{(MN)}_J(\Omega) \) with \( suppT \subset K \), then there exists \( P(D) = \sum_{\gamma \in \Z^n} a_{\gamma} D^{\gamma} \) an ultradifferential operator of class \( \{ M_pN_p \}_{p \in \Z^n}, C > 0 \), and continuous functions \( f_\gamma \) with \( supp f_\gamma \subset K, \forall \gamma \in \Z^n \), and \( \sup_{\gamma \in \Z^n, \epsilon \in K} |f_\gamma(x)| \leq C \), such that

\[
T = \sum_{\gamma \in \Z^n} a_{\gamma} D^{\gamma} f_\gamma.
\]

We have

\[
T \ast \phi_\epsilon(x) = \sum_{\gamma \in \Z^n} a_{\gamma} \frac{(-1)^{|\gamma|}}{|\gamma|!} \int f_\gamma(x + \epsilon y) D^{\gamma} \phi(y) dy.
\]

Let \( \alpha \in \Z^n \), then

\[
|\partial^\alpha (T \ast \phi_\epsilon(x))| \leq \sum_{\gamma \in \Z^n} a_{\gamma} \frac{1}{\epsilon^{\gamma + \alpha}} \int |f_\gamma(x + \epsilon y)| |D^{\gamma + \alpha} \phi(y)| dy.
\]

From (2) and condition \((H2)\), we have \( \exists A > 0, \exists H > 0, \forall h > 0, \exists \epsilon > 0 \), such that

\[
|\partial^\alpha (T \ast \phi_\epsilon(x))| \leq c \sum_{\gamma \in \Z^n} \frac{h^{\gamma} |\gamma|!}{M_{|\gamma|} N_{|\gamma|}} \frac{1}{\epsilon^{\gamma + \alpha}} \int |f_\gamma(x + \epsilon y)| |D^{\gamma + \alpha} \phi(y)| dy
\]

\[
\leq c \sum_{\gamma \in \Z^n} \frac{h^{\gamma} (b^{\gamma + \alpha} N_{|\gamma|})^{\gamma + \alpha}}{M_{|\gamma|} N_{|\gamma|}} \frac{1}{\epsilon^{\gamma + \alpha}} \int |f_\gamma(x + \epsilon y)| |D^{\gamma + \alpha} \phi(y)| dy
\]

\[
\leq cA \sum_{\gamma \in \Z^n} \frac{h^{\gamma} (b^{\gamma + \alpha} H^{\gamma + \alpha})^{\gamma + \alpha} M_{|\alpha|} N_{|\alpha|}}{M_{|\gamma + \alpha|}} \frac{1}{\epsilon^{\gamma + \alpha}} C \| \phi \|_{b,N}
\]

then,

\[
\frac{(2h)^{|\alpha|}}{M_{|\alpha|} N_{|\alpha|}} |\partial^\alpha (T \ast \phi_\epsilon(x))| \leq cCA \| \phi \|_{b,N} \sum_{\gamma \in \Z^n} 2^{-|\gamma|} \frac{(2hHb)^{\gamma + \alpha}}{M_{|\gamma + \alpha|}} \frac{1}{\epsilon^{\gamma + \alpha}}
\]

\[
\leq c \exp \left( M \left( \frac{k}{\epsilon} \right) \right),
\]
i.e.
\begin{equation}
\left| \partial^\alpha (T \ast \phi_\varepsilon (x)) \right| \leq c(\alpha) \exp \left( M \left( \frac{k}{\varepsilon} \right) \right),
\end{equation}

where \( k = 2hHb \).

Suppose that \((T \ast \phi_\varepsilon)_\varepsilon \in A^{(M)}(\Omega)\), then for every compact \( L \) of \( \Omega \), \( \forall k > 0, \exists \varepsilon_0 \in I \),

\begin{equation}
\left| T \ast \phi_\varepsilon (x) \right| \leq c \exp \left( -M \left( \frac{k}{\varepsilon} \right) \right), \forall x \in L, \forall \varepsilon \leq \varepsilon_0
\end{equation}

Let \( \chi \in D^{(MN)}(\Omega) \) and \( \chi = 1 \) in a neighborhood of \( K \), then \( \forall \psi \in E^{(MN)}(\Omega) \),

\[ \langle T, \psi \rangle = \lim_{\varepsilon \to 0} \int \langle (T \ast \phi_\varepsilon) (x) \chi(x) \psi(x) \rangle \, dx \]

Consequently, from (12), we obtain

\[ \left| \int (T \ast \phi_\varepsilon) (x) \chi(x) \psi(x) \right| \, dx \leq c \exp \left( -M \left( \frac{k}{\varepsilon} \right) \right), \forall \varepsilon \leq \varepsilon_0, \]

which gives \( \langle T, \psi \rangle = 0 \). \( \Box \)

**Notation 1.** If \( M = (M_p)_{p \in \mathbb{Z}_+} \) and \( N = (N_p)_{p \in \mathbb{Z}_+} \) are two sequences, then \( MN^{-1} := (M_p N_{p}^{-1})_{p \in \mathbb{Z}_+} \).

In order to show the commutativity of the following diagram of embeddings

\[ D^{(MN^{-1})}(\Omega) \rightarrow G^{(M)}(\Omega) \]

we have to prove the following fundamental result.

**Proposition 4.** Let \( f \in D^{(MN^{-1})}(\Omega) \) and \( \phi \in \sum^{(N)} \), then

\[ \left( f - (f \ast \phi_\varepsilon) / \varepsilon \right) \in A^{(M)}(\Omega). \]

**Proof.** Let \( f \in D^{(MN^{-1})}(\Omega) \), then there exists a constant \( c > 0 \), such that

\[ |\partial^\alpha f(x)| \leq c^{\alpha+1} \frac{M_p}{N_p}, \forall \alpha \in \mathbb{Z}_+^n, \forall x \in \Omega. \]

Let \( \alpha \in \mathbb{Z}_+^n \), the Taylor’s formula and the properties of \( \phi_\varepsilon \) give

\[ \delta^\alpha (f \ast \phi_\varepsilon - f) (x) = \sum_{|\beta|=N} \int \frac{(\varepsilon y)^\beta}{\beta!} \delta^{\alpha+\beta} f (\xi) \phi(y) \, dy, \]
where \( x \leq \xi \leq x + \varepsilon y \). Consequently, for any \( b > 0 \), we have

\[
|\partial^\alpha (f * \phi_\varepsilon - f)(x)| \leq \varepsilon^N \sum_{|\beta| = N} \frac{|y|^{|\beta|}}{|\beta|!} |\partial^{\alpha+\beta} f(\xi)| |\phi(y)| dy
\]

\[
\leq \varepsilon^N \sum_{|\beta| = N} \frac{b^{|\beta|}N_{|\beta|}M_{(\alpha+|\beta|)}}{|\beta|!N_{(\alpha+|\beta|)}} \times
\]

\[
\times \int \frac{N_{|\alpha+|\beta|}}{M_{(\alpha+|\beta|)} (\alpha + |\beta|)!} \left| \partial^{\alpha+\beta} f(\xi) \right| \left| \frac{|y|^{|\beta|}}{b^{|\beta|}N_{|\beta|}} |\phi(y)| \right| dy
\]

\[
\leq A \|\phi\|_{b,N} c \sum_{|\beta| = N} \frac{c |\beta| H_{|\beta|} M_{|\beta|}}{N_{|\alpha|}} \varepsilon^N \sum_{|\beta| = N} b^{|\beta|}H_{|\beta|} M_{|\beta|} c^{|\beta|}.
\]

Let \( k > 0 \) and \( T > 0 \), then

\[
|\partial^\alpha (f * \phi_\varepsilon - f)(x)| \leq c(\alpha) \varepsilon^N M_N (kT)^{-N} \sum_{|\beta| = N} (kTbHc)^{|\beta|}
\]

where \( c(\alpha) = A \|\phi\|_{b,N} c \frac{c |\beta| H_{|\beta|} M_{|\beta|}}{N_{|\alpha|}} \). Taking \( kTbHc \leq \frac{1}{2a} \), with \( a > 1 \), we obtain

\[
|\partial^\alpha (f * \phi_\varepsilon - f)(x)| \leq c(\alpha) \varepsilon^N M_N (kT)^{-N} a^{-N} \sum_{|\beta| = N} \left( \frac{1}{2} \right)^{|\beta|}
\]

(13)

Let \( \varepsilon_0 \in I \) such that \( \varepsilon_0 M_I < 1 \) and take \( T \geq \frac{M_{p^{-1}M_{p-1}}}{M_M} \), \( \forall p \geq 1 \).

Then, see [11], there exists \( N = N(\varepsilon) \in \mathbb{Z}^+ \), such that

\[
1 \leq \varepsilon \left( M_N \right)^N \leq T
\]

which gives

\[
a^{-N} \leq \exp \left( -M \left( \frac{k}{\varepsilon} \right) \right) \quad \text{and} \quad \varepsilon^N M_N (kT)^{-N} < 1,
\]

if we take \( a \geq 2 \). Finally, from (13), we have

\[
|\partial^\alpha (f * \phi_\varepsilon - f)(x)| \leq c \exp \left( -M \left( \frac{k}{\varepsilon} \right) \right),
\]

(14)

i.e. \( (f * \phi_\varepsilon - f)(x) \in K(M) (\Omega) \).

REMARK 3. Microlocal analysis suitable for the algebra of generalized Roumieu ultradistributions will be discussed in a separate paper.
An Algebra of generalized Roumieu ultradistributions

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Khaled BENMERIEM
Department of Mathematics, University of Mascara
Mascara, 29000, ALGERIA
e-mail: benmeriemkhaled@yahoo.fr

Chikh BOUZAR
Department of Mathematics, University of Oran
Oran, 31000, ALGERIA
e-mail: bouzar@yahoo.com

Chikh BOUZAR and Mohammed Taha KHALLADI

LINEAR DIFFERENTIAL EQUATIONS IN THE ALGEBRA OF ALMOST PERIODIC GENERALIZED FUNCTIONS

Abstract. The paper introduces and studies an algebra of almost periodic generalized functions generalizing trigonometric polynomials, classical almost periodic functions as well as almost periodic Schwartz distributions. Then we study a linear system of ordinary differential equations in this algebra of almost periodic generalized functions.

1. Introduction

Trigonometric polynomials are the basic examples of the uniformly almost periodic functions introduced and studied by H. Bohr, see [1]. There exist three equivalent definitions of uniformly almost periodic functions, the first definition of H. Bohr, S. Bochner’s definition and the definition based on the approximation property. Bochner’s definition is more suitable for extension to distributions. L. Schwartz in [6] introduced the basic elements of almost periodic distributions.

The algebra of generalized functions of Colombeau [3] give an answer to the problem of multiplication of distributions, this algebra contains the space of Schwartz distributions, and is currently the subject of many scientific works, see [5].

We introduce and study an algebra of almost periodic generalized functions generalizing trigonometric polynomials, classical almost periodic functions as well as almost periodic Schwartz distributions. Then we study a linear system of ordinary differential equations in this algebra of almost periodic generalized functions, namely we give results on the existence of generalized solutions of the linear system of ordinary differential equations

\[ u(t) = Au(t) + f(t), \]

where \( f \) is an almost periodic generalized function and \( A \) is a matrix of classical complex numbers.

2. Almost periodic functions and distributions

We consider functions and distributions defined on the whole one dimensional space \( \mathbb{R} \).

Let \( \mathcal{C}_b \) be the space of bounded and continuous complex valued functions on \( \mathbb{R} \) endowed with the norm \( \| \cdot \|_\infty \) of uniform convergence on \( \mathbb{R} \), \( (\mathcal{C}_b, \| \cdot \|_\infty) \) is a Banach algebra.

Definition 1. (S. Bochner) A complex valued function \( f \) defined and continuous on \( \mathbb{R} \) is called almost periodic, if for any sequence of real numbers \( (h_n)_n \) one can
extract a subsequence \(\{h_n\}_k\) such that \((f_i + h_n)\) converges in \((C_b, \|\cdot\|_\infty)\). Denote by \(C_{ap}\) the space of almost periodic functions.

To recall Schwartz almost periodic distributions, we need some function spaces, see [6]. Let \(p \in [1, +\infty]\), the space \(\mathcal{D}_L^p := \left\{ \varphi \in C^\infty : \varphi^{(j)} \in L^p, \forall j \in \mathbb{Z}_+ \right\}\) endowed with the topology defined by the countable family of norms \(|\varphi|_{k,p} := \sum_{j \leq k} \|\varphi^{(j)}\|_{L^p}, k \in \mathbb{Z}_+\), is a differential Frechet subalgebra of \(C^\infty\). The topological dual of \(\mathcal{D}_L^1\), denoted by \(\mathcal{D}'_{L^\infty}\), is called the space of bounded distributions.

Let \(h \in \mathbb{R}\) and \(T \in \mathcal{D}'\), the translate of \(T\) by \(h\), denoted by \(\tau_h T\), is defined as \(<\tau_h T, \varphi> = <T, \tau_{-h} \varphi>, \varphi \in \mathcal{D}\), where \(\tau_{-h} \varphi(x) = \varphi(x + h)\).

The definition and characterizations of an almost periodic distribution are summarized in the following results.

**Theorem 1.** For any bounded distribution \(T \in \mathcal{D}'_{L^\infty}\), the following statements are equivalent:

i) The set \(\{\tau_h T, h \in \mathbb{R}\}\) is relatively compact in \(\mathcal{D}'_{L^\infty}\).

ii) \(T * \varphi \in C_{ap}, \forall \varphi \in \mathcal{D}\).

iii) \(\exists \ (f_j)_{j \leq k} \subset C_{ap}, T = \sum_{j=1}^{k} f_j^{(j)}\).

\(T \in \mathcal{D}'_{L^\infty}\) is said almost periodic if it satisfies any (hence every) of the above conditions.

**Definition 2.** The space of almost periodic distributions is denoted by \(B'_{ap}\).

Let us introduce the space of regular almost periodic functions.

**Definition 3.** The space of almost periodic infinitely differentiable functions on \(\mathbb{R}\) is defined and denoted by \(B_{ap} = \left\{ \varphi \in \mathcal{D}_L^\infty : \varphi^{(j)} \in C_{ap}, \forall j \in \mathbb{Z}_+ \right\}\).

Some, easy to prove, properties of \(B_{ap}\) are given in the following proposition.

**Proposition 1.** We have

i) \(B_{ap}\) is a closed differential subalgebra of \(\mathcal{D}_L^\infty\).

ii) If \(T \in B_{ap}'\) and \(\varphi \in B_{ap}\), then \(\varphi T \in B_{ap}'\).

iii) \(B_{ap} * L^1 \subset B_{ap}\).

iv) \(B_{ap} = \mathcal{D}_L^\infty \cap C_{ap}\).

As a consequence of (iv), we have the following result.

**Corollary 1.** If \(v \in \mathcal{D}_L^\infty\) and \(v * \varphi \in C_{ap}, \forall \varphi \in \mathcal{D}\), then \(v \in B_{ap}\).

**Remark 1.** It is important to mention that \(B_{ap} \subset C^\infty \cap C_{ap}\).
3. Almost periodic generalized functions

Let \( I = [0, 1] \) and

\[
\mathcal{M}_{I^*} := \left\{ (u_ε)_ε \in (\mathcal{D}_I)^I, \forall k \in \mathbb{Z}^+, \exists m \in \mathbb{Z}^+, |u_ε|_{k, \infty} = O(ε^{-m}), \varepsilon \to 0 \right\}
\]

\[
\mathcal{N}_{I^*} := \left\{ (u_ε)_ε \in (\mathcal{D}_I)^I, \forall k \in \mathbb{Z}^+, \forall m \in \mathbb{Z}^+, |u_ε|_{k, \infty} = O(ε^m), \varepsilon \to 0 \right\}
\]

DEFINITION 4. The algebra of bounded generalized functions, denoted by \( G_{I^*} \), is defined by the quotient \( G_{I^*} = \frac{\mathcal{M}_{I^*}}{\mathcal{N}_{I^*}} \)

Define

\[
\mathcal{M}_{ap} = \left\{ (u_ε)_ε \in (\mathcal{B}_{ap})^I, \forall k \in \mathbb{Z}^+, \exists m \in \mathbb{Z}^+, |u_ε|_{k, \infty} = O(ε^{-m}), \varepsilon \to 0 \right\}
\]

\[
\mathcal{N}_{ap} = \left\{ (u_ε)_ε \in (\mathcal{B}_{ap})^I, \forall k \in \mathbb{Z}^+, \forall m \in \mathbb{Z}^+, |u_ε|_{k, \infty} = O(ε^m), \varepsilon \to 0 \right\}
\]

The properties of \( \mathcal{M}_{ap} \) and \( \mathcal{N}_{ap} \) are summarized in the following proposition.

PROPOSITION 2. i) The space \( \mathcal{M}_{ap} \) is a subalgebra of \( (\mathcal{D}_I)^I \).

ii) The space \( \mathcal{N}_{ap} \) is an ideal of \( \mathcal{M}_{ap} \).

Proof. i) It follows from the fact that \( \mathcal{B}_{ap} \) is a differential algebra.

ii) Let \( (u_ε)_ε \in \mathcal{N}_{ap} \) and \( (v_ε)_ε \in \mathcal{M}_{ap} \), we have

\[
\forall k \in \mathbb{Z}^+, \exists m' \in \mathbb{Z}^+, \exists c_1 > 0, \exists c_0 \in I, \forall \varepsilon \in (0, e_0), |v_ε|_{k, \infty} < c_1 ε^{-m'}. \]

Take \( m \in \mathbb{Z}^+ \), then for \( m'' = m + m', \exists c_2 > 0 \) such that \( |u_ε|_{k, \infty} < c_2 ε^{-m''} \). Since the family of the norms \( |u_ε|_{k, \infty} \) is compatible with the algebraic structure of \( \mathcal{D}_I \), then \( \forall k \in \mathbb{Z}^+, \exists c_3 > 0 \) such that

\[
|u_ε v_ε|_{k, \infty} \leq c_k |u_ε|_{k, \infty} |v_ε|_{k, \infty},
\]

consequently

\[
|u_ε v_ε|_{k, \infty} < c_k c_2 ε^{-m'} c_1 ε^{-m''} \leq C ε^m, \text{ where } C = c_1 c_2 c_k.
\]

Hence \( (u_ε v_ε)_ε \in \mathcal{N}_{ap} \).

The following definition introduces the algebra of almost periodic generalized functions.

DEFINITION 5. The algebra of almost periodic generalized functions is the quotient algebra

\[
\mathcal{G}_{ap} = \frac{\mathcal{M}_{ap}}{\mathcal{N}_{ap}}
\]
We have a characterization of elements of $G_{ap}$ similar to the result (ii) of theorem 1 for almost periodic distributions.

**Theorem 2.** Let $u = [(u_\varepsilon)_\varepsilon] \in G_{L^\infty}$, the following assertions are equivalent:

i) $u$ is almost periodic.

ii) $u_\varepsilon \ast \varphi \in B_{ap}$, $\forall \varepsilon \in I, \forall \varphi \in D$.

**Proof.** i) $\implies$ ii) If $u \in G_{ap}$, so for every $\varepsilon \in I$ we have $u_\varepsilon \in B_{ap}$, then $u_\varepsilon \ast \varphi \in B_{ap}, \forall \varepsilon \in I, \forall \varphi \in D$.

ii) $\implies$ i) Let $(u_\varepsilon)_\varepsilon \in M_{L^\infty}$ and $u_\varepsilon \ast \varphi \in B_{ap}, \forall \varepsilon \in I, \forall \varphi \in D$, therefore $u \in B_{ap}$

follows from Theorem 2.2 (ii); it suffices to show that

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k; \varepsilon} = O(\varepsilon^{-m}), \varepsilon \to 0,$$

which follows from the fact that $u \in G_{L^\infty}$.

**Remark 2.** The characterization (ii) does not depend on representatives.

**Definition 6.** Denote by $\Sigma$ the set of functions $p \in S$ satisfying $\int p(x) dx = 1$ and $\int x^2 p(x) dx = 0$, $\forall k = 1, 2, ...$. Set $\rho_\varepsilon(\cdot) := \frac{1}{\varepsilon} p\left(\frac{x}{\varepsilon}\right), \varepsilon > 0$.

**Proposition 3.** Let $p \in \Sigma$, the map

$$l_{ap} : B'_{ap} \rightarrow M_{ap},$$

$u \mapsto (u \ast \rho_\varepsilon)_\varepsilon \ast N_{ap}$,

is a linear embedding which commutes with derivatives.

**Proof.** Let $u \in B'_{ap}$, by a characterization of almost periodic distributions we have $u = \sum_{\beta \leq m} c_{\beta}^p$, where $f_\beta \in C_{ap}$, so $\forall \alpha \in \mathbb{Z},$

$$\left| \left( u^{\alpha} \ast \rho_\varepsilon \right)(x) \right| \leq \sum_{\beta \leq m} \frac{1}{\varepsilon^{m+\beta}} \int |f_\beta(x-\varepsilon y)| p^{\alpha+\beta}(y) dy,$$

consequently, there exists $c > 0$ such that

$$\sup_{x \in \mathbb{R}} \left| \left( u^{\alpha} \ast \rho_\varepsilon \right)(x) \right| \leq \sum_{\beta \leq m} \frac{1}{\varepsilon^{m+\beta}} \|f_\beta\|_{L^\infty(\mathbb{R})} \int |p^{\alpha+\beta}(y)| dy \leq \frac{c}{\varepsilon^{m+\beta}},$$

i.e.

$$|u \ast \rho_\varepsilon|_{m', \varepsilon} = \sum_{\alpha \leq m' \varepsilon \in \mathbb{R}} \sup_{x \in \mathbb{R}} \left| \left( u^{\alpha} \ast \rho_\varepsilon \right)(x) \right| \leq \frac{c}{\varepsilon^{m+\beta}} \varepsilon', \varepsilon' = \sum_{\alpha \leq m} c^{\alpha},$$

this shows that $(u \ast \rho_\varepsilon)_\varepsilon \in M_{ap}$. Let $(u \ast \rho_\varepsilon)_\varepsilon \in N_{ap}$, then $\lim_{\varepsilon \to 0} u \ast \rho_\varepsilon = 0$ in $D'_{L^\infty}$, but $\lim_{\varepsilon \to 0} u \ast \rho_\varepsilon = u$ in $D'_{L^\infty}$, this shows that $l_{ap}$ is an embedding. Finally we note that $l_{ap}$
is linear, this results from the fact that the convolution is linear and that \( i_{ap}(w^{(j)}) = (w * \rho) \), \( \rho \in \Sigma \).

The space \( B_{ap} \) is embedded into \( G_{ap} \) canonically, i.e.

\[
\sigma_{ap}: B_{ap} \rightarrow G_{ap}, f \rightarrow [f] = (f)_e + N_{ap}
\]

There are two ways to embed \( f \in B_{ap} \) into \( G_{ap} \). Actually, we have the same result.

**Proposition 4.** The following diagram

\[
\begin{array}{ccc}
B_{ap} & \xrightarrow{\sigma_{ap}} & G_{ap} \\
\downarrow{\iota_{ap}} & & \downarrow{\iota_{ap}} \\
B_{ap} & & G_{ap}
\end{array}
\]

is commutative.

**Proof.** Let \( f \in B_{ap} \), we prove that \( (f * \rho_e - f)_e \in N_{ap} \). By Taylor’s formula and the fact that \( \rho \in \Sigma \), we obtain

\[
|f * \rho_e - f|_{L^\infty} \leq \varepsilon^m \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \left| \frac{(-y)^m}{m!} f^{(m)}(x - \theta(x)ey) \rho(y) dy \right|,
\]

then \( \exists C_m > 0 \), such that

\[
|f * \rho_e - f|_{L^\infty} \leq \varepsilon^m C_m \left\| f^{(m)} \right\|_{L^\infty} \left\| \rho \right\|_{L^1}.
\]

The same result is obtained for all derivatives of \( f \). Hence \( (f * \rho_e - f)_e \in N_{ap} \).

The Colombeau algebra of tempered generalized functions on \( \mathbb{C} \) is denoted \( G_T(\mathbb{C}) \), for more details on \( G_T(\mathbb{C}) \) see [3].

**Proposition 5.** Let \( u \in G_{ap} \) and \( F \in G_T(\mathbb{C}) \), then \( F \circ u = [(F \circ u)_e]_e \) is a well defined element of \( G_{ap} \).

**Proof.** It follows from the classical case of composition in the context of Colombeau algebra, we have \( F \circ u \in B_{ap} \) in view of the classical results of composition and convolution.

We recall a characterization of integrable distributions.
DEFINITION 7. A distribution \( v \in \mathcal{D}' \) is said an integrable distribution, denoted \( v \in \mathcal{D}'_{L^1} \), if and only if \( v = \sum_{i \in I} f_i^{(i)} \), where \( f_i \in L^1 \).

PROPOSITION 6. If \( u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap} \) and \( v \in \mathcal{D}'_{L^1} \), then the convolution \( u * v \) defined by \( (u * v)(x) = \left( \int_{\mathbb{R}} u_\varepsilon(x - y) v(y) \, dy \right)_\varepsilon + \mathcal{N}_{ap} \) is a well defined almost periodic generalized function.

Proof. Let \( (u_\varepsilon)_\varepsilon \in \mathcal{M}_{ap} \) be a representative of \( u \), then \( \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \exists C > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \leq \varepsilon_0 |u_\varepsilon|_{k,m} < C \varepsilon^{-m} \), since \( v \in \mathcal{D}'_{L^1} \), then \( v = \sum_{i \in I} f_i^{(i)} \), where \( f_i \in L^1 \). For each \( \varepsilon \in I \), \( u_\varepsilon * v \) is an almost periodic infinitely differentiable function. By Young inequality there exists \( C > 0 \) such that \( \| (u_\varepsilon * v)^{(j)} \|_{L^\infty} \leq C \sum \| f_i^{(i)} \|_{L^1} \| u_\varepsilon^{(j)} \|_{L^\infty} \), consequently \( |u_\varepsilon * v|_{k,m} = O(\varepsilon^{-m}), \varepsilon \rightarrow 0 \), this shows that \( (u_\varepsilon * v)_\varepsilon \in \mathcal{M}_{ap} \). Suppose that \( (w_\varepsilon)_\varepsilon \in \mathcal{M}_{ap} \) is another representative of \( u \), then there exists \( C > 0 \) such that

\[
\| (u_\varepsilon * v - w_\varepsilon * v) \|_{L^\infty} \leq \sum_{i \in I} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \left| (u_\varepsilon - w_\varepsilon)^{(j)}(x - y) \right| f_i^{(i)}(y) \, dy \\
\leq C \sum_{i \in I} \| f_i^{(i)} \|_{L^1} \| (u_\varepsilon - w_\varepsilon)^{(j)} \|_{L^\infty},
\]

as \( (u_\varepsilon - w_\varepsilon)_\varepsilon \in \mathcal{N}_{ap} \), so \( \forall m \in \mathbb{Z}_+, |(u_\varepsilon * v - w_\varepsilon * v)(x)| = O(\varepsilon^m), \varepsilon \rightarrow 0 \). We obtain the same result for \( (u_\varepsilon * v - w_\varepsilon * v) \varepsilon \). Hence \( (u_\varepsilon * v - w_\varepsilon * v)_\varepsilon \in \mathcal{N}_{ap} \). \( \square \)

If \( u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap} \), taking the integral of each element \( u_\varepsilon \) on a compact, we obtain an element of \( \mathcal{C}^1 \).

DEFINITION 8. Let \( u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap} \) and \( x_0 \in \mathbb{R} \), define the primitive of \( u \) by

\[
U(x) = \left( \int_{x_0}^x u_\varepsilon(t) \, dt \right)_\varepsilon + \mathcal{N}[\mathbb{C}].
\]

We give a generalized version of the classical Bohl-Bohr theorem.

PROPOSITION 7. The primitive of an almost periodic generalized function is almost periodic if and only if it is a bounded generalized function.

Proof. We will obtain this result as a corollary of our study of linear ordinary differential equations in the algebra of almost periodic generalized functions. This study is done in the next section. \( \square \)
4. A system of ordinary differential equations in $\mathcal{G}_{ap}$

Consider the linear system of ordinary differential equations

$$u = Au + f,$$

where $f = [(f_1)_\varepsilon] = [(f_{1,1}, \ldots, f_{n,n})_\varepsilon]$ with components almost periodic generalized functions and $A = (a_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(\mathbb{C})$ is a square matrix of complex numbers. The unknown function $u = [(u_{1,1}, \ldots, u_{n,n})_\varepsilon]$.

**Remark 3.** Note that a generalized function $u = [(u_\varepsilon)_\varepsilon] = [(u_{1,1}, \ldots, u_{n,n})_\varepsilon]$ is called bounded (resp. almost periodic) if all components $u_{i,\varepsilon}, 1 \leq i \leq n$, are bounded (resp. almost periodic).

**Definition 9.** A generalized function $u \in (\mathcal{G}_{ap})^n$ is called solution of the differential equation (2) if it satisfies

$$(u_\varepsilon)_\varepsilon - (Au_\varepsilon)_\varepsilon - (f_\varepsilon)_\varepsilon \in \mathcal{N}_{ap},$$

where $(u_\varepsilon)_\varepsilon$ and $(f_\varepsilon)_\varepsilon$ are respectively representatives of $u$ and $f$.

We will need the following classical lemma.

**Lemma 1.** If $A = (a_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(\mathbb{C})$, then there exists $C = (c_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(\mathbb{C})$ such that

(i) $\det C \neq 0$

(ii) $C^{-1}AC$ is a triangular matrix, i.e.

$$C^{-1}AC = \begin{pmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & \lambda_2 & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $\lambda_i, 1 \leq i \leq n$, are the eigenvalues of $A$, and $b_{ij}, 1 \leq i < j \leq n$, are complex numbers.

The first result is a generalization of Bohr-Neugebauer theorem.

**Theorem 3.** The solution $u$ of (2) is an almost periodic generalized function if and only if it is a bounded generalized function.

**Proof.** Since $\mathcal{G}_{ap} \subset \mathcal{G}_{lp}$, it remains to prove that if the solution $u$ of (2) is a bounded generalized function, i.e. $u \in \mathcal{G}_{lp}$, then it is almost periodic, i.e. $u \in \mathcal{G}_{ap}$. Let $C = (c_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(\mathbb{C})$. By taking $u = Cv$, the system of differential equations

$$u(t) = Au(t) + f(t)$$
is equivalent in $\mathcal{G}(\mathbb{R})^n$ to the following linear system of differential equations

$$
\begin{align*}
\dot{v}_1(t) &= \lambda_1 v_1(t) + b_{12} v_2(t) + \cdots + b_{1n} v_n(t) + g_1(t) \\
\dot{v}_2(t) &= \lambda_2 v_2(t) + b_{22} v_2(t) + \cdots + b_{2n} v_n(t) + g_2(t) \\
&\vdots \\
\dot{v}_n(t) &= \lambda_n v_n(t) + g_n(t)
\end{align*}
$$

(4)

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$ and the generalized function $g$ is given by $Cg = f$. Thus, it is sufficient to study the scalar linear ordinary differential equation

$$
\dot{v}(t) = \lambda v(t) + g(t),
$$

(5)

where $g = [(g_0)] \in \mathcal{G}_{ap}$ and $\lambda \in \mathbb{C}$. Any solution of this equation is given by the class of the generalized function

$$
(v_\epsilon(t)) = \left( e^{\lambda t} C_\epsilon + \int_0^t e^{-\lambda s} g_\epsilon(s) \, ds \right) \epsilon,
$$

where $(C_\epsilon)_{\epsilon} \in \tilde{\mathbb{C}}$ is an arbitrary generalized complex constant. From the assumption that the solution $u$ de (2) is a bounded generalized function, i.e. $u \in \mathcal{G}_{le}$, we distinguish the following cases: (1) $\text{Re}\lambda > 0$, (2) $\text{Re}\lambda < 0$, (2) $\text{Re}\lambda = 0$.

(1) If $\text{Re}\lambda > 0$, then we obtain that

$$
v_\epsilon(t) = -\int_t^{+\infty} e^{\lambda(t-s)} g_\epsilon(s) \, ds.
$$

Moreover, we have

$$
\sup_{t \in \mathbb{R}} |v_\epsilon(t + \tau) - v_\epsilon(t)| = \frac{1}{\text{Re}\lambda} \sup_{t \in \mathbb{R}} |g_\epsilon(t + \tau) - g_\epsilon(t)|,
$$

which shows the almost periodicity of $v_\epsilon$. Similarly, we prove that $\forall j \in \mathbb{Z}_+, \nu^{(j)}_\epsilon$ is almost periodic.

(2) The case $\text{Re}\lambda < 0$ is analogous to the case $\text{Re}\lambda > 0$.

(3) If $\text{Re}\lambda = 0$, then $e^{\lambda t} = e^{i\theta t}$, and the generalized solution of (5) is of the form

$$
v_\epsilon(t) = e^{i\theta t} \left( C_\epsilon + \int_0^t e^{-i\theta s} g_\epsilon(s) \, ds \right), \epsilon \in I,
$$

where $C_\epsilon \in \mathcal{Z}_{ap}[\mathbb{C}]$ is an arbitrary generalized complex constant. The almost periodicity of $(v_\epsilon)_\epsilon$ results from the almost periodicity of $(\int_0^t e^{-i\theta s} g_\epsilon(s) \, ds)_\epsilon$, which is a bounded primitive of an almost periodic generalized function, see [2].

\qed
REMARK 4. The previous theorem does not give the existence of solutions.

We have the following result.

THEOREM 4. If the matrix $A$ has eigenvalues $\lambda$ satisfying $\Re \lambda \neq 0$, then there exists a solution $u = [u_{1,\varepsilon}, \ldots, u_{n,\varepsilon}]^t$ bounded generalized function (and then almost periodic) of (2).

Proof. From the proof of theorem (3), it suffices to show that the system (4), where the eigenvalues $\lambda$ satisfy $\Re \lambda \neq 0$ has a bounded generalized solution. The system (4) shows that by defining $v_{n,\varepsilon}$ as

$$v_{n,\varepsilon}(t) = -\int_{-\infty}^{t} e^{\lambda_{n}(t-s)} g_{n,\varepsilon}(s) \, ds, \text{ if } \Re \lambda_{n} > 0$$

or

$$v_{n,\varepsilon}(t) = \int_{t}^{\infty} e^{\lambda_{n}(t-s)} g_{n,\varepsilon}(s) \, ds, \text{ if } \Re \lambda_{n} < 0$$

and then replacing the solution $v_{n,\varepsilon}$ in the equation $(n-1)$ of (4), we obtain for $v_{n-1,\varepsilon}$ an equation of the same type as (5). Since $\Re \lambda_{n-1} \neq 0$, we obtain the same possible cases (6), (7) for $v_{n-1,\varepsilon}$, and so on. Consequently, it remains to show that the generalized solution of the system (4) thus constructed $(v_{1,\varepsilon}, \ldots, v_{n,\varepsilon})$ is a bounded generalized function. Indeed, let $\mu > 0$ be such that

$$|\Re \lambda_{i}| \geq \mu, \ 1 \leq i \leq n.$$ 

From (6), (7), we have $\forall \varepsilon \in I,$

$$\|v_{n,\varepsilon}(t)\|_{L^\infty} \leq \frac{\|g_{n,\varepsilon}\|_{L^\infty}}{\mu}$$

From the equation

$$v_{n-1,\varepsilon}(t) = \lambda_{n-1} v_{n-1,\varepsilon}(t) + b_{n-1} v_{n,\varepsilon}(t) + g_{n-1,\varepsilon}(t),$$

and by giving $v_{n-1,\varepsilon}$ one of the formulas (6), (7), we obtain $\forall \varepsilon \in I,$

$$\|v_{n-1,\varepsilon}\|_{L^\infty} \leq \frac{(\|b_{n-1}\|_{\mu} + \|g_{n,\varepsilon}\|_{L^\infty} + \|g_{n-1,\varepsilon}\|_{L^\infty})}{\mu}$$

and so on, for $i = n-2, n-3, \ldots, 1,$ we obtain $\exists C > 0, \forall \varepsilon \in I,$

$$\max_{1 \leq i \leq n} \|v_{i,\varepsilon}\|_{L^\infty} \leq C \max_{1 \leq j \leq n} \|g_{j,\varepsilon}\|_{L^\infty}.$$
So it clear how to obtain the following estimates

\[ \exists C = C(A) > 0, \forall k \in \mathbb{Z}_+, \forall \epsilon \in I, \left( |(v_1, \ldots, v_n, \epsilon)|_{k, \infty} \right) \leq C \left( |(g_1, \ldots, g_n, \epsilon)|_{k, \infty} \right) \]

which give the result.

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Chikh BOUZAR
Laboratoire d’Analyse Mathématique et Applications, Université d’Oran,
Oran, ALGERIA
e-mail: bouzar@yahoo.com

Mohammed Taha KHALLADI
Département de Mathématiques et Informatique, Université d’Adrar,
Adrar, ALGERIA
e-mail: ktaha2007@yahoo.fr

ON DISTRIBUTIONAL POINT VALUES AND BOUNDARY VALUES OF ANALYTIC FUNCTIONS

Abstract. We give the following version of Fatou’s theorem for distributions that are boundary values of analytic functions. We prove that if \( f \in D'((a, b) \times (0, R)) \) is the distributional limit of the analytic function \( F \) defined in a region of the form \((a, b) \times (0, R)\), if the one sided distributional limit exists, \( f(x_0 + 0) = \gamma \), and if \( f \) is distributionally bounded at \( x = x_0 \), then the Lojasiewicz point value exists, \( f(x_0) = \gamma \) distributionally, and in particular \( F(z) \to \gamma \) as \( z \to x_0 \) in a non-tangential fashion.

1. Introduction

The study of boundary values of analytic functions is an important subject in mathematics. In particular, it plays a vital role in the understanding of generalized functions [1, 2, 4]. As well known, the behavior of an analytic function at the boundary points is intimately connected with the pointwise properties of the boundary generalized function [7, 9, 19, 18, 20] and the study of this interplay has often an Abelian-Tauberian character. There is a vast literature on Abelian and Tauberian theorems for distributions (see the monographs [8, 13, 14, 20] and references therein).

In this article we present sufficient conditions for the existence of Lojasiewicz point values [12] for distributions that are boundary values of analytic functions. The pointwise notions for distributions used in this paper are explained in Section 2. The following result by one of the authors is well known [7]:

Suppose that \( f \in D'([R]) \) is the boundary value of a function \( F \), analytic in the upper half-plane, that is, \( f(x) = F(x + i0) \); if the distributional lateral limits \( f(x_0 \pm 0) = \gamma \) both exist, then \( \gamma^+ = \gamma^- = \gamma \), and the distributional limit \( f(x_0) \) exists and equals \( \gamma \).

On the other hand, the results of [5] imply that there are distributions \( f(x) = F(x + i0) \) for which one distributional limit exists but not the other. In Theorem 2 we show that the existence of one of the distributional limits may be removed from the previous statement if an additional Tauberian-type condition is assumed, namely, if the distribution is distributionally bounded at the point. We also show that when the distribution \( f \) is a bounded function near the point, then the distributional point value is of order 1. Furthermore, we give a general result of this kind for analytic functions that have distributional limits on a contour.

As an immediate consequence of our results, we shall obtain the following version of Fatou’s theorem [11, 15] for distributions that are boundary values of analytic functions.

Corollary 1. Let \( F \) be analytic in a rectangular region of the form \((a, b) \times \)
Suppose that \( f(x) = \lim_{y \to 0^+} F(x + iy) \) in \( \mathcal{D}'(a, b) \), that \( f \) is a bounded function near \( x_0 \in (a, b) \), and that the following average lateral limit exists
\[
\lim_{x \to x_0^+} \frac{1}{(x - x_0)} \int_{x_0}^{x} f(t) \, dt = \gamma .
\]

Then,
\[
\lim_{z \to x_0^+} F(z) = \gamma \text{ (angularly)}.
\]

Finally, we remark that Theorem 4 below generalizes some of our Tauberian results from [19].

2. Preliminaries

We explain in this section several pointwise notions for distributions. There are several equivalent ways to introduce them. We start with the useful approach from [3]. Define the operator \( \mu_a \) on locally integrable complex valued functions in \( \mathbb{R} \) as
\[
\mu_a \{ f(t) ; x \} = \frac{1}{x - a} \int_{a}^{x} f(t) \, dt , \ x \neq a ,
\]
while the operator \( \partial_a \) is the inverse of \( \mu_a \),
\[
\partial_a (g) = ((x - a) g(x))^\prime .
\]
Suppose first that \( f_0 = f \) is real. Then if it is bounded near \( x = a \), we can define
\[
\overline{f_0}(a) = \limsup_{x \to a} f(x) , \ \ f_0(a) = \liminf_{x \to a} f(x) .
\]
Then \( f_1 = \mu_a (f) \) will be likewise bounded near \( x = a \) and actually
\[
f_0(a) \leq f_1(a) \leq \overline{f_1}(a) \leq \overline{f_0}(a)
\]
and, in particular, if \( f(a) = f_0(a) \) exists, then \( f_1(a) \) also exists and \( f_1(a) = f_0(a) \).

**Definition 1.** A distribution \( f \in \mathcal{D}'(\mathbb{R}) \) is called distributionally bounded at \( x = a \) if there exist \( n \in \mathbb{N} \) and \( f_n \in \mathcal{D}'(\mathbb{R}) \), continuous and bounded in a pointed neighborhood \((a - \varepsilon, a) \cup (a, a + \varepsilon)\) of \( a \), such that \( f = \partial_a^n f_n \).

If \( f_0 \) is distributionally bounded at \( x = a \), then there exists a unique distributionally bounded distribution near \( x = a \), \( f_1 \), with \( f_0 = \partial_a f_1 \). Therefore, \( \partial_a \) and \( \mu_a \) are isomorphisms of the space of distributionally bounded distributions near \( x = a \). Given \( f_0 \) we can form a sequence of distributionally bounded distributions \( \{ f_n \}_{n=-\infty}^{\infty} \) with \( f_n = \partial_a f_{n+1} \) for each \( n \in \mathbb{Z} \).

We say that \( f \) has the distributional point value \( \gamma \) in the sense of Lojasiewicz [12, 10] and write
\[
f(a) = \gamma \text{ (L)} ,
\]
if there exists \( n \in \mathbb{N} \), the order of the point value, such that \( f_n \) is continuous near \( x = a \) and \( f_n(a) = \gamma \).

It can be shown [3, 8, 12, 14] that \( f(a) = \gamma \ (L) \) if and only if
\[
\lim_{\varepsilon \to 0^+} f(a + \varepsilon x) = \gamma,
\]
distributionally, that is, if and only if
\[
\lim_{\varepsilon \to 0^+} \langle f(a + \varepsilon x), \phi(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi(x) \, dx,
\]
for each \( \phi \in \mathcal{D}(\mathbb{R}) \). On the other hand, if \( f \) is distributionally bounded at \( x = a \) then \( \langle f(a + \varepsilon x), \phi(x) \rangle \) is bounded as \( \varepsilon \to 0 \).

We can also consider distributional lateral limits [12, 17]. We say that the distributional lateral limit \( f(a + 0) \ (L) \) as \( x \to a \) from the right exists and equals \( \gamma \), and write
\[
f(a + 0) = \gamma \ (L),
\]
if (1) holds for all \( \phi \in \mathcal{D}(\mathbb{R}) \) with support contained in \((0, \infty)\). The distributional lateral limit from the left \( f(a - 0) \ (L) \) is defined in a similar fashion.

Observe also that if \( f = \partial_\alpha f_n \), and \( f_n \) is bounded near \( x = a \), then \( f(a + 0) \ (L) \) exists, and equals \( \gamma \), if and only if \( f_n(a + 0) = \gamma \ (L) \).

These notions have straightforward extensions to distributions defined in a smooth contour of the complex plane. A natural extension of these pointwise notions for distributions is the so called quasiasymptotic behavior of distributions, explained, e.g., in [14, 16, 20].

3. **Boundary values and distributional point values**

We shall need the following well known fact [1]. We shall use the notation \( \mathbb{H} \) for the half plane \( \{ z \in \mathbb{C} : \text{Im} \, z > 0 \} \).

**Lemma 1.** Let \( F \) be analytic in the half plane \( \mathbb{H} \), and suppose that the distributional limit \( f(x) = F(x + i0) \) exists in \( \mathcal{D}'(\mathbb{R}) \). Suppose that there exists an open, non-empty interval \( I \) such that \( f \) is equal to the constant \( \gamma \) in \( I \). Then \( f = \gamma \) and \( F = \gamma \).

Actually using the theorem of Privalov [15, Cor 6.14] it is easy to see that if \( F \) is analytic in the half plane \( \mathbb{H} \), \( f(x) = F(x + i0) \) exists in \( \mathcal{D}'(\mathbb{R}) \), and there exists a subset \( X \subset \mathbb{R} \) of non-zero measure such that the distributional point value \( f(x_0) \) exists and equals \( \gamma \) if \( x_0 \in X \), then \( f = \gamma \) and \( F = \gamma \).

Our first result is for bounded analytic functions.

**Theorem 1.** Let \( F \) be analytic and bounded in a rectangular region of the form \( (a, b) \times (0, R) \). Set \( f(x) = \lim_{y \to 0^+} F(x + iy) \) in \( \mathcal{D}'(a,b) \), so that \( f \in L^\infty(a,b) \). Let \( x_0 \in (a,b) \) be such that
\[
f(x_0 + 0) = \gamma \ (L)
\]
exists. Then the distributional point value

\[ f(x_0) = \gamma \quad (L) \]

also exists. In fact, the point value is of the first order, and thus

\[ \lim_{x \to x_0} \frac{1}{x - x_0} \int_{x_0}^{x} f(t) \, dt = \gamma. \]

**Proof.** We shall first show that it is enough to prove the result if the rectangular region is the upper half-plane \( \mathbb{H} \). Indeed, let \( C \) be a smooth simple closed curve contained in \((a, b) \times [0, R]\) such that \( C \cap (a, b) = [x_0 - \eta, x_0 + \eta] \), and which is symmetric with respect to the line \( \text{Re} z = x_0 \). Let \( \phi \) be a conformal bijection from \( \mathbb{H} \) to the region enclosed by \( C \) such that the image of the line \( \text{Re} z = x_0 \) is contained in \( \text{Re} z = x_0 \), so that, in particular, \( \phi(x_0) = x_0 \). Then (2)–(4) hold if and only if the corresponding equations hold for \( f \circ \phi \).

Therefore we may assume that \( a = -\infty \), and \( b = R = \infty \). In this case, \( f \) belongs to the Hardy space \( H^\infty \), the closed subspace of \( L^\infty(\mathbb{R}) \) consisting of the boundary values of bounded analytic functions on \( \mathbb{H} \). Let \( f_\varepsilon(x) = f(x_0 + \varepsilon x) \). Clearly, the set \( \{ f_\varepsilon : \varepsilon > 0 \} \) is weak* bounded (as a subset of the dual space \( (L^1(\mathbb{R}))' = L^\infty(\mathbb{R}) \)) and, consequently, a relatively weak* compact set. If \( \{ \varepsilon_n \}_{n=0}^\infty \) is a sequence of positive numbers with \( \varepsilon_n \to 0 \) such that the sequence \( \{ f_{\varepsilon_n} \}_{n=0}^\infty \) is weak* convergent to \( g \in L^\infty(\mathbb{R}) \), then \( g \equiv \gamma \), since \( g \in H^\infty \), and \( g(x) = \gamma \) for \( x > 0 \). In fact, the condition (2) means that

\[
\int_0^\infty g(x) \psi(x) \, dx = \lim_{n \to \infty} \int_0^\infty f_{\varepsilon_n}(x) \psi(x) \, dx = \gamma \int_0^\infty \psi(x) \, dx,
\]

for all \( \psi \in \mathcal{D}(0, \infty) \), which yields the claim. Since any sequence \( \{ f_{\varepsilon_n} \}_{n=0}^\infty \) with \( \varepsilon_n \to 0 \) has a weak* convergent subsequence, and since that subsequence converges to the constant function \( \gamma \), we conclude that \( f_\varepsilon \to \gamma \) in the weak* topology of \( L^\infty(\mathbb{R}) \). Furthermore, (4) follows by taking \( x = x_0 + \varepsilon \) and \( \phi(t) = \chi_{[0,1]}(t) \), the characteristic function of the unit interval, in the limit \( \lim_{\varepsilon \to 0} \langle f_\varepsilon(t), \phi(t) \rangle = \gamma \int_{-\infty}^\infty \phi(t) \, dt \).

We can now prove our main result, a distributional extension of Theorem 1.

**Theorem 2.** Let \( F \) be analytic in a rectangular region of the form \((a, b) \times (0, R)\). Suppose \( f(x) = \lim_{y \to 0^+} F(x + iy) \) in the space \( \mathcal{D}'(a, b) \). Let \( x_0 \in (a, b) \) such that \( f(x_0 + 0) = \gamma \quad (L) \). If \( f \) is distributionally bounded at \( x = x_0 \) then \( f(x_0) = \gamma \quad (L) \). Furthermore, \( F(z) \to \gamma \) as \( z \to x_0 \) in an angular fashion.

**Proof.** There exists \( n \in \mathbb{N} \) and a function \( f_n \) bounded in a neighborhood of \( x_0 \) such that \( f = \partial_n^0 f_n \); notice that \( f(x_0) = \gamma \quad (L) \) if and only if \( f_n(x_0) = \gamma \quad (L) \). But \( f_n(x) = F_n(x + i0) \) distributionally, where \( F_n \) is analytic in \((a, b) \times (0, R)\); here \( F_n \) is the only angularly bounded solution of \( F(z) = \partial_n^0 F_n(z) \) (derivatives with respect to \( z \)). Clearly, \( f_n(x) = F_n(x + i0) \). Since \( f_n \) is bounded near \( x = x_0 \), \( F_n \) is also bounded in a rectangular region of the form \((a_1, b_1) \times (0, R_1)\), where \( x_0 \in (a_1, b_1) \). Clearly \( f_n(x_0 + 0) = \gamma \quad (L) \), so the Theorem 1 yields \( f_n(x_0) = \gamma \quad (L) \), as required. Finally, the fact that \( F(z) \to \gamma \) as \( z \to x_0 \), angularly, is a consequence of the existence of the distributional point value, as shown in [6, 16].
Observe that in general the result (4) does not follow if $f$ is not bounded but just distributionally bounded near $x_0$.

We may use a conformal map to obtain the following general form of the Theorem 2.

**Theorem 3.** Let $C$ be a smooth part of the boundary $\delta\Omega$ of a region $\Omega$ of the complex plane. Let $F$ be analytic in $\Omega$, and suppose that $f \in \mathcal{D}'(\mathbb{C})$ is the distributional boundary limit of $F$. Let $z_0 \in C$ and suppose that the distributional lateral limit $\lim_{z \to z_0} f(z) = \gamma$ exists and $f$ is distributionally bounded at $z = z_0$, then $f(z_0) = \gamma$ and $F(z)$ has non-tangential limit $\gamma$ at the boundary point $z_0$.

We also immediately obtain the following Tauberian theorem. As mentioned at the Introduction, it generalizes some Tauberian results by the authors from [19].

**Theorem 4.** Let $F$ be analytic in a rectangular region of the form $(a,b) \times (0,R)$. Suppose $f(x) = \lim_{y \to 0^+} F(x + iy)$ in the space $\mathcal{D}'((a,b))$. Let $x_0 \in (a,b)$ such that the distributional limit $\lim_{y \to 0^+} F(x_0 + iy) = \gamma$ exists. If $f$ is distributionally bounded at $x = x_0$, then $f(x_0) = \gamma$ and the angular (ordinary) limit exists: $\lim_{z \to x_0} F(z) = \gamma$.

*Proof.* If we consider the curve $C$ to be the union of the segments $(a,x_0]$ and $[x_0,iR)$, then the distributional lateral limit of the boundary value of $F$ on $C$ exists and equals $\gamma$ as we approach $x_0$ from the right along $C$ and so the Theorem 3 yields that the distributional limit from the left, which is nothing but $f(x_0 - 0)$, also exists and equals $\gamma$. Then the Theorem 2 gives us that $f(x_0) = \gamma$. The existence of the angular limit of $F(z)$ as $z \to x_0$ then follows.

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Ricardo ESTRADA  
Department of Mathematics, Louisiana State University  
Baton Rouge, LA 70803, USA  
e-mail: restrada@math.lsu.edu

Jasson VINDAS  
Department of Mathematics, Ghent University  
Krijgslaan 281, B 9000 Gent, BELGIUM  
e-mail: jvindas@cage.ugent.be

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Ghislain R. Franssens

THE CONVOLUTION AND MULTIPLICATION OF
ONE-DIMENSIONAL ASSOCIATED HOMOGENEOUS
DISTRIBUTIONS

Abstract. The set of Associated Homogeneous Distributions (AHDs) with support in $\mathbb{R}$ is an important subset of the tempered distributions, since it contains the majority of the (one-dimensional) distributions typically encountered in physics applications. In previous work of the author, a convolution and multiplication product for AHDs on $\mathbb{R}$ was defined and fully investigated. The aim of this paper is to give an easily readable introduction to these new distributional algebras.

The constructed algebras are internal to Schwartz’ theory of distributions and, when one restricts to AHDs, provide a simple alternative for any of the larger generalized function algebras, now used in non-linear models. Our approach belongs to the same class as certain methods of renormalization used in quantum field theory and which are known in the distributional literature as multi-valued methods. Products of AHDs on $\mathbb{R}$, based on our definition, are generally multi-valued only at critical degrees of homogeneity. Unlike other definitions proposed earlier in this class of methods, the multi-valuedness of our products is canonical in the sense that it involves at most one arbitrary constant.

As a simple example of the convolution and multiplication products that are obtained by our method, the product tables of HDs of integer degree are presented.

Keywords: Generalized function, Associated homogeneous distribution, Convolution, Multiplication.

1. Introduction

Homogeneous Distributions (HDs) are the distributional analogue of homogeneous functions, such as $|x|^z : \mathbb{R} \to \mathbb{C}$, which is homogeneous with complex degree $z$. Associated to homogeneous functions are power-log functions, which arise when taking the derivative with respect to the degree of homogeneity $z$. The set of Associated Homogeneous Distributions (AHDs) with support in the line $\mathbb{R}$, and which we denote by $\mathcal{H}'(\mathbb{R})$, is the distributional analogue of the set of power-log functions with domain in $\mathbb{R}$. For compactness of notation, we will drop $(\mathbb{R})$ in $\mathcal{H}'(\mathbb{R})$ and other sets.

The set $\mathcal{H}'$ is an interesting and important subset of the distributions of slow growth (or tempered distributions), $S'$, [22], [25]. For instance, $\mathcal{H}'$ is, just as $S'$, closed under Fourier transformation. In addition, $\mathcal{H}'$ contains the majority of the (one-dimensional) distributions one typically encounters in physics applications, such as the delta distribution $\delta$, the eta distribution $\eta \triangleq \frac{1}{\pi} x^{-1}$ (a normalized Cauchy’s principal value $\text{Pv} \frac{1}{x}$), the Heaviside step distributions $1_+$, several so called pseudo-functions generated by taking Hadamard’s finite part of certain divergent integrals, associated Riesz kernels, generalized Heisenberg distributions, all their generalized derivatives and primitives, and many familiar others, [16].
One-dimensional AHDs were first considered in [14]. The various (and sometimes inconsistent) definitions under which they have been introduced in the literature were closely examined in [23]. In a series of papers, [11]–[12], the present author undertook a detailed study of the set $\mathcal{H}'$, resulting in the construction of a convolution product and an isomorphic multiplication product. The multiplication product for AHDs on $R$ provides a non-trivial example of how multiplication can be defined for a useful subset of distributions, containing a derivation and the delta distribution, and how this construction is influenced by Schwartz’ “impossibility theorem”, [21].

Our definition results in a convolution product and a multiplication product that is non-commutative and non-associative. However, these products deviate from commutativity and associativity only at critical degrees of homogeneity and in a minimal and interesting way, as explained below. The resulting product structures $(\mathcal{H}', *)$ and $(\mathcal{H}', .)$ are derivation magmas with identity.

Our approach towards defining distributional products for AHDs on $R$ belongs to what in the distribution literature is called the class of multi-valued products, [18]. It remains entirely within the scope of Schwartz’ distribution theory and thus provides a simpler alternative for the multiplication of AHDs on $R$ than the larger generalized function algebras, such as e.g., [3]. It is related, but not identical to and in a sense more natural than, certain methods of renormalization in quantum field theory, such as in [15], [1], [24], [17].

In the following sections, we start with giving the definition for AHDs on $R$ and list some of their immediate properties. Then, we introduce a few distributional concepts that were found useful in the treatment of AHDs on $R$. Thereafter, we proceed to show how a closed convolution product on $\mathcal{H}'$ can be defined, first for non-critical products and then for critical products. Based on the generalized convolution theorem, the definition of a multiplication product on $\mathcal{H}'$ is then stated. We discuss the non-commutativity of critical products and the non-associativity of critical triple products and exhibit the elegant nature of this lack of commutativity and associativity.

In order to guide the reader who wishes to consult or verify the underlying calculations we provide the following road map to the previous papers:

(i) Notation, definitions, properties and an extensive list of properties of basis AHDs on $R$ are derived and collected in [11].

(ii) Structure theorems for AHDs on $R$, being complex holomorphic in their degree of homogeneity in some $\Omega \subseteq \mathbb{C}$, are proved in [7].

(iii) The construction of the convolution product for AHDs on $R$, in case the resulting degree of homogeneity is not a natural number, called a non-critical product, is developed in [5]. Here is also proved the associativity of non-critical triples under convolution.

(iv) The completion of the construction of the convolution product for AHDs on $R$, in case the resulting degree of homogeneity is a natural number, called a critical product, is given in terms of a functional extension process in [6]. Combining the results from [5] and [6] then shows that $(\mathcal{H}', *)$ is closed.

(v) The general convolution product formula for AHDs on $R$ is derived in [8].
Here is also proved the particular form of the non-associativity of critical triple convolution products.

(vi) The general multiplication product formula for AHDs on $R$ is derived in [10]. Here is also proved the particular form of the non-associativity of critical triple multiplication products.

(vii) The structures $(\mathcal{H}', +)$ and $(\mathcal{H}', .)$ contain various interesting abstract algebraic substructures. These are investigated in [12]. The most important substructure of $(\mathcal{H}', +)$ is a particular subgroup, which contains AHDs that, when used as kernels of convolution operators, allow to define integration operators of complex degree over the whole line $R$.

We use the notation and definitions introduced in [11].

2. Associated Homogeneous Distributions on $R$

2.1. Definition

A distribution $f_0^z \in \mathcal{D}'$ is called a (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ iff it satisfies for any $r > 0$,

\[(1) \quad \langle f_0^z, \varphi(x/r) \rangle = r^{z+1} \langle f_0^z, \varphi(x) \rangle, \forall \varphi \in \mathcal{D},\]

or, using [11, eq. (63)], \( (f_0^z)_{(r)} = r^z (f_0^z)_{(s)} \). A homogeneous distribution is also called an associated homogeneous distribution of order $m = 0$.

A distribution $f_m^z \in \mathcal{D}'$ is called an associated (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ and order of association $m \in \mathbb{Z}_+$, iff there exists a sequence of associated homogeneous distributions $f_{m-l}^z$ of degree of homogeneity $z$ and associated order $m-l$, $\forall l \in \mathbb{Z}_{[1,m]}$, not depending on $r$ and with $f_0^z \neq 0$, satisfying,

\[(2) \quad \langle f_m^z, \varphi(x/r) \rangle = r^{z+1} \left( f_m^z + \sum_{l=1}^{m} \frac{(\ln r)^{l}}{l!} f_{m-l}^z, \varphi(x) \right), \forall \varphi \in \mathcal{D}.\]

Differentiate (2) $l$ times with respect to $r$, put $r = 1$, use the definitions [11, eqs. (29) and (32)] of the generalized derivatives, and [11, eq. (38)]. This yields the system, $\forall l \in \mathbb{Z}_{[1,m+1]}$,

\[(3) \quad X_l f_m^z = f_{m-l}^z,\]

wherein we set $f_{-1}^z = 0$ and

\[(4) \quad X_z \triangleq X \cdot D - z \text{Id},\]

wherein $X \cdot D$ is the generalized Euler operator and $\text{Id}$ the identity operator. The system (3) can be used as an equivalent for definition (2) and generalizes Euler’s theorem on homogeneous functions to AHDs on $R$. 


2.2. General properties

The following properties of AHDs on $R$ easily follow from (2) or (3).

(i) AHDs of the same order $m$, but of different degrees $\{z_1, \ldots, z_k\}$, are linearly independent.

(ii) AHDs of the same degree $z$, but of different orders $\{m_1, \ldots, m_k\}$, are linearly independent. Any such linear combination is again an AHD of degree $z$ and of order $m \leq \max \{m_1, \ldots, m_k\}$.

(iii) Let $f^m_z$ be an AHD of order $m$ which is complex analytic in its degree $z \in \Omega \subset \mathbb{C}$. If $f^m_z$ has an analytic extension $(f^m_z)_{a,e}$ to a region $\Omega_1 \supset \Omega$, then $(f^m_z)_{a,e}$ is a unique AHD of degree $z$ and order $m$ due to the uniqueness of the process of analytic continuation, [14, p. 150].

(iv) AHDs are distributions of slow growth: $\mathcal{H}^r \subset \mathcal{S}^\prime$. A proof for homogeneous distributions can be found in [4, pp. 154–155]. By using in addition [7, Theorem 1], the fact that $D_z f^m_z$ is associated of order $m + 1$ and of the same degree $z$, linearity and induction, it follows that any AHD is a distribution of slow growth.

(v.1) If $f^p_a g^q_b$ exists (as a distribution) and neither $f^p_a$ nor $g^q_b$ is a zero divisor, then this multiplication product is associated of order $m = p + q$ and of degree $a + b$. Under these conditions, an injective multiplication operator with a homogeneous kernel of degree 0 is a map from $\mathcal{H}^m \rightarrow \mathcal{H}^m_m$. In particular, the parity reversal transformation (i.e., the multiplication operator $S \triangleq -i \text{sgn}$., see [11, eq. (86)]) then preserves the degree of homogeneity and order of association.

(v.2) If $f^{p-1}_b g^{q-1}_b$ exists (as a distribution) and neither $f^{p-1}_b$ nor $g^{q-1}_b$ is a zero divisor, then this convolution product is associated of order $m = p + q$ and of degree $a + b - 1$. Under these conditions, an injective convolution operator with a homogeneous kernel of degree $-1$ is a map from $\mathcal{H}^m \rightarrow \mathcal{H}^m_m$. In particular, the Hilbert transformation (i.e., the convolution operator $H \triangleq \eta^*$, see [11, eq. (172)]) then preserves the degree of homogeneity and order of association.

(vi) The Fourier transformation $\mathcal{F}$ maps any AHD to an AHD, such that:

(a) the order of association $m$ is preserved,

(b) the degree of homogeneity $z$ is mapped to $-(z + 1)$,

(c) the parity of the distribution is preserved.

A more detailed overview of the properties of AHDs on $R$ and many specific properties of several important basis AHDs on $R$ can be found in [11].

3. Method

3.1. Preliminaries

DEFINITION 1. A partial distribution is a linear and sequentially continuous functional that is only defined on a proper subset $\mathcal{D}_r \subset \mathcal{D}$. Similarly, a partial tempered distribution is a linear and sequentially continuous functional that is only defined on a proper subset $\mathcal{S}_r \subset \mathcal{S}$.
**Definition 2.** An extension \( f_e \) from \( D_e \) to \( D \) of a partial distribution \( f \) is a distribution \( f_e \in D' \), defined \( \forall \psi \in D \), such that \( \langle f_e, \psi \rangle = \langle f, \psi \rangle \), \( \forall \psi \in D_e \subset D \). Similarly, an extension \( f_e \) from \( S_e \) to \( S \) of the partial tempered distribution \( f \) is a tempered distribution \( f_e \in S' \), defined \( \forall \psi \in S \), such that \( \langle f_e, \psi \rangle = \langle f, \psi \rangle \), \( \forall \psi \in S_e \subset S \).

\( D \) is a sequentially complete, locally convex, Hausdorff topological linear space, [20, p. 152], [2, pp. 427–431]. Schwartz’ space \( S \) is a Fréchet space, [20, p. 184], [13, Appendix], and also a Fréchet space is locally convex [20, p. 9]. Since \( D (S) \) is locally convex the continuous extension version of the Hahn-Banach theorem applies, [19, p. 56]. This theorem ensures that an extension \( f_e \) of the partial distribution \( f \), only defined on \( D_e \subset D (S_e \subset S) \), exists as a sequentially continuous linear functional on \( D (S) \) and that \( \langle f_e, \psi \rangle = \langle f, \psi \rangle \), \( \forall \psi \in D_e (S_e) \), [20, p. 61]. If \( D_e (S_e) \) is dense in \( D (S) \), then the extension \( f_e \) is unique. Otherwise, an extension \( f_e \) may or may not be unique, [20, p. 56], [2, p. 424]. Let \( D'_e (S'_e) \) denote the continuous dual of \( D_e (S_e) \). The subset of \( D'_e (S'_e) \) which maps \( D_e (S_e) \) to zero is called the annihilator of \( D_e (S_e) \) and denoted by \( D_e^\perp (S'_e) \). Any two extensions \( f_{e,1} \) and \( f_{e,2} \) from \( D_e (S_e) \) to \( D (S) \) differ by a generalized function \( g \in D_e^\perp (S'_e) \).

The classical process of the regularization of divergent integrals, as given in [14] and which goes back to Hadamard, is here placed in the more general context of a functional extension process, justified by the Hahn-Banach theorem. We will refer to this method as “extension of partial distributions”. The advantage of regarding regularization as an extension process is that it clearly exhibits the unavoidable non-uniqueness of regularization.

The Hahn-Banach theorem does not say how an extension is to be constructed. A natural extension process however is the following, inspired by the regularization of divergent integrals, [14, p. 10 and p. 45]. This procedure consists in the construction of a projection operator \( T : D \rightarrow D_e \) and to replace an integral, convergent \( \forall \psi \in D_e \) but divergent for test functions \( \psi \in D \backslash D_e \),

\[
\int_{-\infty}^{+\infty} f(x) \psi(x) \, dx,
\]

by the integral

\[
\int_{-\infty}^{+\infty} f(x) (T \psi)(x) \, dx,
\]

which now by construction is convergent \( \forall \psi \in D \). The integral (6) is then said to be a regularization of the integral (5). Equivalently, we could say that (6) defines a distribution \( f_e \) which is an extension of the partial distribution \( f \) defined by (5). The non-uniqueness of this particular regularization method stems from the non-uniqueness of the projection operator \( T \). Particular choices for the projection operator \( T \) give also rise to: (i) analytic continuations of AHDs on \( R \), see [11, eq. (100)], and (ii) the special extensions, denoted with subscript 0, which coincide with M. Riesz’ finite analytic part. For details, see [11].
DEFINITION 3. The equivalence class of extensions \([f_e]\) from \(S_r\) to \(S\) of a partial distribution \(f \in \mathcal{H}'\) is the set of all extensions \(f_e \in \mathcal{H}'\) from \(S_r\) to \(S\) of \(f\), together with the equivalence relation given in (7) or in (8).

In the theory of AHDs on \(R\), regularization is only required at integer degrees of homogeneity. In addition, it turns out that partial distributions in \(\mathcal{H}'\) are defined on only (i) \(S_{\{k\}}\) (the subspace of \(S\) whose members have zero \(k\)-th order moment) or (ii) \(S_{\{-l\}}\) (the subspace of \(S\) whose members have zero \([-l+1]\)-th order derivative at the origin), with \(k \in \mathbb{N}\) and \(l \in \mathbb{Z}_+\). Therefore, only the following two equivalence classes, \(\sim_{\mathbb{N}}\) and \(\sim_{\mathbb{Z}_-}\), of extensions in \(\mathcal{H}'\) are needed,

\[
(7) \quad [f^k] \triangleq \{ f^k + cx^k \in \mathcal{H}' : f^k + cx^k \sim_{\mathbb{N}} f^k, \forall c \in \mathbb{C} \}, \forall k \in \mathbb{N},
\]

\[
(8) \quad [f^{-l}] \triangleq \{ f^{-l} + c\delta^{(l-1)} \in \mathcal{H}' : f^{-l} + c\delta^{(l-1)} \sim_{\mathbb{Z}_-} f^{-l}, \forall c \in \mathbb{C} \}, \forall l \in \mathbb{Z}_+.
\]

Finally, we will need the following.

DEFINITION 4. The convolution product of any two AHDs on \(R\) of degrees \(a - 1\) and \(b - 1\) is called a critical (convolution) product, iff the resulting degree \(a + b - 1 \triangleq k \in \mathbb{N}\).

The multiplication product of any two AHDs on \(R\) of degrees \(a\) and \(b\) is called a critical (multiplication) product, iff the resulting degree \(a + b \triangleq -l \in \mathbb{Z}_-\).

3.2. Convolution

Let \(\mathcal{D}'_R\) denote the distributions based on \(R\) with support bounded on the left and \(\mathcal{D}'_L\) denote the distributions based on \(R\) with support bounded on the right, [22, vol II, p. 28-30]. One of our structure theorems (the normalized half-lines representation) states that any AHD on \(R\) is the sum of an AHD in \(\mathcal{D}'_L\) and an AHD in \(\mathcal{D}'_R\), [7, Theorem 1].

Our definition of the convolution product of AHDs on \(R\) then comprises three cases.

Any degree

Case 1. The factors in the convolution product have one-sided support, bounded at the same side. In this case we use the standard definition, involving the direct product, e.g., [25, p. 123, eq. (2) and Theorem 5.4-1].

Non-critical degree

Case 2. The factors in the convolution product have one-sided support, bounded at different sides. In this case, the convolution \(f \ast g\), with \(f \in \mathcal{D}'_R\) and \(g \in \mathcal{D}'_L\), can not straightforwardly be defined in terms of a direct product, because \(\text{supp}(f) \cap \text{supp}(g \in \mathcal{D}(R^2))\) is generally non-compact. In [5] however it is shown that if \(f^a\) and \(g^b\) are AHDs on \(R\) with degrees of homogeneity \(a - 1\) and \(b - 1\), whose support is...
bounded on different sides, one can still construct their convolution, provided the resulting degree of homogeneity \(a + b - 1\) is not a natural number. This is a consequence of the existence of the mixed-sided convolution product \(D^m_n \Phi^\pm \ast D^m_n \Phi^\pm\), \(\forall m, n \in \mathbb{N}\) and \(\forall a, b \in \mathbb{C}\) such that \(a + b - 1 \notin \mathbb{N}\). Here are the normalized half-line basis AHDs \(D^m_n \Phi^\pm \in \mathcal{D}'_1\) and \(D^m_n \Phi^\pm \in \mathcal{D}'_2\) defined as \(\Phi^\pm_a \triangleq x_+^{a-1}/\Gamma(z)\), [14], [11].

Let \(T \triangleq \{(a, b) \in \mathbb{C}^2 : 0 < \text{Re}(a), 0 < \text{Re}(b) \text{ and } \text{Re}(a + b) < 1\}\). In \(T\), \(D^m_n \Phi^\pm\) and \(D^m_n \Phi^b\) are regular distributions. A direct calculation, using the standard convolution integral, shows that \(D^m_n \Phi^\pm \ast D^m_n \Phi^\pm\) exists in \(T\) and is also a regular distribution. Let \(R \triangleq \{(a, b) \in \mathbb{C}^2 : a + b - 1 \notin \mathbb{N}\}\). The distribution \(D^m_n \Phi^\pm \ast D^m_n \Phi^\pm\) is subsequently defined in \(R\) by analytic continuation. Its explicit form was derived in [5].

**Critical degree**

**Case 3.** Let \(f^{a-1}\) and \(g^{b-1}\) be AHDs on \(R\) of degree \(a - 1\) and \(b - 1\), respectively, and \(a + b - 1 = k \in \mathbb{N}\). The standard technique of the preceding subsection, when applied to a critical convolution product of AHDs, does not generate a distribution. It was shown in [6] however that:

(i) The convolution product \(f^{a-1} \ast g^{b-1}\) is now in general a partial distribution \((f^{a-1} \ast g^{b-1})_{p.d.}\), defined only on \(S_{(k)}\).

(ii) It is natural to consider as a particular extension of \((f^{a-1} \ast g^{b-1})_{p.d.}\) from \(S_{(k)}\) to \(S\), the analytic finite part. However, this finite part, being a limit in \(\mathbb{C}^2\), was also found to be non-unique in general. Fortunately, a detailed investigation revealed that this non-uniqueness also vanishes on \(S_{(k)}\). This can be read off from the explicit expression of the analytic finite part, \((f^{a-1} \ast g^{b-1})_0\), given in [6, eq. (27)].

Now, any critical convolution product of AHDs on \(R\), of a pair \(f^{a-1}\) and \(g^{b-1}\) with \(a + b - 1 = k \in \mathbb{N}\) and which results in the partial distribution \((f^{a-1} \ast g^{b-1})_{p.d.}\), is defined as any extension of \((f^{a-1} \ast g^{b-1})_{p.d.}\) from \(S_{(k)}\) to \(S\). For instance, \(x^k \ast x^l \triangleq (x^k \ast x^l)_{p.d.} = 0 + cx^{k+l-1}\), with \(c \in \mathbb{C}\) arbitrary, \(\forall k, l \in \mathbb{N}\).

The general formula for the convolution product of two arbitrary AHDs on \(R\) is given in [8, Theorem 6].

**3.3. Multiplication**

Let \(f^{a}\) and \(g^{b}\) be AHDs on \(R\) of degree \(a\) and \(b\), respectively. Then, \((f^{-1}_{2l} f^{a})\) and \((f^{-1}_{2l} g^{b})\) are also AHDs on \(R\). Owing to the results from [5] and [6], \((f^{a-1}_{2l} f^{a}) \ast (f^{b-1}_{2l} g^{b})\) exists, so we define

\[
\int f^a g^b \triangleq f^{a-1}_{2l} \left( (f^{-1}_{2l} f^{a}) \ast (f^{-1}_{2l} g^{b}) \right),
\]

and \(f^a g^b\) is again an AHD on \(R\) of degree \(a + b\).

The multiplication product \(f^a g^b\) in the critical case, i.e., when \(a + b = -l \in \mathbb{Z}_-\), is a partial distribution only defined on \(S_{(-1)}\). Any critical multiplication product
of AHDs on $R$, $f^a$ and $f^b$, which results in a partial distribution $f^a \cdot f^b$, is then defined as any extension from $\mathcal{S}_{[-1]}$ to $\mathcal{S}$ of the partial distribution $f^a \cdot f^b$. For instance, $\delta^{(k-1)} \cdot \delta^{(l-1)} \triangleq (\delta^{(k-1)} \cdot \delta^{(l-1)}) = 0 + c\delta^{(k+l-1)}$, with $c \in \mathbb{C}$ arbitrary, $\forall k, l \in \mathbb{Z}_+$.

The general formula for the multiplication product of two arbitrary AHDs on $R$ is given in [10, Theorem 6].

Having a multiplication product for AHDs on $R$, now also allows us to judge the appropriateness of the suggestive notation commonly used for these distributions. For instance, with $l, m \in \mathbb{Z}_+$, the distributions $x^{-l}_a \ln^m |x|$ are often tacitly interpreted as the distributional multiplication products $x^{-l}_a \ln^m |x|$, which is correct in this case.

On the other hand, the notation $(x \pm i0)^{-l} \ln^m (x \pm i0)$, used in e.g., [14, pp. 96-98] for the distributions $D^0_{\pm} (x \pm i0)^{-l}$ at $z = -l$, is prone to be read as the distributional multiplication products $(x \pm i0)^{-l} \ln^m (x \pm i0)$, but which is incorrect (subject to our definition of multiplication), see [11, eq. (229)].

### 3.4. Non-commutativity

**Convolution**

The non-commutativity of the convolution product is as follows.

(i) Non-critical convolution products are always commutative, [5].

(ii) Critical convolution products are generally non-commutative as a result of their definition as any extension of a partial distribution.

Let $f_m^{a-1} \cdot s_n^{b-1} \in \mathcal{H}'$ and $a + b - 1 = k \in \mathbb{N}$. Then, $f_m^{a-1} \cdot s_n^{b-1}$ and $s_n^{b-1} \cdot f_m^{a-1}$ exist as the partial distributions $(f_m^{a-1} \cdot s_n^{b-1})_{p.d.}$ and $(s_n^{b-1} \cdot f_m^{a-1})_{p.d.}$, respectively, and $(f_m^{a-1} \cdot s_n^{b-1})_{p.d.} = (s_n^{b-1} \cdot f_m^{a-1})_{p.d.}$. It is natural to define the distributions

\[
\begin{align*}
    f_m^{a-1} \cdot s_n^{b-1} & \triangleq \text{any extension } (f_m^{a-1} \cdot s_n^{b-1})_{e} \text{ of } (f_m^{a-1} \cdot s_n^{b-1})_{p.d.}, \\
    s_n^{b-1} \cdot f_m^{a-1} & \triangleq \text{any extension } (s_n^{b-1} \cdot f_m^{a-1})_{e} \text{ of } (s_n^{b-1} \cdot f_m^{a-1})_{p.d.}.
\end{align*}
\]

This makes that the distributions $f_m^{a-1} \cdot s_n^{b-1}$ and $s_n^{b-1} \cdot f_m^{a-1}$ are generally different, since the arbitrary constants, in both extensions, do not necessarily have to be equal. Hence, in general $f_m^{a-1} \cdot s_n^{b-1} \neq s_n^{b-1} \cdot f_m^{a-1}$ is evaluated, since we can always compensate for the effect that a change of order induces, by a change of extension of the final result. In other words, if $a + b - 1 \in \mathbb{N}$, calculating $f_m^{a-1} \cdot s_n^{b-1}$ in different orders merely results in different extensions of the critical product.

This can equivalently be stated as: the convolution product on $\mathcal{H}' / \sim_{\partial}$ is non-commutative.
Multiplication

The non-commutativity of the multiplication product is isomorphic to the non-commutativity of the convolution product.

(i) Non-critical multiplication products are always commutative.

(ii) Critical multiplication products are generally non-commutative as a result of their definition as any extension of a partial distribution.

Let \( f_m^a \delta^n b \in \mathcal{H}' \) and \( a + b = -l \in \mathbb{Z} \). Then, \( f_m^a \delta^n b - b_m^b f_m = e \delta^{(l-1)} \) with \( c \in \mathbb{C} \) arbitrary, or equivalently: the multiplication product on \( \mathcal{H}'/\sim_{\mathbb{Z}} \) is commutative.

3.5. Non-associativity

Convolution

The non-associativity of triple convolution products is as follows.

(i) Non-critical triple convolution products are always associative, [5, Theorem 9].

(ii) Critical triple convolution products are generally non-associative.

From [8, Theorem 3] combined with linearity and since critical convolution products are defined as any extension, it follows that the critical triple products \( (f_m^{a-1} * f_m^{b-1}) \cdot f_m^{c-1} \), \( f_m^{a-1} * (f_m^{b-1} * f_m^{c-1}) \) and \( (f_m^{a-1} * f_m^{b-1}) * f_m^{c-1} \), \( \forall f_m^{a-1}, f_m^{b-1}, f_m^{c-1} \in \mathcal{H}' \):

Thus, if \( a + b + c - 1 \in \mathbb{N} \), we do not need to pay attention to the order in which the product \( f_m^{a-1} * f_m^{b-1} * f_m^{c-1} \) is evaluated, since we can always compensate for the effect that a change of order induces, by a change of extension of the final result. In other words, if \( a + b + c - 1 \in \mathbb{N} \), calculating \( f_m^{a-1} * f_m^{b-1} * f_m^{c-1} \) in different orders merely results in different extensions of the critical triple product. This can equivalently be stated as: the convolution product on \( \mathcal{H}'/\sim_{\mathbb{Z}} \) is associative.

Multiplication

The (non-)associativity of triple multiplication products immediately follows from the (non-)associativity of triple convolution products, since the multiplication product is homomorphic to the convolution product under Fourier transformation. We thus have the following.

(i) Non-critical triple multiplication products are associative.

(ii) Critical triple multiplication products are generally non-associative.

In this case the critical triple products \( f_m^a \cdot (f_m^b, f_m^c) \), \( f_m^b \cdot (f_m^a, f_m^c) \) and \( (f_m^a, f_m^b) \cdot f_m^c \), \( \forall f_m^a, f_m^b, f_m^c \in \mathcal{H}' : a + b + c = -l \in \mathbb{Z} \), differ by a distribution of the form \( r \delta^{(l-1)} \), with \( r \in \mathbb{C} \) arbitrary, or equivalently: the multiplication product of triples
Table 4.1: Some particular convolution products of homogeneous distributions.

<table>
<thead>
<tr>
<th>( \delta(k) )</th>
<th>( \eta(l) )</th>
<th>( \delta(k+l) )</th>
<th>( \eta(k+l) )</th>
<th>( 1 \frac{x^j}{x^l} )</th>
<th>( 1 \frac{x^j \text{sgn}}{x^l} )</th>
<th>( 1 \frac{x^j \text{sgn}}{x^l} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(k) )</td>
<td>( \eta(l) )</td>
<td>( \delta(k+l) )</td>
<td>( \eta(k+l) )</td>
<td>( 1 \frac{x^j}{x^l} )</td>
<td>( 1 \frac{x^j \text{sgn}}{x^l} )</td>
<td>( 1 \frac{x^j \text{sgn}}{x^l} )</td>
</tr>
<tr>
<td>( \delta(k) )</td>
<td>( \eta(l) )</td>
<td>( \delta(k+l) )</td>
<td>( \eta(k+l) )</td>
<td>( 1 \frac{x^j}{x^l} )</td>
<td>( 1 \frac{x^j \text{sgn}}{x^l} )</td>
<td>( 1 \frac{x^j \text{sgn}}{x^l} )</td>
</tr>
<tr>
<td>( \delta(k) )</td>
<td>( \eta(l) )</td>
<td>( \delta(k+l) )</td>
<td>( \eta(k+l) )</td>
<td>( 1 \frac{x^j}{x^l} )</td>
<td>( 1 \frac{x^j \text{sgn}}{x^l} )</td>
<td>( 1 \frac{x^j \text{sgn}}{x^l} )</td>
</tr>
</tbody>
</table>

Table 4.2: Some special multiplication products of homogeneous distributions (part A).

<table>
<thead>
<tr>
<th>( \delta^j )</th>
<th>( x^l )</th>
<th>( x^{k+l} )</th>
<th>( \text{sgn} )</th>
<th>( (-1)^{\frac{\delta^j}{x^l}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^k )</td>
<td>( x^{k+l} )</td>
<td>( \text{sgn} )</td>
<td>( (-1)^{\frac{\delta^j}{x^l}} )</td>
<td></td>
</tr>
<tr>
<td>( x^l )</td>
<td>( x^{k+l} )</td>
<td>( \text{sgn} )</td>
<td>( (-1)^{\frac{\delta^j}{x^l}} )</td>
<td></td>
</tr>
<tr>
<td>( x^2 )</td>
<td>( x^{k+l} )</td>
<td>( \text{sgn} )</td>
<td>( (-1)^{\frac{\delta^j}{x^l}} )</td>
<td></td>
</tr>
</tbody>
</table>

4. Selected results

4.1. Convolution products

A direct calculation, using the general results obtained in [5] and [6], produces the convolution products of HDs of integer degree of homogeneity, as summarized in the Table 4.1, \( \forall k, l \in \mathbb{N} \) and wherein \( c \in \mathbb{C} \) is an arbitrary constant.

This table is symmetric, modulo the choice for the constant \( c \). More general and new convolution products were obtained in [8].

4.2. Multiplication products

Starting from Table 4.1 and using results [10, eqs. (1)–(5)] leads to the multiplication products of HDs on \( R \) of integer degree of homogeneity, as summarized in the Tables 4.2, 4.3, \( \forall k, l \in \mathbb{N} \) and wherein \( c \in \mathbb{C} \) is an arbitrary constant.

Again, this table is symmetric, modulo the choice for the constant \( c \). More general and new multiplication products were obtained in [10].
Table 4.3: Some special multiplication products of homogeneous distributions (part B).

<table>
<thead>
<tr>
<th>( x^k )</th>
<th>((-1)^j \pi_\gamma^{(j)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{k} x^k \text{sgn} )</td>
<td>( 1_{k&lt;0} x^k - 1 + 1_{k=0} (-1)^{j-k} \pi_\gamma^{(j-k)} (\text{M.4.4}) )</td>
</tr>
<tr>
<td>( \frac{1}{k} x^k \text{sgn} )</td>
<td>( 1_{k&lt;0} (-1)^{j-k} \pi_\gamma^{(j-k)} \text{sgn}(\eta_\gamma^{(j-k)}) + 1_{k=0} \frac{1}{k} x^k - 1 \text{sgn}(\text{M.4.4}) )</td>
</tr>
</tbody>
</table>

References


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Ghislain FRANSSENS
Belgian Institute for Space Aeronomy
Ringlaan 3, B-1180 Brussels, BELGIUM
e-mail: ghislain.franssens@aeronomy.be

A. Kamiński and S. Sorek

REMARKS ON PROOFS OF DIAGONAL THEOREM AND ITS APPLICATIONS IN THE THEORY OF DISTRIBUTIONS

Abstract. We recall and discuss main ideas of an elementary proof of the equivalence of the functional and sequential theories of distributions presented in [5], offering certain clarifications of its details. In particular, we present a modified proof of the theorem on the equivalence of the strong and weak boundedness in sequential Köthe spaces and discuss some of its nuances to avoid any doubts concerning its completeness (see Remark 2). We also present an alternative proof of Diagonal Theorem (see Remark 1) playing the main role in showing the latter equivalence.

1. Introduction

That the functional theory of distributions, initiated by S. L. Sobolev in [16] and created as a whole by L. Schwartz in his famous book [15], and the more elementary sequential approach, offered in [14] by J. Mikusiński and R. Sikorski (and extended by them in collaboration with P. Antosik in the book [5]), are equivalent is a consequence of the fact that the linear topologies of spaces considered in the functional theory are sequential. An elementary proof of the equivalence of the two approaches, without considering various topologies but the corresponding types of convergence of sequences of distributions, is given in [5]. The whole proof is very elegant and its beauty justifies one’s wish to clarify all details and to resolve any potential readers’ doubts concerning completeness of the reasoning.

The main idea of the proof consists in the use of Hermite expansions in the space of tempered distributions and the replacement of the two types of convergence in this space by the corresponding convergences of matrices of Hermite coefficients of the considered tempered distributions. This reduces the problem to the equivalence of strong and weak boundedness and, consequently, of strong and weak convergence in sequential Köthe spaces. In section 4, we present an essential modification of the original proof of the latter equivalence given in [5] (see also [11] and [12]) which, in our opinion, contains a subtle gap discussed in Remark 2 (at the end of section 4) and filled in due to our modification.

The chief tool in the proof of the equivalence of strong and weak boundedness in Köthe spaces is a version of Diagonal Theorem which has turned out to be very useful in proving numerous theorems in measure theory and functional analysis (see [7], [18] and some references thereof) as well as in the theory of generalized functions (see [10] for a recent application). In section 3, we recall this beautiful theorem in the form given in [1] (see also [5, pp. 217-219]), presenting its alternative proof and discussing some of its details (see Remark 1 in section 3).

The first author during the conference on generalized functions GF 2011 in Mar-
tinique presented several ideas connected with the equivalence of the two approaches to the theory of distributions. In this note we confine ourselves to an exact formulation of the principal result (Theorem 1 in section 2) and discussions concerning the proofs of the two theorems used in [5] as the chief tools in the proof of the equivalence of the approaches: Diagonal Theorem and the equivalence of the two types of boundedness in Köthe spaces.

2. Theories of distributions and Köthe spaces

Distributions, studied by L. Schwartz in [15] in terms of functional analysis, are defined in the sequential approach (see [14] and [5]) as equivalence classes, Mikusiński-Sikorski distributions, of so-called fundamental sequences of (continuous or smooth) functions on an open set \( \Omega \subseteq \mathbb{R}^d \) (see [5, pp. 6–10, 63–65]). Consider the space \( \mathcal{M}(\Omega) \) of all such classes with the sequentially strong \( \overset{\text{ss}}{\longrightarrow} \) and sequentially weak \( \overset{\text{sw}}{\longrightarrow} \) convergences (see [5, pp. 86, 232]), the space \( \mathcal{T} \) of tempered distributions in the sense of [5, p. 165] and the inner product \( (f, \phi)_\Omega \) of \( f \in \mathcal{M}(\Omega) \) and \( \phi \in \mathcal{D}(\Omega) \), defined in [5, p. 174]. The equivalence of the functional and sequential theories of distributions discussed in [5, pp. 180–235] (cf. [17]) can be formulated as follows:

**Theorem 1.** Let \( \Omega \subseteq \mathbb{R}^d \) be an open set. For every Schwartz distribution \( T \in \mathcal{D}'(\Omega) \) there is a unique \( f = f_T \in \mathcal{M}(\Omega) \) given by \( (f, \phi)_\Omega := \langle T, \phi \rangle \) for \( \phi \in \mathcal{D}(\Omega) \). Moreover, if \( T_n \in \mathcal{D}'(\Omega) \) for \( n \in \mathbb{N}_0 \), then

\[
T_n \to T_0 \text{ in } \mathcal{D}'(\Omega) \quad \iff \quad f_{T_n} \overset{\text{ss}}{\longrightarrow} f_{T_0} \text{ in } \Omega \quad \iff \quad f_{T_n} \overset{\text{sw}}{\longrightarrow} f_{T_0} \text{ in } \Omega.
\]

Conversely, if \( f \in \mathcal{M}(\Omega) \), then there is a unique Schwartz distribution \( T = T_f \in \mathcal{D}'(\Omega) \), given by \( \langle T, \phi \rangle := (f, \phi)_\Omega \). If \( f_n \in \mathcal{M}(\Omega) \) \((n \in \mathbb{N}_0)\), then

\[
f_n \overset{\text{ss}}{\longrightarrow} f_0 \text{ in } \Omega \quad \iff \quad f_n \overset{\text{sw}}{\longrightarrow} f_0 \text{ in } \Omega \quad \iff \quad T_{f_n} \to T_{f_0} \text{ in } \mathcal{D}'(\Omega).
\]

The mappings \( \mathcal{M}(\Omega) \ni f \mapsto T_f \in \mathcal{D}'(\Omega) \) and \( \mathcal{D}'(\Omega) \ni T \mapsto f_T \in \mathcal{M}(\Omega) \) are isomorphisms.

As shown in [5, pp. 215-235] (cf. [17]), Theorem 1 results from a similar theorem on isomorphism of the spaces \( \mathcal{T} \) and \( \mathcal{S}' \) following from the equivalence of strong and weak boundedness in Köthe spaces. We give in section 4 a complete proof of this equivalence (see Remark 2). We recall briefly elements of the Köthe theory, in a simplified form but sufficient for our aims (cf. [5], chapter 10).

Let \( \Lambda \) be a fixed countable set. A mapping \( A : \Lambda \to \mathbb{R} \) will be called a vector and \( a_\lambda := A(\lambda) \), for a given \( \lambda \in \Lambda \), its \( \lambda \)-th coordinate; we will write \( A = [a_\lambda] \). We call a vector \( A \) positive if its all coordinates are positive. By \( e_\lambda \) denote the vector whose \( \lambda \)-th coordinate is 1 and the remaining ones are 0. Denote the set of all (positive) vectors by \( \mathcal{R} \) (by \( \mathcal{R}_+ \)). It is clear that \( \mathcal{R} \) is a linear space over \( \mathbb{R} \) with the usual coordinatewise operations. For a given \( A = [a_\lambda] \in \mathcal{R}_+ \), we define the vector \( A^{-1} := [a_\lambda^{-1}] \in \mathcal{R}_+ \). For
A = \{a_i\} \in \mathcal{R}, we define the vector \(|A| := |a_i| \in \mathcal{R}\) and consider the two norms:

\[ ||A||_1 := \sum_{k \in A} |a_k|; \quad ||A||_\infty := \sup_{k \in A} |a_k| \]

(denoted in a different way in [5, p. 216]: by \(|A|\) and \(|A|\), respectively). If moreover \(B = \{b_j\} \in \mathcal{R}\), we define \(AB := |a_i b_j| \in \mathcal{R}\) and the scalar product:

\[ (A, B) := \sum_{k \in A} a_k b_k := \lim_{n \to \infty} \sum_{i=1}^n a_k b_k, \]

whenever \(||AB||_1 = \sum_{k \in A} |a_k b_k| < \infty\).

**Definition 1.** Let \(\{V_j\}_{j \in \mathbb{N}}\) be a sequence of positive vectors such that

\[
(V) \quad ||V_i V_j^{-1}||_\infty < \infty \quad \text{for } i \in \mathbb{N}.
\]

A vector \(A \in \mathcal{R}\) is called rapidly decreasing if \(||V_i A||_1 < \infty\) for all \(i \in \mathbb{N}\) and tempered if \(||V_i^{-1} A||_\infty < \infty\) for some \(i_0 \in \mathbb{N}\). The sets of all rapidly decreasing and all tempered vectors are denoted by \(\mathcal{S}\) and \(\mathcal{T}\), respectively.

Under condition \((V)\), for every \(\eta > 0\) we may assume that \(||V_i V_j^{-1}||_\infty < \eta^i\) for \(i, j \in \mathbb{N}\). In fact, if \(||V_i V_j^{-1}||_\infty =: \eta_{i_1}\), then denoting \(V_j := \eta^{-i_1} \cdots \eta_1 V_i\), we have \(||V_j V_i^{-1}||_\infty < \eta^j\) for \(i, j \in \mathbb{N}\) and the sets \(\mathcal{S}\) and \(\mathcal{T}\) defined by the sequences \(\{V_i\}\) and \(\{V_j\}\) are identical. Thus we may and will assume that

\[
(1) \quad ||V_i V_j^{-1}||_\infty < 2^{-j}, \quad i, j \in \mathbb{N}, i < j.
\]

**Definition 2.** A set \(\mathcal{A} \subseteq \mathcal{T}\) of tempered vectors is called \(1^\circ\) strongly bounded if there are \(i_0 \in \mathbb{N}\) and \(\alpha > 0\) so that \(||V_i^{-1} A||_\infty < \alpha\) for all \(A \in \mathcal{A}\); \(2^\circ\) weakly bounded if the set \(\{(A, S) : A \in \mathcal{A}\}\) is bounded for all \(S \in \mathcal{S}\).

**Definition 3.** Let \(S_n \in \mathcal{S}\) for \(n \in \mathbb{N}_0\). We say that \(S_n\) converges to \(S_0\) in \(\mathcal{S}\) and write \(S_n \xrightarrow{\mathcal{S}} S_0\), whenever \(||V_i (S_n - S_0)||_1 \to 0\) for all \(i \in \mathbb{N}\).

### 3. Diagonal Theorem

To prove Theorem 3 on the equivalence of two types of boundedness in Köthe spaces we need Diagonal Theorem, a formalization of the known sliding-hump technique. It was first shaped by Jan Mikusiński in [13] and then reformulated by Piotr Antosik in [1] (see also [5, p. 217]). The theorem and its later versions are very convenient for applications in measure theory and functional analysis (see e.g. [13], [7], [3], [18]). We will recall and discuss Diagonal Theorem in the version given in [1], for quasi-normed groups, with care over its formulation (see the comments below) and its proof. Notice that this version, due to a nice result proved in [8], implies the theorem formulated for topological groups (cf. [2]). An interesting extension of Diagonal Theorem was given by M. Florencio, P. J. Paúl and J. M. Virués in [9] (see also [6] and [4]). We begin with introducing some definitions and notations.
Definition 4. By a quasi-normed group \((X, +, |·|)\) we mean an Abelian group \((X, +)\) endowed with a functional \(|·|\), a quasi-norm, satisfying the conditions:

\((N_1)\) \( |0| = 0; \)

\((N_2)\) \( |−x| = |x|, \quad x \in X; \)

\((N_3)\) \( |x + y| \leq |x| + |y|, \quad x, y \in X. \)

From conditions \((N_1) - (N_3)\) it follows that \(|x| \geq 0\) for all \(x \in X\) and

\[
\min\{|x + y|, |x - y|\} \geq |x| - |y|, \quad x, y \in X.
\]

If we put \(d(x, y) := |x - y|\) for a given quasi-norm \(|·|\) and all \(x, y \in X\), then \(d\) is clearly not a metric, in general, but a quasi-metric in \(X\), i.e. it satisfies the symmetry and triangle conditions and \(d(x, x) = 0\) for \(x \in X\); in [1], the second sentence on page 306 is an obvious mistake. If the convergence \(x_n \to x\) in \(X\) means that \(d(x_n, x) = |x_n - x| \to 0\) as \(n \to \infty\), then limits of convergent sequences are evidently not unique, in general.

Let \(X = (X, +, |·|)\) be a quasi-normed group (not necessarily complete) and let \(\{x_n\}\) be a sequence in \(X\). Assuming, for a fixed infinite subset \(J\) of \(\mathbb{N}\), the condition:

\[
\sum_{j \in J} |x_j| < \infty,
\]

we introduce the notation:

\[
|\sum_{j \in J} x_j| := \lim_{p \to \infty} \left| \sum_{k=1}^{p} x_{j_k} \right|,
\]

where \(\{j_k\}\) is the increasing sequence of all elements of \(J\). No matter that limits in \(X\) may be not unique and \(X\) not complete, the limit in (4) exists and is unique, so the notation makes sense. This is because the sequence \(\{y_p\}\), where \(y_p := \sum_{k=1}^{p} x_{j_k}\) for \(p \in \mathbb{N}\), is a numerical Cauchy sequence, in view of the inequalities (resulting from (2)):

\[
|y_r - y_p| \leq \left| \sum_{k=p+1}^{r} x_{j_k} \right| \leq \sum_{k=p+1}^{r} |x_{j_k}|, \quad p < r
\]

and due to condition (3). The same applies to the numerical sequence \(\{z_p\}\), where \(z_p := |\sum_{k=1}^{p} x_{j_{\pi(k)}}|\) for an arbitrary bijection \(\pi: \mathbb{N} \to \mathbb{N}\) and \(p \in \mathbb{N}\). Moreover we have

\[
\lim_{p \to \infty} \left| \sum_{k=1}^{p} x_{j_k} \right| = \lim_{p \to \infty} \left| \sum_{k=1}^{p} x_{j_{\pi(k)}} \right|.
\]

Identity (5), which can be shown in a standard way (see e.g. [17]), justifies the use of the same symbol as in (4) for the following more general notation:

\[
|\sum_{j \in J} x_j| := \lim_{n \to \infty} \left| \sum_{i \in J_n} x_i \right|,
\]

where \(\{J_n\}\) is an arbitrary nondecreasing sequence of finite subsets of \(J\) such that \(\bigcup_{n=1}^{\infty} J_n = J\). Notation (6) is consistent also in case \(J\) is finite.
Remarks on proofs of diagonal theorem and its applications in the theory of distributions  143

THEOREM 2 (cf. [1] and [5, pp. 217–219]). Let $x_{i,j} \in X$ ($i, j \in \mathbb{N}$)

$$(H) \quad \lim_{j \to \infty} |x_{i,j}| = 0 \quad \text{for } i \in \mathbb{N},$$

where $(X, +, \cdot, \cdot)$ is a quasi-normed group. Then there are an infinite set $I \subseteq \mathbb{N}$ and a set (finite or infinite) $J \subseteq I$ such that

$$(A1) \quad \sum_{j \in \Delta} |x_{i,j}| < \infty \quad (i \in I) \quad \text{and} \quad (A2) \quad \sum_{j \in \Delta} x_{i,j} \geq \frac{|x_{i,j}|}{2} \quad (i \in I).$$

Proof. 1. We may assume that, for any finite $\Delta \subseteq \mathbb{N}$, the implication holds:

$$(*) \quad \sum_{j \in \Delta} |x_{i,j}| \geq \frac{|x_{i,j}|}{2} \quad \text{for } i \in \Delta \quad \Rightarrow \quad \exists i_0 \geq \Delta \quad \sum_{j \in \Delta} |x_{i,j}| < \frac{|x_{i,j}|}{2} \quad (i \in \Delta).$$

For, if $(*)$ does not hold for some finite $\Delta$, then there is an increasing sequence (depending on $\Delta$) of indices $i_n$ such that $\sum_{j \in \Delta} x_{i,j} \geq \frac{1}{2} |x_{i,j}|$ for $i \in \Delta \cup \Delta_0$, where $\Delta_0 := \{ i_n : n \in \mathbb{N} \}$, i.e. assertions $(A1)$ and $(A2)$ hold for $J := \Delta$ and $I := \Delta \cup \Delta_0$. From now we will thus assume that implication $(*)$ is true.

II. Clearly, the left hand side of the implication is satisfied for $\Delta := \{1\}$. Hence, by $(*)$, there is an index $i_1 > 1$ such that

$$(7) \quad |x_{i_1,j}| > 2 \sum_{j \in \Delta} x_{i,j} \geq 0 \quad \text{for } i \geq i_1.$$

We will inductively construct an increasing sequence of indices $i_n \in \mathbb{N}$ with $i_1$ as above (and so of the sets $\Delta_n := \{ i_1, \ldots, i_n \}$) and a sequence of numbers $\epsilon_n \in (0, 1/2]$ with $\epsilon_1 := 1/2$ satisfying the two conditions:

$$(8) \quad \sum_{j \in \Delta_{r-1}} x_{i,j} = \left( \frac{1}{2} - \epsilon_r \right) |x_{i,j}| \quad \text{for } r \in \mathbb{N} \setminus \{1\},$$

$$(9) \quad \sum_{q=p}^r |x_{i_1,q}| < \epsilon_p |x_{i_1,p}| \quad \text{for } p \in \mathbb{N}, \quad p < r \in \mathbb{N} \setminus \{1\}$$

and the third one:

$$(10) \quad \sum_{j \in \Delta_r} x_{i,j} > \frac{1}{2} |x_{i,j}| \quad \text{for } i \in \Delta_r, \quad r \in \mathbb{N},$$

which is the antecedent of implication $(*)$ for $\Delta = \Delta_r$.

III. By $(7)$, inequality $(10)$ for $r = 1$ is obvious. Hence, by the consequent of implication $(*)$ for $\Delta = \Delta_1$, there is a natural index $i' > i_1$ such that

$$(11) \quad |x_{i,i'}| < \frac{1}{2} |x_{i,j}| \quad \text{for } i \geq i'.$$
In view of (H) and (7), we can find a natural \( i_2 \geq i_1' > i_1 \) such that

\[
|x_{i_1,i_2}| < \frac{1}{2} |x_{i_1,i_1}|
\]

i.e. condition (9) is fulfilled for \( p = 1, r = 2 \). Defining

\[
\varepsilon_2 := \left( \frac{1}{2} |x_{i_2,i_2}| - |x_{i_2,i_1}| \right) |x_{i_2,i_2}|^{-1},
\]

we see that (8) holds for \( r = 2 \) and \( \varepsilon_2 \in (0, 1/2] \), by (11) and (13). Moreover, we have

\[
|x_{i_1,i_1} + x_{i_1,i_2}| > \frac{1}{2} |x_{i_1,i_1}| \quad \text{and} \quad |x_{i_2,i_1} + x_{i_2,i_2}| > \frac{1}{2} |x_{i_2,i_2}|,
\]
due to (12) and (11), respectively, which means that (10) holds for \( r = 2 \).

Suppose that, given a natural \( s > 2 \), we have selected positive integers \( i_1 < \ldots < i_{s-1} \) and numbers \( \varepsilon_1, \ldots, \varepsilon_{s-1} \in (0, 1/2] \) so that (10) holds for all natural \( r \leq s - 1 \), (8) holds for all \( r \in \mathbb{N} \) such that \( 1 < r \leq s - 1 \) and (9) holds for all \( p, r \in \mathbb{N} \) such that \( p < r \leq s - 1 \). By (10) for \( r = s - 1 \), the antecedent of implication (\( \ast \)) and so its consequent hold for \( \Delta = \Delta_{s-1} \). Hence, by (H), (7) and (9), assumed for \( r = s - 1 \) and \( p < s - 1 \), there is a common index \( i_s > i_{s-1} \) such that \( \sum_{q=p+1}^{s} |x_{i_q,i_q}| < \varepsilon_p |x_{i_p,i_p}| \) for \( 1 \leq p < s \) and

\[
|\sum_{j \in \Delta_{s-1}} x_{i,j}| < \frac{1}{2} |x_{i,i_s}| \quad \text{for} \quad i \geq i_s.
\]

Hence (9) holds for \( p < r = s \). Define

\[
\varepsilon_s := \left( \frac{1}{2} |x_{i_s,i_s}| - \left| \sum_{j \in \Delta_{s-1}} x_{i_s,j} \right| \right) |x_{i_s,i_s}|^{-1}.
\]

By (14) and (15), \( 0 < \varepsilon_s \leq 1/2 \) and (8) for \( r = s \) follows from (15). Also

\[
|\sum_{j \in \Delta_s} x_{i_s,j}| \geq |x_{i_s,i_s}| - \left| \sum_{j \in \Delta_{s-1}} x_{i_s,j} \right| > \frac{1}{2} |x_{i_{s+1},i_s}|,
\]
in view of (14), and

\[
|\sum_{j \in \Delta_s} x_{i_s,j}| \geq |x_{i_s,i_s}| - \sum_{j \in \Delta_{s-1}} x_{i_s,j} = \sum_{q=p+1}^{s} |x_{i_q,i_q}| > \frac{1}{2} |x_{i_p,i_p}|,
\]

whenever \( 1 \leq p < s - 1 \) (with the sum over \( \Delta_{p+1} \) vanishing for \( p = 1 \)), by (8) for \( r = p < s \) and (9) for \( p < r = s \), i.e. condition (10) holds for \( r = s \). By induction, conditions (10), (8) and (9) hold for all \( p, r \in \mathbb{N} \) as indicated.

IV. We define \( I = J := \{ i_n : n \in \mathbb{N} \} \), where \( \{ i_n \} \) is the sequence just constructed. We conclude from (9) that \( \sum_{q=p+1}^{s} |x_{i_q,i_q}| \leq \varepsilon_p |x_{i_p,i_p}| < \infty \) for \( p \in \mathbb{N} \), i.e. (A1) holds. By (8) and (9), we have, for all \( p, n \in \mathbb{N}, p < n \),

\[
|\sum_{q=1}^{n} x_{i_p,i_q}| \geq |x_{i_p,i_p}| - \sum_{j \in \Delta_{p-1}} x_{i_p,j} = \sum_{q=p+1}^{n} |x_{i_p,i_q}| > \frac{1}{2} |x_{i_p,i_p}|,
\]
where the sum over $\Delta_{p-1}$ is 0 in case $p = 1$. Hence, as $n \rightarrow \infty$,

$$\left| \sum_{j \in J} x_{i_p,j} \right| = \lim_{n \rightarrow \infty} \left| \sum_{q=1}^n x_{i_p,q} \right| \geq \frac{1}{2} |x_{i_p,j}|, \quad p \in \mathbb{N},$$

i.e. also assertion (A2) of the theorem holds.

**Remark 1.** After reducing the above proof to the case where implication (*) is true, we used (*) to construct inductively sequences $\{i_n\}$ and $\{e_n\}$ satisfying conditions (8), (9) and (10). Of these three conditions only (8) and (9) were used in the final part IV of the proof to show assertions (A1) and (A2) for $I = J = \{i_1, i_2, \ldots\}$. Condition (10) was needed merely to guarantee that the antecedent of implication (*) is satisfied (and so (*) can be properly used) for every $n \in \mathbb{N}$ in the construction. As a matter of fact, condition (10) can be deduced, for arbitrary $r \in \mathbb{N} \setminus \{1\}$, directly from (8) and (9), due to inequality (16) (with $s$ replaced by $r$), which is a consequence of (N1) and just conditions (8) and (9). This is another way to conclude (10) than it is shown above.

The above comment means that the original proof of Diagonal Theorem given in [1] (see also [5], section 10.5) does not contain a gap, though nothing is mentioned there that the set $\Delta = \Delta_p = \{i_1, \ldots, i_p\}$ satisfies the antecedent of implication (*) for every $p \in \mathbb{N} \setminus \{1\}$.

The example below shows the situation where the set $J$ cannot be taken to be equal $I$ in the assertion of Theorem 2 (cf. [13], [1] and [5, p. 217]).

**Example 1.** Consider the infinite matrix defined as follows: $x_{i,j} = 2, x_{i,j} = -1$ if $j < i$ and $x_{i,j} = 0$ if $j > i$ (for $i, j \in \mathbb{N}$). Obviously, assumption (H) is satisfied and the assertion holds for an arbitrary infinite $I \subset \mathbb{N}$ and a respective finite $J \subset I$. In fact, if $I := \{i_n \in \mathbb{N} : n \in \mathbb{N}\}$, where $i_n \uparrow \infty$, then (A1) and (A2) hold for $J := \{i_1\}$. On the other hand, for each infinite $I \subset \mathbb{N}$ and $J := I$, written in the form $I = J := \{i_n \in \mathbb{N} : n \in \mathbb{N}\}$ with $i_n \uparrow \infty$, there exists an $i \in I$, namely $i := i_1$, such that $\left| \sum_{j \in J} x_{i,j} \right| = 0$, so (A2) is not satisfied for the indicated $i$.

4. Boundedness of sets in Köthe spaces

**Theorem 3.** Every countable set $\mathcal{A}$ of tempered vectors is strongly bounded if and only if it is weakly bounded.

**Proof.** If the set $\mathcal{A} = \{A_n : n \in \mathbb{N}\} \subset \mathbb{F}$ is strongly bounded, then $\|V_j^{-1}A_n\|_\alpha \leq \alpha$ for some $j \in \mathbb{N}$, $\alpha > 0$ and for all $n \in \mathbb{N}$. If $S \in \mathcal{S}$, then $\|V_iS\|_\beta < \beta < \infty$ for all $i \in \mathbb{N}$. Hence $\|A_n,S\| \leq \alpha \beta$ for all $n \in \mathbb{N}$, i.e. the set $\mathcal{A}$ is weakly bounded. Let $V_i := [v_{i,j}]$ and $A_n := [a_{n,j}]$ ($i, j, n \in \mathbb{N}$). Suppose that $\mathcal{A}$ is weakly but not strongly bounded, i.e. (a) the sequences $\{\|V_j^{-1}A_n\|_\alpha\}$ are unbounded for all $i \in \mathbb{N}$, and (b) for any $i, r \in \mathbb{N}$ there are constants $\alpha_{i,r} \geq 1$ such that

$$\|(A_n, V_j^{-1}e_r)\| = |v_{i,r}^{-1} a_{n,r}| = \|V_j^{-1}e_r A_n\|_\alpha \leq \alpha_{i,r}$$

(17)
for all \( n \in \mathbb{N} \), since \( V_i^{-1} e_r \in \mathcal{S} \). Using (a) and (b), we will construct inductively increasing sequences \( \{ n_i \} \) and \( \{ r_i \} \) of positive integers such that the following inequalities, given below in two equivalent forms, are true:

\[
\| V_i^{-1} e_r A_{n_i} \|_\infty > \| V_i^{-1} A_{n_i} \|_\infty - 1 > \alpha_i, \quad i \in \mathbb{N}
\]

and

\[
| v_i^{-1} a_{n_i, r_i} | > \sup_{j \in \mathbb{N}} | v_i^{-1} a_{n_i, j} | - 1 > \alpha_i, \quad i \in \mathbb{N},
\]

where \( \alpha_i \) are defined, via constants \( \alpha_{i, r} \) satisfying (17), as follows:

\[
\alpha_1 := \alpha_{1,1}; \quad \alpha_r := \max \{ \alpha_{i, r} : r \leq r_i - 1 \} + \alpha_{i-1}, \quad i > 1.
\]

Clearly, since \( \alpha_{i, r} \geq 1 \) for all \( i, r \in \mathbb{N} \), the definition in (20) guarantees that \( \alpha_n \uparrow \infty \). Find, by (a), an \( n_1 \in \mathbb{N} \) to fulfill the second inequality and then, by the definition of \( \| \cdot \|_\infty \), an \( r_1 \in \mathbb{N} \) to satisfy, for \( i = 1 \), the first inequality in (18) or, equivalently, in (19). Assume that indices \( n_1 < \ldots < n_p \) and \( r_1 < \ldots < r_p \) satisfying (18) for \( i = 1, \ldots, p \) are chosen. Apply (a) for \( i = p + 1 \) to find an index \( n_{p+1} > n_p \) such that \( \| V_{p+1}^{-1} A_{n_{p+1}} \|_\infty > \alpha_{p+1} + 1 \).

By the definition of \( \| \cdot \|_\infty \), there is an \( r_{p+1} \in \mathbb{N} \) such that the first inequality in (18) holds for \( i = p + 1 \), i.e. \( \| V_{p+1}^{-1} e_{r_{p+1}} A_{n_{p+1}} \|_\infty > \alpha_{p+1} \). But, by (17) and (20), we have

\[
\| V_{p+1}^{-1} e_{r_{p+1}} A_{n_{p+1}} \|_\infty \leq \alpha_{p+1} + 1 \quad \text{for all} \ r \leq r_p,
\]

i.e. the index \( r_{p+1} \) just found cannot be among indices \( r \leq r_p \). Consequently, it must be \( r_{p+1} > r_p \). Thus the inductive construction of increasing sequences \( \{ n_i \} \) and \( \{ r_i \} \) satisfying (18) and (19) is completed. Put \( x_i j := (A_{ni}, V_i^{-1} e_{r_i}) \in \mathbb{R} \) for \( i, j \in \mathbb{N} \). Since \( A_{ni} \) are tempered, there are \( p_1 \in \mathbb{N} \) and \( \beta_i > 0 \) so that \( \| V_{p_1}^{-1} A_{ni} \|_\infty \leq \beta_i < \infty \) for all \( i \in \mathbb{N} \). Hence

\[
| x_i j | \leq \| V_{p_1}^{-1} A_{ni} \|_\infty \cdot \| V_{p_1} V_j^{-1} \|_\infty \leq \beta_i \cdot 2^{p_1-j}, \quad i, j \in \mathbb{N},
\]

due to (1), and so \( \lim_{j \to \infty} | x_i j | = 0 \) for every \( i \in \mathbb{N} \). It follows from Diagonal Theorem that there exist an infinite set \( I \subseteq \mathbb{N} \) and its subset \( J \) (finite or infinite) such that

\[
\sum_{j \in J} (A_{ni}, V_j^{-1} e_{r_j}) < \infty \quad \text{and we have, for all} \ i \in I,
\]

(21)

\[
\frac{1}{2} \sum_{j \in J} (A_{ni}, V_j^{-1} e_{r_j}) \leq \lim_{n \to \infty} \left| \sum_{j \in J_n} (A_{ni}, V_j^{-1} e_{r_j}) \right|,
\]

where \( J_n \) are finite sets forming a nondecreasing sequence such that \( \bigcup_{n=1}^{\infty} J_n = J \) (i.e. \( J_n = J \) for sufficiently large \( n \) in case \( J \) is finite). We are now in a position to define the vector \( R \) whose \( r_j \)-th coordinate coincides with the \( r_j \)-th coordinate of \( V_j^{-1} \) for all \( j \in J \) and the remaining coordinates are equal to 0 (we have 0 \( \leq R \leq V_j^{-1} \)), i.e.

(22)

\[
R := \sum_{j \in J} e_{r_j} V_j^{-1} = \lim_{n \to \infty} R_n,
\]

where \( R_n := \sum_{j \in J_n} e_{r_j} V_j^{-1} \) and the limit is meant coordinatewise (convergence in a
one applies just the definition of the norm $J$ and which implies that the sequence $V_i$ grow to infinity. However, the definition of the vector $R$ in (22) (in case the set $J$ is infinite) is incorrect if, for a certain $r \in \mathbb{N}$, the equation $r_i = r$ holds for infinitely many indices $i \in \mathbb{N}$, i.e. if the sequence $\{r_i\}$ contains constant subsequences. Fortunately, this bad possibility was eliminated in the above proof by imposing on the inductively constructed sequences $\{n_i\}$ and $\{r_i\}$ of positive integers the additional requirement in the form of the second inequality in (18) (or, equivalently, in (19)) with the constants $\alpha_j$ defined by formula (20), forcing the strict increase of $\{r_i\}$. Consequently, the requirement (satisfied due to conditions $(a)$ and $(b)$) guaranteed the strict increase of both sequences $\{n_i\}$ and $\{r_i\}$ and the correctness of the definition of the vector $R$.

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References


Remarks on proofs of diagonal theorem and its applications in the theory of distributions

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Andrzej KAMIŃSKI, Sławomir SOREK,
Institute of Mathematics, University of Rzeszów
Rejtana 16A, 35-310 Rzeszów, POLAND
e-mail: akaminsk@univ.rzeszow.pl, ssorek@univ.rzeszow.pl

Abstract. We examine 2-dimensional integral equations of Volterra type with two singular interior lines corresponding to \(x = a\) and \(y = b\). The non-homogeneous integral equation that we can consider involves functions \(A(x), B(y), C(x, y)\). Given certain inequalities for \(A(a)\) and \(B(b)\), it always has solutions on suitable domains that contain arbitrary functions of one variable. With other hypotheses, the equation has a unique solution in some domain.

1. Introduction and preliminaries

Consider the rectangle
\[ D_0 = \{a_0 < x < a_1, b_0 < y < b_1\}, \]
and the straight lines
\[ \Gamma_1 = \{a_0 < x < a, y = b\}, \]
\[ \Gamma_2 = \{a < x < a_1, y = b\}, \]
\[ \Gamma_3 = \{x = a, b_0 < y < b\}, \]
\[ \Gamma_4 = \{x = a, b < y < b_1\}, \]
where \(a_0 < a < a_1, b_0 < b < b_1\).

In the domain \(D = D_0 \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4)\), we consider the 2-dimensional integral equation

\[
\begin{align*}
&u(x, y) + \int_a^x A(t)u(t, y) \frac{dt}{|t-a|^\alpha} - \int_b^y B(s)u(x, s) \frac{ds}{|b-s|^\beta} \\
&+ \int_a^x dt \int_s^b C(t, s)u(t, s) \frac{ds}{|b-s|^\beta} = f(x, y),
\end{align*}
\]

where \(A(x), B(y), C(x, y)\) are given functions in \(D_0\), \(f(x, y) \in C(D)\), and both \(\alpha\) and \(\beta\) are positive constants.

The solution of many problems having a significance in applications can be figured out by the help of integral equations in explicit form. For that reason, this article is dedicated to this area.

For the equation (1), we find the solution of a second-order hyperbolic equation with two super-singular lines in the domain \(D_2 = \{a < x < a_0, b_0 < y < b\}\) and for types of functions, approaching infinity on singular lines.

Problems concerning 2-dimensional Volterra-type integral equations

\[
\begin{align*}
&u(x, y) + \lambda \int_a^x \frac{u(t, y)}{(t-a)^\alpha} dt - \mu \int_b^y \frac{u(x, s)}{(b-s)^\beta} ds + \delta \int_a^x \frac{dt}{(t-a)^\alpha} \int_s^b \frac{u(t, s)}{(b-s)^\beta} ds = f(x, y)
\end{align*}
\]
with two boundary singular and super-singular lines in the domain $D_2$, are investigated in [1, 2, 4].

Integral equations of type (1) with boundary singular and super-singular lines, and cases

$$\alpha = 1, \beta > 1; \quad \alpha > 1, \beta = 1; \quad \alpha < 1, \beta > 1$$

are investigated in [3, 5].

References [6, 7] are dedicated to the problem of finding continuous solutions of second-order hyperbolic equation with two boundary singular or super-singular lines $\Gamma_1$ and $\Gamma_2$ corresponding to the study of integral equation (1) in the domain $D_2$ with $\alpha \geq 1$ and $\beta \geq 1$.

Finally, [8, 9] deal with the integral equation

$$u(x, y) + \int_a^z K_1(x, y; t)u(y, y) \frac{dt}{(t-a)^{\alpha}} - \int_y^b K_2(x, y; s)u(x, s) \frac{ds}{(b-s)^{\beta}} + \int_a^z \frac{dt}{(t-a)^{\alpha}} \int_y^b K_3(x, y; t, s)u(t, s) \frac{ds}{(b-s)^{\beta}} = f(x, y),$$

in the domain $D_2$ in the cases $\alpha = 1$, $\beta = 1$ where

$$\lambda := K_1(a, b; a), \quad \mu := K_2(a, b; b), \quad \delta := K_3(a, b; a, b) = -\lambda \mu.$$

For $\alpha > 1, \beta > 1$ we set

$$A(t) = K_1(a, b; t), \quad B(s) = K_2(a, b; s), \quad C(t, s) = K_3(a, b; t, s)$$

and require $C_1(t, s) := C(t, s) + A(t)B(s)$ not to be identically zero.

In this paper, we find the solution of the 2-dimensional Volterra type linear integral equation with interior singularities for exponents $\alpha = 1$ and $\beta = 1$ in the kernels in (1), when $C(x, y) \neq A(x)B(y)$.

In this case we shall prove that, when the coefficients of the integral equation are related in a determined way, the homogeneous integral equation (1) has infinitely many linear independent solutions given conditions on $A(a), B(b)$. For other values of these quantities, the homogeneous integral equation (1) has non-zero solution in some of the domains $D_j$.

These solutions are found by resolving known integral equations of Volterra type with weak singularity lines.

In the domain $D_0$, if we fix lines $x = a$ and $y = b$, the domain $D_0$ is divided into four domains as

$$D_1 = \{a_0 < x < a, 0 < y < b,\},$$
$$D_2 = \{a < x < a_1, 0 < y < b,\},$$
$$D_3 = \{a_0 < x < a, b < y < b_1,\},$$
$$D_4 = \{a < x < a_1, b < y < b_1,\}.$$

If $(x, y) \in D_1$, then we integrate over $s$ and $t$ that satisfy $a_0 < x < t < a$ and $b_0 < y < s < b$. The left-hand side of (1) takes the form:

$$u(x, y) = \int_x^a \frac{A(t)u(t, y)}{a-t}dt - \int_y^b \frac{B(s)u(x, s)}{b-s}ds = \int_x^a \frac{dt}{a-t} \int_y^b \frac{C(t, s)u(t, s)}{b-s}ds.$$
Similarly, if \((x,y) \in D_2, D_3, D_4\) then the left-hand side of (1) becomes respectively

\[
\begin{align*}
  u(x,y) &= \frac{x}{a-t} \int_a^x A(t)u(t,y) \, dt - \int_y^b B(s)u(x,s) \, ds + \int_y^x \frac{dt}{a-t} \int_y^b C(t,s)u(t,s) \, ds, \\
  u(x,y) &= \frac{x}{a-t} \int_a^x A(t)u(t,y) \, dt + \int_b^y B(s)u(x,s) \, ds - \int_y^x \frac{dt}{a-t} \int_b^y C(t,s)u(t,s) \, ds, \\
  u(x,y) &= \frac{x}{t-a} \int_a^x A(t)u(t,y) \, dt + \int_b^y B(s)u(x,s) \, ds - \int_y^x \frac{dt}{t-a} \int_b^y C(t,s)u(t,s) \, ds.
\end{align*}
\]

In this way, the study of these four integral equations in the respective domains \(D_1, D_2, D_3, D_4\) is investigated in [1, 2].

2. A first theorem

Our first result is the following statement that deals with the case in which \(A(a) < 0\) and \(B(b) > 0\).

**Theorem 1.** Given (1), suppose that \(A(x)\) and \(B(y)\) define continuous functions on \(\Gamma_1 \cup \Gamma_2\) and \(\Gamma_3 \cup \Gamma_4\) respectively, and that they satisfy the Hölder condition, \(A(a) < 0\) and \(B(b) > 0\). Suppose that \(C(x,y)\) defines a continuous function on \(D_0\) and that

\[
C_1(x,y) = C(x,y) + A(x)B(y)
\]

is continuous on \(\overline{D_0}\) and not identically zero. Suppose that \(C_1(x,y)\) vanishes on \(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\), with

\[
C_1(x,y) = O\left([a-x]^\xi[b-y]^\eta\right) \quad \text{as} \quad x \to a \pm 0, \quad y \to b \pm 0, \quad \varepsilon > 0
\]

Moreover, suppose that \(f(x,y)\) is a continuous function on \(\overline{D_0}\) that satisfies \(f(a,b) = 0\) and has the following asymptotic behavior:

\[
f(x,y) = \begin{cases} 
  o\left([x-a]^\delta_1\right) & \text{as} \quad x \to a + 0, \quad \delta_1 > |A(a)|, \\
  o\left((a-x)^\tau\right) & \text{as} \quad x \to a - 0, \\
  o\left((b-y)^\gamma_1\right) & \text{as} \quad y \to b - 0, \quad \gamma_1 > B(b), \\
  o\left((y-b)^\mu\right) & \text{as} \quad y \to b + 0.
\end{cases}
\]

Then (1) always admits a solution with \(u \in C(\overline{D_0})\) and \(u(x,y) \to 0\) as \((x,y) \to \Gamma_j\) for \(j = 1, 2, 3, 4\), and its general solution contains four arbitrary functions each of one variable.

We claim that the solution is given by means of following formulas:
are all resolvents of known integral equation of Volterra type with weak singularity lines,

\[ w_a^{-1}(x) = \int_a^x \frac{A(t) - A(a)}{t - a} dt, \quad w_a^{-1}(x) = \int_a^x \frac{A(a) - A(t)}{a - t} dt, \]
The non-homogeneous integral equation (1) in the class $C$ points $x$

We now state three theorems analogous to the first, corresponding to the other possible signs of $A(a)$ and $B(b)$.

**Theorem 2.** In equation (1), let $A(x) \in C(\Gamma_1 \cup \Gamma_2)$, $B(y) \in C(\Gamma_3 \cup \Gamma_4)$ for the points $x = a$, $y = b$ they satisfy the Helder’s conditions, $A(a) > 0$, $B(b) < 0$, $C(x, y) \in C(D_0)$, $C_1(x, y) = C(x, y) + A(x)B(y) \neq 0$, $C_1(x, y) \in C(D_0)$ and $C_1(x, y) = 0$ with following asymptotic behavior on the boundaries $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$:

$$C_1(x, y) = o[|a - x|^\delta_2 |b - y|^\delta_3] \quad \text{as } x \to a \pm o, \quad y \to b \pm o$$

Moreover, let $f(x, y) \in C(D_0)$, $f(a, b) = 0$ with following asymptotic behavior:

$$f(x, y) = \begin{cases} 
  o[|a - x|^\delta_2] & \text{as } x \to a - o, \quad \delta_2 > |A(a)|, \\
  o[|x - a|^\delta_3] & \text{as } x \to a + o, \\
  o[|y - b|^\gamma_3] & \text{as } y \to b - o, \\
  o[|y - b|^\gamma_3] & \text{as } y \to b + o. \quad \gamma_3 > |B(b)|.
\end{cases}$$

The non-homogeneous integral equation (1) in the class $C(D_0)$, approaching zero in $\Gamma_j$, $j = 1, 2, 3, 4$, is always solvable and its general solution contains four arbitrary functions with one variable.

The solutions are in fact given by means of following formulas:

$$u(x, y) = \exp[-w_a^{-1}(x)](a - x)^{A(a)} \psi_1(y) + K_{a,b}^{-}(f(x, y))$$

$$- \exp[-w_a^{-1} - w_b^{-1}(y)](a - x)^{A(a)}(b - y)^{B(b)}$$

$$\times \int_a^x dt \int_y^b \Gamma_{21}(x, y; t, s)E_5[\psi_1(s), f(t, s)] ds.$$

$$u(x, y) = K_{a,b}^{-}(f(x, y)) - \exp[-w_a^{-1}(x) - w_b^{-1}(y)](x - a)^{-A(a)}(b - y)^{B(b)}$$

$$\times \int_a^x dt \int_y^b \Gamma_{22}(x, y; t, s)E_6[f(t, s)] ds.$$

$$(x, y) \in D_1$$

$$(x, y) \in D_2$$
\[
    u(x,y) = \exp[-w_a^{-1}(y)](a-x)^{A(a)}\psi_2(y) \\
    + \exp[-w_b^{-1}(y)](y-b)^{-B(b)} \left[ \psi_1(x) + \int_a^x \exp[w_a^{-1}(t)-w_a^{-1}(x)] \times \right. \\
    \left. \times \left( \frac{a-x}{a-t} \right)^{A(a)} \frac{A(t)}{a-t} \psi_1(t) dt \right] + K_{a,b}^{+} (f(x,y)) - \exp[w_a^{-1}(x)-w_b^{-1}(y)] \times \\
    \times (a-x)^{A(a)}(y-b)^{-B(b)} \int_a^x \int_b^y \Gamma_23(x,y)\; t,s E_7[\psi_1(t),\psi_2(s), f(t,s)] ds. \\
    (x,y) \in D_3 \\
    u(x,y) = \exp[-w_a^{-1}(y)](y-b)^{-B(b)} \left[ \psi_2(x) - \int_a^x \left( \frac{t-a}{x-a} \right)^{A(a)} \times \right. \\
    \left. \times \exp[w_a^{-1}(t)-w_a^{-1}(x)]\frac{A(t)}{t-a} \psi_2(t) dt \right] \\
    + K_{a,b}^{+} (f(x,y)) + \exp[-w_a^{-1}(x)-w_b^{-1}(y)](x-a)^{-A(a)}(b-y)^{-B(b)} \\
    \times \int_a^x dt \int_b^y \Gamma_24(x,y)\; t,s E_6[\psi_2(t), f(t,s)] ds. \\
    (x,y) \in D_4 \\
    
    Here, \\
    K_{a,b}^{-} (f(x,y)), \quad K_{a,b}^{+} (f(x,y)), \quad K_{a,b}^{--} (f(x,y)), \quad K_{a,b}^{++} (f(x,y)) \\
    
    are known integral operators, and \\
    \Gamma_21(x,y; t,s), \quad \Gamma_22(x,y; t,s), \quad \Gamma_23(x,y; t,s), \quad \Gamma_24(x,y; t,s) \\
    
    are resolvents of known integral equation of Volterra type with weak singularity lines, \\
    \psi_j(x), \psi_j(y), \; j = 1,2 \; are \; arbitrary \; continuous \; functions \; for \; the \; boundaries \; \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4. \\
    Moreover, at \; x \to a, \; y \to b, \; \psi_j(x) \; and \; \psi_j(y) \; approach \; zero \; and \; their \; behavior \; is \; determined \; by \; the \; following \; asymptotic \; formulas \\
    \psi_1(x) = o((a-x)^{\delta_1}) \quad \text{as} \; x \to a - o, \quad \delta_1 > A(a) \\
    \psi_2(x) = o((x-a)^{\gamma_2}) \quad \text{as} \; x \to a + o, \\
    \psi_1(y) = o((b-y)^{\delta_2}) \quad \text{as} \; y \to b - o, \\
    \psi_2(y) = o((y-b)^{\gamma_2}) \quad \text{as} \; y \to b + o, \quad \gamma_2 > |B(b)|. \\
    
    THEOREM 3. Let in equation (1) \; A(x) \in C(\Gamma_1 \cup \Gamma_2), \; B(y) \in C(\Gamma_3 \cup \Gamma_4) \; for \; the \; points \; x = a, \; y = b \; they \; satisfy \; the \; Helder's \; condition, \; A(a) > 0, B(b) > 0, C(x,y) \in
Cauchy type problems

\[ C(\mathcal{D}_0), C_1(x,y) = C(x,y) + A(x)B(y) \neq 0, C_1(x,y) \in C(\mathcal{D}_0) \text{ and } C_1(x,y) = o \text{ with following asymptotic behavior on boundaries } \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 : \]

\[ C_1(x,y) = o[|a - x|^n|b - y|^m] \text{ at } x \to a \pm 0, y \to b \pm 0. \]

Moreover, let \( f(x,y) \in C(\mathcal{D}_0), f(a,b) = 0 \) with

\[
\begin{align*}
&f(x,y) = o[(a-x)^\alpha], \quad \alpha > A(a) \text{ at } x \to a - 0, \\
&f(x,y) = o[(x-a)^\alpha], \quad x \to a + 0, \\
&f(x,y) = o[(b-y)^\gamma], \quad \gamma > B(b) \text{ at } y \to b - 0, \\
&f(x,y) = o[(y-b)^\gamma], \quad y \to b + 0.
\end{align*}
\]

Then the non homogeneous integral equation (1) in the class \( C(\mathcal{D}_0) \), approaching zero in \( \Gamma_j, j = 1, 2, 3, 4 \), is always solvable and its general solution contains four arbitrary functions of one variable.

The solution is given by means of the following formulas:

\[
\begin{align*}
u(x,y) &= (a-x)^{A(a)} \exp[-w_1^{-1}(x)]q_1(y) \\
&+ \exp[-w_1^{-1}(y)](b-y)^{B(b)} \left[ q_1(x) + \int_a^x \exp[w_1^{-1}(t)] - w_1^{-1}] \right] \\
&\times \left( \frac{a-x}{a-t} \right)^{A(a)} \frac{A(t)}{a-t} q_1(t) dt \\
&+ K_{x,b}^- (f(x,y)) - \exp[-w_1^{-1}(x) - w_1^{-1}(y)](a-x)^{A(a)}(b-y)^{B(b)} \\
&\times \int_a^a \int_y^y \Gamma_3(x,y,t,s)E_0[q_1(t),q_1(s),f(t,s)] ds. \quad \text{when } (x,y) \in D_1,
\end{align*}
\]

\[
\begin{align*}
u(x,y) &= \exp[-w_1^{-1}(y)](b-y)^{B(b)} \left[ q_2(x) - \int_a^x (t-a)^{A(a)} \\
&\times \exp[-w_1^{-1}(t)] \frac{A(t)}{t-a} q_2(t) dt \\
&+ K_{x,b}^+ (f(x,y)) - \exp[-w_1^{-1}(x) - w_1^{-1}(y)](x-a)^{A(a)}(b-y)^{B(b)} \\
&\times \int_a^x \int_y^y \Gamma_3(x,y,t,s)E_1[q_2(t),f(t,s)] ds. \quad \text{when } (x,y) \in D_2
\end{align*}
\]

\[
\begin{align*}
u(x,y) &= \exp[-w_1^{-1}(x)](a-x)^{A(a)}q_2(y) \\
&+ K_{x,b}^- (f(x,y)) - \exp[-w_1^{-1}(x) - w_1^{-1}(y)](a-x)^{A(a)}(y-b)^{B(b)} \\
&\times \int_a^a \int_y^y \Gamma_3(x,y,t,s)E_1[q_2(s),f(t,s)] ds. \quad \text{when } (x,y) \in D_3,
\end{align*}
\]

\[
\begin{align*}
u(x,y) &= K_{x,b}^+ (f(x,y)) - \exp[-w_1^{-1} - w_1^{-1}(y)](x-a)^{A(a)}(y-b)^{B(b)} \\
&\times \int_a^a \int_y^y \Gamma_3(x,y,t,s)E_3(x,y,t,s)E_3(f(t,s)] ds. \quad \text{when } (x,y) \in D_4.
\end{align*}
\]
\[ K_{a,b}^-(f(x,y)), \quad K_{a,b}^+(f(x,y)), \quad K_{1,a}^-(f(x,y)), \quad K_{a,b}^+(f(x,y)) \]

and

\[ E_0[\psi_1(x), \psi_1(y), f(x,y)], \quad E_{10}[\psi_1(x), f(x,y)], \quad E_{11}[\psi_2(y), (f(x,y))], \quad E_{12}[f(x,y)] \]

are known integral operators, and

\[ \Gamma_1(x; y; t, s), \quad \Gamma_2(x; y; t, s), \quad \Gamma_3(x; y; t, s), \quad \Gamma_4(x; y; t, s) \]

are resolvents of known integral equation of Volterra type with weak singularity lines, \( \psi_j(x), \psi_j(y), j = 1, 2 \) are arbitrary continuous function for the boundaries \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \). Moreover, at \( x \to a, y \to b, \psi_j(x) \) and \( \psi_j(y) \) approach zero and their behavior is determined by the following asymptotic formula

\[ \psi_1(x) = o((a - x)^{\delta_6}) \quad \text{as} \quad x \to a - o, \quad \delta_6 > A(a), \]

\[ \psi_2(x) = o((x - a)^{\gamma_6}) \quad \text{as} \quad x \to a + o, \]

\[ \psi_1(y) = o((b - y)^{\delta_6}) \quad \text{as} \quad y \to b - o, \quad \gamma_6 > B(b), \]

\[ \psi_2(y) = o((y - b)^{\gamma_6}) \quad \text{as} \quad y \to b + o. \]

**Theorem 4.** Let in equation (1) \( A(x) \in C(\Gamma_1 \cup \Gamma_2), B(y) \in C(\Gamma_3 \cup \Gamma_4) \) for the points \( x = a, y = b \) they satisfy the Helder's condition, \( A(a) < 0, B(b) < 0, C(x, y) \in C(\overline{D}_0), C_1(x, y) = C(x, y) + A(x)B(y) \neq 0, C_1(x, y) \in C(\overline{D}_0) \) and \( C_1(x, y) = 0 \) with following asymptotic behavior on boundaries \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 : \)

\[ C_1(x, y) = o(|a - x|^\delta |b - y|^\gamma) \quad \text{as} \quad x \to a \pm o, \quad y \to b \pm o \]

Moreover, let \( f(x, y) \in C(\overline{D}_0), f(a, b) = 0 \) with following asymptotic behavior:

\[ f(x, y) = o(|a - x|^\delta) \quad \text{as} \quad x \to a - o, \]

\[ f(x, y) = o((a - x)^{\delta_7}) \quad \text{as} \quad x \to a + o, \quad \delta_7 > |A(a)|, \]

\[ f(x, y) = o(|b - y|^\gamma) \quad \text{as} \quad y \to b - o, \]

\[ f(x, y) = o((b - y)^{\gamma_7}) \quad \text{as} \quad y \to b + o. \quad \gamma_7 > |B(b)|. \]

Then the non-homogeneous integral equation (1) in the class \( C(\overline{D}_0) \), approaching zero in \( \Gamma_j \) for \( j = 1, 2, 3, 4 \), is always solvable and its general solution contain four arbitrary functions of one variable.

The solution is given by means of following formulas:
\[
\begin{align*}
\begin{cases}
\begin{aligned}
u(x, y) & = K_{a,b}^{-}(f(x, y)) - \exp[-w_A^{-1}(x) - w_B^{-1}(y)](a - x)^{-A(a)}(b - y)^{B(b)} \\
& \times \int_a^b \int_y^b \Gamma_{41}(x, y; t, s) E_{13}[f(t, s)] ds, \quad (x, y) \in D_1
\end{aligned}
\end{cases} \\
\begin{cases}
\begin{aligned}
u(x, y) & = \exp[-w_A^{-1}(x)](x - a)^{-A(a)} \psi_1(y) \\
& + K_{a,b}^{-+}(f(x, y)) - \exp[-w_A^{-1}(x) - w_B^{-1}(y)](a - x)^{-A(a)}(b - y)^{B(b)} \\
& \times \int_a^b \int_y^b \Gamma_{42}(x, y; t, s) E_{14}[\psi_1(t), f(t, s)] ds, \quad (x, y) \in D_2
\end{aligned}
\end{cases} \\
\begin{cases}
\begin{aligned}
u(x, y) & = \exp[-w_B^{-1}(y)](y - b)^{-B(b)} \left[ \psi_1(x) - \frac{a}{a - t} \right] \\
& \times \exp[w_A^{-1}(t) - w_B^{-1}(y)] A(t) \left[ \frac{A(t)}{a - t} \psi_1(t) dt \right] \\
& + K_{a,b}^{++}(f(x, y)) - \exp[-w_A^{-1}(x) - w_B^{-1}(y)](a - x)^{-A(a)}(y - b)^{-B(b)} \\
& \times \int_a^b \int_y^b \Gamma_{43}(x, y; t, s) E_{15}[\psi_1(t), f(t, s)] ds, \quad (x, y) \in D_3
\end{aligned}
\end{cases} \\
\begin{cases}
\begin{aligned}
u(x, y) & = \exp[-w_A^{-1}(y)](x - a)^{-A(a)} \psi_2(y) \\
& + \exp[-w_B^{-1}(y)](y - b)^{-B(b)} \left[ \psi_2(x) - \int_a^x \exp[w_A^{-1}(t) - w_A^{-1}(x)] \\
& \times \left[ \frac{t - a}{x - a} \right] A(t) \left[ \frac{A(t)}{t - a} \psi_2(t) dt \right] \right] \\
& + K_{a,b}^{++}(f(x, y)) - \exp[-w_B^{-1}(y)](x - a)^{-A(a)}(y - b)^{-B(b)} \\
& \times \int_a^b \int_y^b \Gamma_{44}(x, y; t, s) E_{16}[\psi_2(t), \psi_2(s), f(t, s)] ds, \quad (x, y) \in D_4
\end{aligned}
\end{cases}
\end{align*}
\]

Here,

\[K_{a,b}^{-}(f(x, y)), \quad K_{a,b}^{-+}(f(x, y)), \quad K_{a,b}^{++}(f(x, y)), \quad K_{a,b}^{++}(f(x, y))\]

and

\[E_{13}[f(x, y)], \quad E_{14}[\psi_1(y), f(x, y)], \quad E_{15}[\psi_1(x), f(x, y)], \quad E_{16}[\psi_2(x), \psi_2(y), f(x, y)]\]

are known integral operators, and

\[\Gamma_{41}(x, y; t, s), \quad \Gamma_{42}(x, y; t, s), \quad \Gamma_{43}(x, y; t, s), \quad \Gamma_{44}(x, y; t, s)\]

are resolvents of known integral equations of Volterra type with weak singularity lines, \(\psi_j(x), \psi_j(y), j = 1, 2\) are arbitrary continuous functions for the boundaries \(\Gamma_1, \Gamma_2, \Gamma_3\)
and $\Gamma_4$. Moreover, at $x \to a$, $y \to b$, $\psi_j(x)$ and $\psi_j(y)$ approach zero and their behavior is determined by the following asymptotic formulas

\[
\begin{align*}
\varphi_1(x) &= o((a-x)^\gamma) \quad \text{as } x \to a - 0, \\
\varphi_2(x) &= o((x-a)^\delta) \quad \text{as } x \to a + 0, \quad \delta > |A(a)|, \\
\psi_1(y) &= o((b-y)^\gamma) \quad \text{as } y \to b - 0, \\
\psi_2(y) &= o((y-b)^\delta) \quad \text{as } y \to b + 0. \quad \delta > |B(b)|. 
\end{align*}
\]

4. Four problems

PROBLEMS Find a solution of the integral equation (1), belonging to the class $C(\overline{D_0})$ and approaching zero on $\Gamma_j$ for $j = 1, 2, 3, 4$, such that it satisfies one of the following conditions. These respect in order the inequalities $(A(a), B(b))$ of Theorems 1.2.3.4.

\[
P_1 : \ A(a) < 0, \ B(b) > 0 \quad \left\{ \begin{array}{c}
(b-y)^{-B(b)}u(x,y) \bigg|_{y \to b - a} = f_1(x) \quad \text{for } x \in \Gamma_1, \\
(b-y)^{-B(b)}u(x,y) \bigg|_{y \to b - a} = f_2(x) \quad \text{for } x \in \Gamma_2, \\
(x-a)^{-A(a)}u(x,y) \bigg|_{x \to a + a} = g_1(y) \quad \text{for } y \in \Gamma_3, \\
(x-a)^{-A(a)}u(x,y) \bigg|_{x \to a + a} = g_2(y) \quad \text{for } y \in \Gamma_4, \\
(b-y)^{-B(b)}u(x,y) \bigg|_{y \to b - a} = 0. \\
\end{array} \right.
\]

\[
P_2 : \ A(a) > 0, \ B(b) < 0 \quad \left\{ \begin{array}{c}
(y-b)^{B(b)}u(x,y) \bigg|_{y \to b + a} = f_1(x) \quad \text{for } x \in \Gamma_1, \\
(y-b)^{B(b)}u(x,y) \bigg|_{y \to b + a} = f_2(x) \quad \text{for } x \in \Gamma_2, \\
(a-x)^{-A(a)}u(x,y) \bigg|_{x \to a - a} = g_1(y) \quad \text{for } y \in \Gamma_3, \\
(a-x)^{-A(a)}u(x,y) \bigg|_{x \to a - a} = g_2(y) \quad \text{for } y \in \Gamma_4, \\
(y-b)^{B(b)}u(x,y) \bigg|_{y \to b + a} = 0. \\
\end{array} \right.
\]

\[
P_3 : \ A(a) > 0, \ B(b) > 0 \quad \left\{ \begin{array}{c}
(b-y)^{-B(b)}u(x,y) \bigg|_{y \to b - a} = f_1(x) \quad \text{for } x \in \Gamma_1, \\
(b-y)^{-B(b)}u(x,y) \bigg|_{y \to b - a} = f_2(x) \quad \text{for } x \in \Gamma_2, \\
(a-x)^{-A(a)}u(x,y) \bigg|_{x \to a - a} = g_1(y) \quad \text{for } y \in \Gamma_3, \\
(a-x)^{-A(a)}u(x,y) \bigg|_{x \to a - a} = g_2(y) \quad \text{for } y \in \Gamma_4, \\
(b-y)^{-B(b)}u(x,y) \bigg|_{y \to b - a} = 0. \\
\end{array} \right.
\]
1) \( f_1(x) \in C(\Gamma_1), \ f_2(x) \in C(\Gamma_2), \ g_1(y) \in C(\Gamma_3), \ g_2(y) \in C(\Gamma_4). \)

2) \( f_1(a) = 0, \ f_2(a) = 0, \ g_1(b) = 0, \ g_2(b) = 0, \) with asymptotic behavior

- \( f_1(x) = o((a-x)^{\delta_2}) \) as \( x \to a - 0, \)
- \( f_2(x) = o((x-a)^{\delta_2}) \) as \( x \to a - 0, \) \( \delta_2 > |A(a)|, \)
- \( g_1(y) = o((b-y)^{\gamma_2}) \) as \( y \to b - 0, \) \( \gamma_2 > B(b), \)
- \( g_2(x) = o((y-b)^{\gamma_2}) \) as \( y \to b + 0. \)

then problem \( P_1 \) has a unique solution and it can be written in the form (2), in the case in which

\[
\begin{align*}
\Psi_1(y) &= g_1(y), & \text{when } y &\in \Gamma_3, \\
\Psi_2(y) &= g_2(y), & \text{when } y &\in \Gamma_4, \\
\varphi_1(x) &= c_1 + f_1(x) - \int_x^a \frac{A(t)}{a-t} f_1(t) \, dt, & \text{when } x &\in \Gamma_1, \\
\varphi_2(x) &= f_2(x) - \int_x^a \frac{A(t)}{a-t} f_2(t) \, dt, & \text{when } x &\in \Gamma_2, \quad c_1 = 0.
\end{align*}
\]

\[ \text{THEOREM 6. Suppose that } A(x), B(y), C(x, y), f(x, y) \text{ satisfy the hypotheses of Theorem 2. If} \]

1) \( f_1(x) \in C(\Gamma_1), \ f_2(x) \in C(\Gamma_2), \ g_1(y) \in C(\Gamma_3), \ g_2(y) \in C(\Gamma_4). \)

2) \( f_1(a) = 0, \ f_2(a) = 0, \ g_1(b) = 0, \ g_2(b) = 0, \) with asymptotic behavior

- \( f_1(x) = o((a-x)^{\delta_4}) \) as \( x \to a - 0, \) \( \delta_4 > |A(a)|, \)
- \( f_2(x) = o((x-a)^{\delta_4}) \) as \( x \to a - 0, \)
- \( g_1(x) = o((b-y)^{\gamma_4}) \) as \( y \to b - 0, \) \( \gamma_4 > |B(b)|. \)
then problem $P_2$ has a unique solution and it can be written in the form (3), in the case when

$$\psi_1(y) = g_1(y) \quad \text{when } y \in \Gamma_3,$$

$$\psi_2(y) = g_2(y) \quad \text{when } y \in \Gamma_4,$$

$$q_1(x) = f_1(x) - \int_a^x \frac{A(t)}{a-t} f_1(t) \, dt, \quad \text{when } x \in \Gamma_1,$$

$$q_2(x) = c_2 + f_2(x) + \int_a^x \frac{A(t)}{t-a} f_2(t) \, dt, \quad \text{when } x \in \Gamma_2, \quad c_2 = 0.$$ 

**Theorem 7.** Suppose that $A(x), B(y), C(x,y), f(x,y)$ satisfy the hypotheses of Theorem 3. If

1) $f_1(x) \in C(\Gamma_1), f_2(x) \in C(\Gamma_2), g_1(y) \in C(\Gamma_3), g_2(y) \in C(\Gamma_4),$

2) $f_1(a) = 0, f_2(a) = 0, g_1(b) = 0, g_2(b) = 0$ with asymptotic behavior

$$f_1(x) = o[(a-x)^{\delta_6}] \quad \text{as } x \to a-o, \quad \delta_6 > A(a),$$

$$f_2(x) = o[(x-a)^{\gamma_6}] \quad \text{as } x \to a-o,$$

$$g_1(x) = o[(b-y)^{\gamma_6}] \quad \text{as } y \to b-o, \quad \gamma_6 > B(b),$$

$$g_2(x) = o[(y-b)^{\delta_6}] \quad \text{as } y \to b+o.$$

then problem $P_3$ has a unique solution and it can be written in the form (4) in the case when

$$\psi_1(y) = g_1(y) \quad \text{when } y \in \Gamma_3,$$

$$\psi_2(y) = g_2(y) \quad \text{when } y \in \Gamma_4,$$

$$q_1(x) = f_1(x) - \int_x^a \frac{A(t)}{a-t} f_1(t) \, dt, \quad \text{when } x \in \Gamma_1,$$

$$q_2(x) = c_3 + f_2(x) + \int_a^x \frac{A(t)}{t-a} f_2(t) \, dt, \quad \text{when } x \in \Gamma_2, \quad c_3 = 0.$$ 

**Theorem 8.** Let the integral equation (1) $A(x), B(y), C(x,y), f(x,y)$ satisfy the hypotheses of Theorem 4. If

1) $f_1(x) \in C(\Gamma_1), f_2(x) \in C(\Gamma_2), g_1(y) \in C(\Gamma_3), g_2(y) \in C(\Gamma_4),$

2) $f_1(a) = 0, f_2(a) = 0, g_1(b) = 0, g_2(b) = 0$, with asymptotic behavior

$$f_1(x) = o[(a-x)^{\delta_8}] \quad \text{as } x \to a-o,$$

$$f_2(x) = o[(x-a)^{\delta_8}] \quad \text{as } x \to a-o, \quad \delta_8 > |A(a)|,$$

$$g_1(x) = o[(b-y)^{\gamma_8}] \quad \text{as } y \to b-o,$$

$$g_2(x) = o[(y-b)^{\gamma_8}] \quad \text{as } y \to b+o. \quad \gamma_8 > |B(b)|.$$ 

then problem $P_4$ has a unique solution and it can be written in the form (5) in the case
when
\[
\begin{align*}
\psi_1(y) &= g_1(y) & \text{when } y \in \Gamma_3, \\
\psi_2(y) &= g_2(y) & \text{when } y \in \Gamma_4, \\
\phi_1(x) &= c_4 + f_1(x) - \int_x^a \frac{A(t)}{a-t} f_1(t) \, dt, & \text{when } x \in \Gamma_1, \\
\phi_2(x) &= f_2(x) + \int_a^x \frac{A(t)}{t-a} f_2(t) \, dt & \text{when } x \in \Gamma_2, c_4 = 0.
\end{align*}
\]

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Lutfiya RADZHABOVA
Tajik Technical University
Dushanbe, TAJIKISTAN
e-mail: lutfya62@mail.ru

M. Ruzhansky\textsuperscript{*} - M. Sugimoto

SMOOTHING PROPERTIES OF INHOMOGENEOUS EQUATIONS VIA CANONICAL TRANSFORMS

Abstract. The paper describes a new approach to global smoothing problems for inhomogeneous dispersive evolution equations based on an idea of canonical transformation. In our previous papers [20, 21], we introduced such a method to show global smoothing estimates for homogeneous dispersive equations. It is remarkable that this method allows us to carry out a global microlocal reduction of equations to some low dimensional model cases. The purpose of this paper is to pursue the same treatment for inhomogeneous equations. Especially, time-global smoothing estimates for the operator $a(D_x)$ with lower order terms are the benefit of our new method.

1. Introduction

This article consists partly of a survey of the arguments developed in author’s recent paper [21] (Sections 2 and 3) and partly of obtaining new results via the extension and continuation of these arguments (Sections 4 and 5).

Let us first consider the following Schrödinger equation:

$$
\begin{cases}
(i \partial_t + \Delta_x) u(t,x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^n, \\
u(0,x) = \phi(x) & \text{in } \mathbb{R}^n.
\end{cases}
$$

We know that the solution operator $e^{it\Delta_x}$ preserves the $L^2$-norm for each fixed $t \in \mathbb{R}$. On the other hand, the extra gain of regularity of order $1/2$ in $x$ can be observed if we integrate the solution in $t$. For example we have the estimate

$$
\left\| \langle x \rangle^{-s} |D_x|^{1/2} e^{it\Delta_x} \phi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n)} \leq C \|\phi\|_{L^2(\mathbb{R}^n)} \quad (s > 1/2)
$$

for $u = e^{it\Delta_x} \phi$ and this estimate was first given by Ben-Artzi and Klainerman [3] ($n \geq 3$). Since the independent pioneering works by Constantin and Saut [10], Sjölin [24] and Vega [27], the local, then the global smoothing estimates for Schrödinger or more general dispersive equations have been intensively investigated. (Smoothing for generalised Korteweg-de Vries equations was already studied by Kato [13].) There has already been a lot of literature on this subject: Ben-Artzi and Devinatz [1, 2], Chihara [9], Hoshiro [11, 12], Kato and Yajima [14], Kenig, Ponce and Vega [4]–[8], Linares and Ponce [18], Simon [23], Sugimoto [25, 26], Walther [28, 29], and many others.

In our previous papers [20, 21], we introduced a new method to show global smoothing estimates for Schrödinger equations, or more generally, those for homoge-
neous dispersive equations:

$$
\begin{cases}
(i\partial_t + a(D_x))u(t,x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\
u(0,x) = \varphi(x) & \text{in } \mathbb{R}_x^n.
\end{cases}
$$

where $a(\xi)$ is a real-valued function of $\xi = (\xi_1, \ldots, \xi_n)$ with the growth of order $m$, and $a(D_x)$ is the corresponding operator. The main idea was to change the equation

$$(i\partial_t + a(D_x))u(t,x) = 0 \quad \text{to} \quad (i\partial_t + \sigma(D_x))v(t,x) = 0,$$

where the operators $a(D_x)$ and $\sigma(D_x)$ are related with each other by the relation

$$a(\xi) = (\sigma \circ \psi)(\xi).$$

Such an idea can be realised by a canonical transformation $T$ in the following way:

$$a(D_x) \circ T = T \circ \sigma(D_x).$$

If now operators $T$ and $T^{-1}$ are bounded in $L^2(\mathbb{R}_x^n)$ and in weighted $L^2(\mathbb{R}_x^n)$ respectively, we can reduce global smoothing estimates for $u = e^{ita(D_x)}\varphi$ to those for $v = e^{it\sigma(D_x)}\varphi$. It is remarkable that the method of canonical transformations described above allows us to carry out a global microlocal reduction of equation (1) to the model cases $a(\xi) = |\xi_n|^m$ (elliptic case) or $a(\xi) = \xi_1|\xi_n|^{m-1}$ (non-elliptic case) under a dispersive-ness condition.

The purpose of this paper is to pursue the same treatment for inhomogeneous equations:

$$
\begin{cases}
(i\partial_t + a(D_x))u(t,x) = f(t,x) & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\
u(0,x) = 0 & \text{in } \mathbb{R}_x^n.
\end{cases}
$$

We will obtain the corresponding results on the global smoothing for solutions to inhomogeneous problems. There are considerably less results on this topic available in the literature. Mostly the Schrödinger equation was treated (e.g. Linares and Ponce [18], Kenig, Ponce and Vega [8]), or the one dimensional case (Kenig, Ponce and Vega [5, 7] or Laurey [17]). Some more general results on the local smoothing for dispersive operators were obtained by Chihara [9] and Hoshiro [12], and for dispersive differential operators by Koch and Saut [15]. In this paper we will extend these results in two directions: we will establish the global smoothing for rather general dispersive equations of different orders in all dimensions. Especially, these kinds of time-global estimate for the operator $a(D_x)$ with lower order terms are the benefit of our new method. The treatment of the inhomogeneous equations may allow one to treat nonlinear equations with lower order terms and with corresponding nonlinearities, see the author’s paper [22] for one example.

We will explain the organisation of this paper. In Section 2, we introduce our main tools established by the authors in [21], which originate in the idea of canonical transformation. In Section 3, we list results of smoothing estimates for homogeneous equations which were partially announced in [20] and will be completely given in [21].
The operators discussed in [21]. We will first review our main tool to reduce general operators
2. Canonical transforms
Let \( \psi : \Gamma \to \tilde{\Gamma} \) be a \( C^\infty \)-diffeomorphism between open sets \( \Gamma \subset \mathbb{R}^n \) and \( \tilde{\Gamma} \subset \mathbb{R}^n \). We always assume that
\[
(2) \quad C^{-1} \leq |\text{det} \partial \psi(\xi)| \leq C \quad (\xi \in \Gamma),
\]
for some \( C > 0 \). We set formally
\[
I_\psi u(x) = \mathcal{F}^{-1} [\mathcal{F} \psi(\xi)] (x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-\xi) \nabla \psi(\xi)} u(y) dy d\xi,
\]
\[
I_\psi^{-1} u(x) = \mathcal{F}^{-1} [\mathcal{F} \psi^{-1} \psi^{-1}(\xi)] (x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-\xi) \nabla \psi^{-1}(\xi)} u(y) dy d\xi.
\]
The operators \( I_\psi \) and \( I_\psi^{-1} \) can be justified by using cut-off functions \( \gamma \in C^\infty(\Gamma) \) and \( \tilde{\gamma} = \gamma \circ \psi^{-1} \in C^\infty(\tilde{\Gamma}) \) which satisfy \( \text{supp} \gamma \subset \Gamma \), \( \text{supp} \tilde{\gamma} \subset \tilde{\Gamma} \). We set
\[
I_{\psi, \tilde{\gamma}} u(x) = \mathcal{F}^{-1} [\tilde{\gamma}(\xi) \mathcal{F} u(\psi(\xi))] (x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-\xi) \nabla \psi(\xi)} \gamma(\xi) u(y) dy d\xi,
\]
\[
I_{\psi, \tilde{\gamma}}^{-1} u(x) = \mathcal{F}^{-1} [\tilde{\gamma}(\xi) \mathcal{F} u(\psi^{-1}(\xi))] (x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-\xi) \nabla \psi^{-1}(\xi)} \gamma(\xi) u(y) dy d\xi.
\]
In the case that \( \Gamma, \tilde{\Gamma} \subset \mathbb{R}^n \setminus 0 \) are open cones, we may consider the homogeneous \( \psi \) and \( \gamma \) which satisfy \( \text{supp} \gamma \cap \mathbb{S}^{n-1} \subset \Gamma \cap \mathbb{S}^{n-1} \) and \( \text{supp} \tilde{\gamma} \cap \mathbb{S}^{n-1} \subset \tilde{\Gamma} \cap \mathbb{S}^{n-1} \), where \( \mathbb{S}^{n-1} = \{ \xi \in \mathbb{R}^n : |\xi| = 1 \} \). Then we have the expressions for compositions
\[
(4) \quad I_{\psi, \tilde{\gamma}} = \gamma(D_x) \cdot I_\psi = I_\psi \cdot \tilde{\gamma}(D_x), \quad I_{\psi, \tilde{\gamma}}^{-1} = \gamma(D_x) \cdot I_\psi^{-1} = I_\psi^{-1} \cdot \tilde{\gamma}(D_x),
\]
and the identities
\[
(5) \quad I_{\psi, \tilde{\gamma}} \cdot I_{\psi, \tilde{\gamma}}^{-1} = \gamma(D_x)^2, \quad I_{\psi, \tilde{\gamma}}^{-1} \cdot I_{\psi, \tilde{\gamma}} = \tilde{\gamma}(D_x)^2.
\]
We have also the formulae
\[ I_{q^*,\gamma} \cdot \sigma(D_x) = (\sigma \circ \psi)(D_x) \cdot I_{q^*,\gamma}, \quad I_{q^*,\gamma}^{-1} \cdot (\sigma \circ \psi)(D_x) = \sigma(D_x) \cdot I_{q^*,\gamma}^{-1}. \]

We also introduce the weighted \( L^2 \)-spaces. For the weight function \( w(x) \), let \( L^2_w(\mathbb{R}^n; w) \) be the set of measurable functions \( f : \mathbb{R}^n \to \mathbb{C} \) such that the norm
\[
\|f\|_{L^2(\mathbb{R}^n; w)} = \left( \int_{\mathbb{R}^n} |w(x)f(x)|^2\,dx \right)^{1/2}
\]
is finite. Then, from the relations (4), (5), and (6), we obtain the following fundamental theorem ([21, Theorem 4.1]):

**Theorem 1.** Assume that the operator \( I_{q^*,\gamma} \) defined by (3) is \( L^2(\mathbb{R}^n; w) \)-bounded. Suppose that we have the estimate
\[ \left\| w(x)p(D_x) e^{i\alpha(D_x)} q(x) \right\|_{L^2(\mathbb{R}^n; w)} \leq C \|q\|_{L^2(\mathbb{R}^n; w)} \]
for all \( \psi \) such that \( \text{supp} \, \tilde{\psi} \subset \text{supp} \, \gamma \), where \( \gamma = \gamma \circ \psi^{-1} \). Assume also that the function
\[ q(\xi) = \frac{\gamma \circ \psi}{\rho \circ \psi}(\xi) \]
is bounded. Then we have
\[ \left\| w(x)\xi(D_x) e^{i\alpha(D_x)} q(x) \right\|_{L^2(\mathbb{R}^n; w)} \leq C \|q\|_{L^2(\mathbb{R}^n; w)} \]
for all \( \psi \) such that \( \text{supp} \, \tilde{\psi} \subset \text{supp} \, \gamma \), where \( \alpha(\xi) = (\sigma \circ \psi)(\xi) \).

Note that \( e^{i\alpha(D_x)} q(x) \) and \( e^{i\sigma(D_x)} q(x) \) are solutions to
\[
\begin{cases}
(i\partial_t + a(D_x)) u(t,x) = 0, & u(0,x) = q(x),
\end{cases}
\]
and
\[
\begin{cases}
(i\partial_t + \sigma(D_x)) v(t,x) = 0, & v(0,x) = g(x),
\end{cases}
\]
respectively. Theorem 1 means that smoothing estimates for equation with \( \alpha(D_x) \) implies those with \( \sigma(D_x) \) if the canonical transformations which relate them are bounded on weighted \( L^2 \)-spaces. The same thing is true for inhomogeneous equations
\[
\begin{cases}
(i\partial_t + a(D_x)) u(t,x) = f(t,x), & \quad \begin{cases}
(i\partial_t + \alpha(D_x)) v(t,x) = f(t,x), & v(0,x) = 0,
\end{cases}
\end{cases}
\]
whose solutions are \( -i \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x)\,d\tau \) and \( -i \int_0^t e^{i(t-\tau)\sigma(D_x)} f(\tau,x)\,d\tau \), respectively. The only difference is that we need the weighted \( L^2 \)-boundedness of the operator \( I_{q^*,\gamma}^{-1} \) instead of just the \( L^2 \)-boundedness of it induced by the boundedness of \( q(\xi) \):
THEOREM 2. Assume that the operator \( I_{q,\gamma} \) defined by (3) is \( L^2(\mathbb{R}^n; w) \)-bounded. Suppose that we have the estimate
\[
\left\| w(x)\rho(D_x) \int_0^\infty e^{i(t-\tau)\omega(D_x)} f(\tau, x) d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_\gamma)} \leq C\|v(x)f(t, x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_\gamma)}
\]
for all \( f \) such that \( \text{supp } \mathcal{F} f(t, \cdot) \subset \text{supp } \tilde{\gamma} \), where \( \tilde{\gamma} = \gamma \circ \psi^{-1} \). Also assume that the operator \( I_{q,\gamma}^{-1} \) defined by (3) with \( q(\xi) = (\gamma \cdot \tilde{\xi})/(\rho \circ \psi)(\xi) \) is \( L^2(\mathbb{R}^n; v) \)-bounded. Then we have
\[
\left\| w(x)\xi(D_x) \int_0^\infty e^{i(t-\tau)\omega(D_x)} f(\tau, x) d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_\gamma)} \leq C\|v(x)f(t, x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_\gamma)}
\]
for all \( f \) such that \( \text{supp } \mathcal{F} f(t, \cdot) \subset \text{supp } \gamma \), where \( a(\xi) = (\sigma \circ \psi)(\xi) \).

The proof of Theorem 1 is given in [21], and that of Theorem 2 is just a slight modification of it, hence here we omit them.

As for the \( L^2(\mathbb{R}^n; w) \)-boundedness of the operator \( I_{q,\gamma} \), we have criteria for some special weight functions. For \( \kappa \in \mathbb{R} \), let \( L^2_\kappa(\mathbb{R}^n) \), \( L^2_\infty(\mathbb{R}^n) \) be the set of measurable functions \( f \) such that the norm
\[
\|f\|_{L^2_\kappa(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |x|^\kappa |f(x)|^2 dx \right)^{1/2}, \quad \|f\|_{L^2_\infty(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |x|^\kappa |f(x)|^2 dx \right)^{1/2}
\]
is finite, respectively. Then we have the following mapping properties ([21, Theorems 4.2 and 4.3]).

THEOREM 3. Suppose \( \kappa \in \mathbb{R} \). Assume that all the derivatives of entries of the \( n \times n \) matrix \( \partial \psi \) and those of \( \gamma \) are bounded. Then the operators \( I_{q,\gamma} \) and \( I_{q,\gamma}^{-1} \) defined by (3) are \( L^2_\kappa(\mathbb{R}^n) \)-bounded.

THEOREM 4. Let \( \Gamma, \tilde{\Gamma} \subset \mathbb{R}^n \setminus 0 \) be open cones. Suppose \( |\kappa| < n/2 \). Assume \( \psi(\lambda \xi) = \lambda \psi(\xi), \gamma(\lambda \xi) = \gamma(\xi) \) for all \( \lambda > 0 \) and \( \xi \in \Gamma \). Then the operators \( I_{q,\gamma} \) and \( I_{q,\gamma}^{-1} \) defined by (3) are \( L^2_\kappa(\mathbb{R}^n) \)-bounded and \( L^2_\infty(\mathbb{R}^n) \)-bounded.

We remark that the following result due to Kurtz and Wheeden [16, Theorem 3] is essentially used to prove Theorem 4:

LEMMA 1. Suppose \( |\kappa| < n/2 \). Assume that \( m(\xi) \in C^0(\mathbb{R}^n \setminus 0) \) and all the derivative of \( m(\xi) \) satisfies \( |\partial^\alpha m(\xi)| \leq C_\alpha|\xi|^{-n} \) for all \( \xi \neq 0 \) and \( |\alpha| \leq n \). Then \( m(D_x) \) is \( L^2_\kappa(\mathbb{R}^n) \) and \( L^2_\infty(\mathbb{R}^n) \)-bounded.
3. Smoothing estimates for homogeneous dispersive equations

In author’s paper [21], it is explained how to derive smoothing estimates for general homogeneous dispersive equations from model estimates as an application of the canonical transformations described in Section 2. We will repeat it here to help readers to understand the later part of this paper concerning estimates for inhomogeneous equations.

Let us consider the solution
\[ u(t, x) = e^{iα(D)}\phi(x) \]

for the homogeneous equation
\[
\begin{align*}
(i\partial_t + a(D))u(t, x) &= 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}^n, \\
u(0, x) &= \phi(x) \quad \text{in } \mathbb{R}^n,
\end{align*}
\]

where we always assume that function \( a(ξ) \) is real-valued. Let \( a_m(ξ) \in C^∞(\mathbb{R}^n \setminus 0) \), the principal part of \( a(ξ) \), be a positively homogeneous function of order \( m \), that is, satisfy \( a_m(λξ) = λ^m a_m(ξ) \) for all \( λ > 0 \) and \( ξ \neq 0 \).

First we consider the case that \( a(ξ) \) has no lower order terms, and assume that \( a(ξ) \) is dispersive:
\[ (H) \quad a(ξ) = a_m(ξ), \quad ∀ a_m(ξ) \neq 0 \quad (ξ ∈ \mathbb{R}^n \setminus 0), \]

where \( V = (\partial_1, \ldots, \partial_n) \) and \( ∂_j = ∂_{ξ_j} \). A typical example is \( a(ξ) = a_m(ξ) = |ξ|^m \). Especially, \( a(ξ) = a_2(ξ) = |ξ|^2 \) is the case of the Schrödinger equation.

The following result ([21, Theorem 5.1]) is a generalisation of the one given by Ben-Artzi and Klainerman [3] which treated the case \( a(ξ) = |ξ|^2 \) and \( n ≥ 3 \):

**Theorem 5.** Assume \((H)\). Suppose \( n ≥ 1, m > 0, \) and \( s > 1/2 \). Then we have
\[
\left\| D_\lambda^{-(m-1)/2} e^{it|D_\lambda|^m} \phi(x) \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \phi(x) \right\|_{L^2(\mathbb{R}^n)}.
\]

We review how to prove Theorem 5. The main idea is reducing them to the special cases \( a(D_n) = |D_n|^m, D_λ = D_n |D_n|^{m-1}, \) where \( D_λ = (D_1, \ldots, D_n) \), by using Theorem 1. The following estimates ([21, Theorem 3.1, Corollary 3.3]) for them act as model ones:

**Proposition 1.** Suppose \( n = 1 \) and \( m > 0 \). Then we have
\[
\left\| D_λ^{(m-1)/2} e^{it|D_λ|^m} \phi(x) \right\|_{L^2(\mathbb{R}_t)} \leq C \left\| \phi(x) \right\|_{L^2(\mathbb{R})}
\]
for all $x \in \mathbb{R}$. Suppose $n = 2$ and $m > 0$. Then we have

$$
\left\| D_x^{(m-1)/2} e^{it D_x} D_t \psi(x, t) \right\|_{L^2(\mathbb{R} \times \mathbb{R}_t)} \leq C \| \psi \|_{L^2(\mathbb{R}^2)}
$$

for all $x \in \mathbb{R}$.

**Corollary 1.** Suppose $n \geq 1$, $m > 0$, and $s > 1/2$. Then we have

$$
\left\| \langle x \rangle^{-s} D_n^{(m-1)/2} e^{it D_n} \psi(x) \right\|_{L^2(\mathbb{R} \times \mathbb{R}_t^n)} \leq C \| \psi \|_{L^2(\mathbb{R}^2)}.
$$

Suppose $n \geq 2$, $m > 0$, and $s > 1/2$. Then we have

$$
\left\| \langle x \rangle^{-s} D_n^{(m-1)/2} e^{it D_n} \psi(x) \right\|_{L^2(\mathbb{R} \times \mathbb{R}_t^n)} \leq C \| \psi \|_{L^2(\mathbb{R}^2)}.
$$

We assume (H). Let $\Gamma \subset \mathbb{R}^n \setminus 0$ be a sufficiently small conic neighbourhood of $e_n = (0, \ldots, 0, 1)$, and take a cut-off function $\gamma(\xi) \in C^\infty(\Gamma)$ which is positively homogeneous of order 0 and satisfies $\text{supp} \gamma \cap S^{n-1} \subset \Gamma \cap S^{n-1}$. By the microlocalisation and the rotation of the initial data $\psi$, we may assume $\text{supp} \hat{\psi} \subset \text{supp} \gamma$. The dispersive assumption $\nabla a_m(e_n) \neq 0$ in this direction implies the following two possibilities:

(i) $\partial_n a_m(e_n) \neq 0$. Then, by Euler’s identity $a_m(\xi) = (1/m) \nabla a_m(\xi) \cdot \xi$, we have $a_m(e_n) \neq 0$. Hence, in this case, we may assume that $a(\xi)(> 0)$ and $\partial_n a(\xi)$ are bounded away from 0 for $\xi \in \Gamma$.

(ii) $\partial_n a_m(e_n) = 0$. Then there exits $j \neq n$ such that $\partial_j a_m(e_n) \neq 0$, say $\partial_1 a_m(e_n) \neq 0$. Hence, in this case, we may assume $\partial_1 a(\xi)$ is bounded away from 0 for $\xi \in \Gamma$. We remark $a(e_n) = 0$ by Euler’s identity.

The estimate with the case $n = 1$ is given by estimate (11) in Corollary 1. In fact, we have $a(\xi) = a(1) |\xi|^m$ for $\xi > 0$ in this case. Hence we may assume $n \geq 2$. We remark that it is sufficient to show theorem with $1/2 < s < n/2$ because the case $s \geq n/2$ is easily reduced to this case. We will use the notation $\xi = (\xi_1, \ldots, \xi_n)$, $\eta = (\eta_1, \ldots, \eta_n)$.

In the case (i), we take

$$
\alpha(\eta) = |\eta_n|^m, \quad \psi(\xi) = (\xi_1, \ldots, \xi_{n-1}, a(\xi)^{1/m}).
$$

Then we have $a(\xi) = (\alpha \circ \psi)(\xi)$ and

$$
\det \partial \psi(\xi) = \begin{vmatrix}
E_{n-1} & 0 \\
(1/m) a(\xi)^{1/m-1} \partial_n a(\xi)
\end{vmatrix},
$$

where $E_{n-1}$ is the identity matrix of order $n - 1$. We remark that (2) is satisfied since

$$
\det \partial \psi(e_n) = (1/m) a(e_n)^{1/m-1} \partial_n a(e_n) \neq 0.
$$

By estimate (11) in Corollary 1, we have
estimate (7) in Theorem 1 with \( \sigma(D_x) = |D_x|^m \), \( w(x) = \langle x \rangle^{-\gamma} \), and \( \rho(\xi) = |\xi|^{(m-1)/2} \). Note here the trivial inequality \( \langle x \rangle^{-\gamma} \leq \langle x_n \rangle^{-\gamma} \). If we take \( \xi(\xi) = |\xi|^{(m-1)/2} \), then \( q(\xi) = \gamma(\xi)^{(m-1)/2} \) defined by (8) is a bounded function. On the other hand, \( f_{\gamma,\gamma} \) is \( L^2_{s,\gamma} \)-bounded for \( 1/2 < s < n/2 \) by Theorem 4. Hence, by Theorem 1, we have estimate (9), that is, estimate (10).

In the case (ii), we take

\[
\sigma(\eta) = \eta_1|\eta_1|^{m-1}, \quad \psi(\xi) = (a(\xi)|\xi_n|^{-m}, \xi_2, \ldots, \xi_n)
\]

Then we have \( a(\xi) = (\sigma \circ \psi)(\xi) \) and

\[
\det \partial \psi(\xi) = \begin{vmatrix}
\partial_1 a(\xi) & |\xi_n|^{-m} \\
0 & \ast
\end{vmatrix}
\]

Since \( \det \partial \psi(\xi) = \delta_{1a}(\xi) \neq 0 \), (2) is satisfied. Similarly to the case (i), the estimate for \( \sigma(D_x) = D_D|D_{\eta}|^m \) is given by estimate (12) in Corollary 1, which implies estimate (10) again by Theorem 1.

As another advantage of the method explained here, we can also consider the case that \( a(\xi) \) has lower order terms, and assume that \( a(\xi) \) is dispersive in the following sense:

\[
\text{(L)} \quad a(\xi) \in C^\infty(\mathbb{R}^n), \quad \nabla a(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0), \quad |\partial(\xi) - a_m(\xi)| \leq C_n|\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices} \ \alpha \text{and all} \ |\xi| \geq 1.
\]

or equivalently

\[
\text{(L)} \quad a(\xi) \in C^\infty(\mathbb{R}^n), \quad |\nabla a(\xi)| \geq C(\xi)^{m-1} \quad (\xi \in \mathbb{R}^n) \quad \text{for some} \ C > 0, \quad |\partial(\xi) - a_m(\xi)| \leq C_n|\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices} \ \alpha \text{and all} \ |\xi| \geq 1.
\]

The last lines of these assumptions simply amount to saying that the principal part \( a_m \) of \( a \) is positively homogeneous of order \( m \) for \( |\xi| \geq 1 \).

The following result ([21, Theorem 5.4]) is also derived from Corollary 1:

**THEOREM 6.** Assume (L). Suppose \( n \geq 1, m > 0, \) and \( s > 1/2 \). Then we have

\[
\| \langle x \rangle^{-s}(D_x)^{(m-1)/2} e^{i\alpha(D_x)} \psi(x) \|_{L^2(\mathbb{R}^n \times \mathbb{R}^s)} \leq C\| \psi \|_{L^2(\mathbb{R}^n)}.
\]

We review how to prove Theorem 6. We sometimes decompose the initial data \( \psi \) into the sum of the low frequency part \( \psi_l \) and the high frequency part \( \psi_h \), where \( \text{supp} \psi_l \subset \{ \xi : |\xi| < 2R \} \) and \( \text{supp} \psi_h \subset \{ \xi : |\xi| > R \} \) with sufficiently large \( R > 0 \). Each part can be realised by multiplying \( \chi(D_x) \) or \( (1 - \chi) (D_x) \) to \( \psi(x) \), hence to \( u(t,x) \), where \( \chi \in C_0^\infty(\mathbb{R}^n) \) is an appropriate cut-off function. For high frequency part, the same
argument as in the proof of Theorem 5 is valid. (Furthermore, we can use Theorem 3 instead of Theorem 4 to assure the boundedness of \( I_{p,q} \), hence we need not assume \( n \geq 2 \).) We show how to get the estimates for low frequency part. Because of the compactness of it, we may assume \( \xi = (\xi_1, \ldots, \xi_n) \) instead of Theorem 4 to assure the boundedness of the argument as in the proof of Theorem 5 is valid. (Furthermore, the assumptions for the smoothing properties of inhomogeneous equations set \( (c) \) and \( \sigma(\eta) \) instead of Theorem 4 to assure the boundedness of the argument as in the proof of Theorem 5 is valid. (Furthermore, the assumptions for the smoothing properties of inhomogeneous equations set \( (c) \) and \( \sigma(\eta) \) are verified if we notice (14). By estimate (11) in Corollary 1, we have estimate (7) in Theorem 1 with \( a(D_x) = |D_x|^{m-1}, \omega(x) = \langle x \rangle^{-s} \) (\( s > 1/2 \)), and \( \rho(x) = |x|^{-m} \) as in the proof of Theorem 5. If we take \( \xi(\xi) = (\langle \xi \rangle^{(m-1)/2} + q(\xi) \xi/ |a(\xi)|^{1/m})^{(m-1)/2} \) defined by (8) is a bounded function. On the other hand, \( I_{p,q} \) is \( L^2 \)-bounded for all \( s > 1/2 \) by Theorem 3. Hence, by Theorem 1, we have estimate (9), that is, estimate (15).

Finally, we introduce an intermediate assumption between (H) and (L), and discuss what happens if we do not have the condition \( \nabla a(\xi) \neq 0 \):

\[
\text{(HL)} \quad a(\xi) = a_m(\xi) + r(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0), \quad r(\xi) \in C^\infty(\mathbb{R}^n)
\]

\[|\partial^\alpha r(\xi)| \leq C|\xi|^{-m-1-|\alpha|} \quad \text{for all multi-indices } \alpha.\]

In view of the proof of Theorem 6, we see that Theorem 5 remains valid if we replace assumption (H) by (HL) and functions \( q(\xi) \) in the estimates by its (sufficiently large) high frequency part \( q_h(\xi) \). However we cannot control the low frequency part \( q_l(\xi) \), and so have only the time local estimates on the whole. We just put such a result ([21, Theorem 5.6]) below without its proof:

**Theorem 7.** Assume (HL). Suppose \( n \geq 1, m > 0, s > 1/2, \) and \( T > 0 \). Then we have

\[
\int_0^T \left\| \left\langle x \right\rangle^{-s} \left| D_x \right|^{(m-1)/2} e^{it a(D_x)} \right\|^2_{L^2(\mathbb{R}^n)} dt \leq C\|q\|^2_{L^2(\mathbb{R}^n)},
\]

where \( C > 0 \) is a constant depending on \( T > 0 \).

We remark that Theorem 6 is the time global version (that is, the estimate with \( T = \infty \)) of Theorem 7, and the extra assumption \( \nabla a(\xi) \neq 0 \) is needed for that. Since the assumption \( \nabla a(\xi) \neq 0 \) for large \( \xi \) is automatically satisfied by assumption (HL), Theorem 6 means that the condition \( \nabla a(\xi) \neq 0 \) for small \( \xi \) assures the time global estimate. In this sense, the low frequency part have a responsibility for the time global smoothing.
We now turn to deal with inhomogeneous equations, for which we also have similar smoothing estimates. Such estimates are necessary for nonlinear applications, and they can be obtained by further developments of the presented methods. Let us consider the solution
\[ u(t,x) = -i \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x) d\tau \]
to the equation
\[ \begin{cases} (i\partial_t + a(D_x)) u(t,x) = f(t,x) & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\ u(0,x) = 0 & \text{in } \mathbb{R}_x^n. \end{cases} \]

We will give model estimates for it below, where we write \( x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n \) and \( D_x = (D_1,D_2,\ldots,D_n) \). We also write \( x = x_1, D_x = D_1 \) in the case \( n = 1 \), and \( (x,y) = (x_1,x_2), (D_x,D_y) = (D_1,D_2) \) in the case \( n = 2 \).

**Proposition 2.** Suppose \( n = 1 \) and \( m > 0 \). Let \( a(\xi) \in C^\infty(\mathbb{R} \setminus 0) \) be a real-valued function which satisfies \( a(\lambda \xi) = \lambda^m a(\xi) \) for all \( \lambda > 0 \) and \( \xi \neq 0 \). Then we have
\[ \left\| a'(D_x) \int_0^t e^{i(t-\tau)a(D_x)} f(\tau,x) d\tau \right\|_{L^2(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} \| f(t,x) \|_{L^2(\mathbb{R}^n)} dx \]
for all \( x \in \mathbb{R} \). Suppose \( n = 2 \) and \( m > 0 \). Then we have
\[ \left\| D_x^{m-1} \int_0^t e^{i(t-\tau)|D_x|^{m-1} f(\tau,x,y) d\tau \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} \| f(t,x,y) \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^n)} dy \]
for all \( y \in \mathbb{R} \).

**Corollary 2.** Suppose \( n \geq 1, m > 0, \) and \( s > 1/2 \). Let \( a(\xi) \in C^\infty(\mathbb{R} \setminus 0) \) be a real-valued function which satisfies \( a(\lambda \xi) = \lambda^m a(\xi) \) for all \( \lambda > 0 \) and \( \xi \neq 0 \). Then we have
\[ \left\| (x_n)^{-s} a'(D_n) \int_0^t e^{i(t-\tau)a(D_n)} f(\tau,x) d\tau \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^n)} \leq C \left\| (x_n)^{-s} f(t,x) \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^n)}. \]
Suppose \( n \geq 2, m > 0, \) and \( s > 1/2 \). Then we have
\[ \left\| (x_1)^{-s} D_n^{m-1} \int_0^t e^{i(t-\tau)|D_n|^{m-1} f(\tau,x) d\tau \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^n)} \leq C \left\| (x_1)^{-s} f(t,x) \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^n)}. \]

Proposition 2 with the case \( n = 1 \) is a unification of the results by Kenig, Ponce and Vega who treated the cases \( a(\xi) = \xi^2 \) ([15, p.258]), \( a(\xi) = |\xi|^s \) ([17, p.160]), and
Corollary 2 is a straightforward result of Proposition 2 and Cauchy–Schwarz’s inequality. They act as model estimates for inhomogeneous equations just like Proposition 1 and Corollary 1 do for homogeneous ones. In [21], Corollary 1 is given straightforwardly from the translation invariance of Lebesgue measure, by using a newly introduced method (comparison principle).

Since we unfortunately do not know the comparison principle for inhomogeneous equations, we will give a direct proof to Proposition 2. Note that we have another expression of the solution to inhomogeneous equation

\[
\begin{cases}
(i \partial_t + a(D_x)) u(t, x) = f(t, x) \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\
u(0, x) = 0 \quad \text{in } \mathbb{R}_x^n,
\end{cases}
\]

using the weak limit \( R(\tau \pm i \varepsilon) \) of the resolvent \( R(\tau \pm i \varepsilon) \) as \( \varepsilon \searrow 0 \), where \( R(\lambda) = (a(D_x) - \lambda)^{-1} \):

\[
u(t, x) = -i \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) d\tau,
\]

\[
= \mathcal{F}_t^{-1} R(\tau - i \varepsilon) \mathcal{F}_t f^+ + \mathcal{F}_t^{-1} R(\tau + i \varepsilon) \mathcal{F}_t f^-
\]

(see Sugimoto [25] and Chihara [9]). Here \( \mathcal{F}_t \) denotes the Fourier Transformation in \( t \) and \( \mathcal{F}_t^{-1} \) its inverse, and \( f^\pm(t, x) = f(t, x) Y(\pm t) \) is the characteristic function \( Y(t) \) of the set \( \{ t \in \mathbb{R} : t > 0 \} \).

**Proof of Estimate (16).** Let us use a variant of the argument of Chihara [9, Section 4]. We set \( R(\lambda) = (a(D_x) - \lambda)^{-1} \) and show the estimate

\[
| a'(D_x) R(s \pm i \varepsilon) g(x) | \leq C \int_\mathbb{R} |g(x)| dx,
\]

where \( C > 0 \) is a constant independent of \( s \in \mathbb{R}, x \in \mathbb{R} \) and \( g \in L^1(\mathbb{R}) \). Then, on account of the expression (18), Plancherel’s theorem, and Minkowski’s inequality, we have the desired result. For this purpose, we consider the kernel

\[
k(s, x) = \mathcal{F}_t^{-1} \left[ a'(\xi) (a(\xi) - (s \pm i \varepsilon))^{-1} \right](x)
\]

and show its uniform boundedness. By the scaling argument, everything is reduced to show the estimates

\[
\sup_{x \in \mathbb{R}} |k(\pm 1, x)| \leq C \quad \text{and} \quad \sup_{x \in \mathbb{R}} |k(0, x)| \leq C.
\]

By using an appropriate partition of unity \( \hat{\phi}_1(\xi) + \hat{\phi}_2(\xi) + \hat{\phi}_3(\xi) = 1 \), we split \( k(\pm 1, x) \) into the corresponding three parts \( k = k_1 + k_2 + k_3 \), where \( \hat{\phi}_1 \) has its support near the origin, \( \hat{\phi}_2 \) near the point \( \xi = \pm 1 \), and \( \hat{\phi}_3 \) away from these points. The estimate for \( k_1 \) is trivial. The other estimates are reduced to the boundedness of

\[
k_0^\pm(x) = \mathcal{F}_t^{-1} \left[ (\xi \pm i \varepsilon) - (s \pm i \varepsilon) \right](x) = \mp i \sqrt{2\pi} Y(\pm x).
\]
In fact,
\[ k_2(\pm 1, x) = \mathcal{F}^{-1} \left[ (\xi - (\alpha \pm i0))^{-1} \hat{\phi}(\xi) \right] (x) = (e^{\alpha a} k_0^x) \ast \psi(x) \]
where \( \alpha \in \mathbb{R} \) is a point which solves \( a(\alpha) = \pm 1 \), and
\[ \hat{\phi}(\xi) = a'(\xi) \frac{\xi - \alpha}{a(\xi) - (\pm 1)} \hat{\phi}_2(\xi) \in C_0^\infty(\mathbb{R}). \]
Furthermore, if we notice
\[ \frac{a'(\xi)}{a(\xi) - s} = m \left( \frac{s}{(a(\xi) - s)\xi} + \frac{1}{\xi} \right), \]
we have
\[ \frac{1}{m} k_3(\pm 1, x) = \pm \mathcal{F}^{-1} \left[ \frac{\hat{\phi}_3(\xi)}{(a(\xi) \mp 1)\xi} \right] (x) + k_0^y (x) - k_0^y \ast (\phi_1(x) + \phi_2(x)). \]
It is easy to deduce the estimates for \( k_2 \) and \( k_3 \). It is also easy to verify
\[ \frac{a'(\xi)}{a(\xi) \pm i0} = \frac{m}{\xi \pm i0} + c\delta \]
with a constant \( c \) and Dirac’s delta function \( \delta \), and have the estimate for \( k(0, x) \).

**Proof of Estimate (17).** We set \( R(\lambda) = (|D_x|^{m-1} D_y - \lambda)^{-1} \) and show the estimate
\[ \left\| |D_x|^{m-1} R(s \pm i0)g(x, y) \right\|_{L^2(\mathbb{R}_x)} \leq C \int \|g(x, y)\|_{L^2(\mathbb{R}_y)} dy, \]
where \( C > 0 \) is a constant independent of \( s \in \mathbb{R} \), \( y \in \mathbb{R} \) and \( g \in L^1(\mathbb{R}^2) \). Then, by the expression (18), Plancherel’s theorem, and Minkowski’s inequality again, we have the desired result.

First we note, we may assume \( \hat{g}(\xi, \eta) = 0 \) for \( \xi < 0 \). Then we have
\[ |D_x|^{m-1} R(s \pm i0)g(x, y) \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{i\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} |\xi|^{m-1} \left( |\xi|^{m-1} \eta - (s \pm i0) \right) \hat{g}(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{i\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} |\xi|^{m-1} \left( |\xi|^{m-1} \eta - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{i\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( |\xi|^{m-1} \eta - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{ia\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( a - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{ia\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( a - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{ia\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( a - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{ia\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( a - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{ia\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( a - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{ia\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( a - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{ia\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( a - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{ia\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( a - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} e^{ia\xi x} \int_{-\infty}^{\infty} e^{i\xi^2 y} \left( a - (s \pm i0) \right) \hat{g}_y(\xi, \eta) d\xi d\eta \]
The following is a counterpart of Theorem 5 which treated homogeneous equations. We have

$$\mathcal{F} \left[ |D_x|^{m-1} R(s \pm i0) g(x, y) \right](b) = \int_{-\infty}^{\infty} e^{-isa} k_\pm^\tau (-a) b^{m-1} \tilde{g}_y(b, ab^{m-1}) \, da$$

for $b \geq 0$, and it vanishes for $b < 0$. Here $g_s(x, \cdot) = g(x, \cdot + y)$, and $\tilde{g}_y$ denotes its partial Fourier transform with respect to the first variable. We have also used here the change of variables $a = \xi^{m-1} \eta$, $b = \xi$ and Parseval’s formula. Note that $\partial_s(a, b)/\partial\eta = b^{m-1}$ and $k_\pm^\tau$ is a bounded function defined by (19). Then we have the estimate

$$|\mathcal{F} \left[ |D_x|^{m-1} R(s \pm i0) g(x, y) \right](b)| \leq \sqrt{2\pi} \int_{-\infty}^{\infty} |b^{m-1} \tilde{g}_y(b, ab^{m-1})| \, da$$

and, by Plancherel’s theorem and Minkowski’s inequality, we have

$$\left\| |D_x|^{m-1} R(s \pm i0) g(x, y) \right\|_{L^2(\mathbb{R}_y)} \leq \sqrt{2\pi} \int_{-\infty}^{\infty} \|g_s(x, a)\|_{L^2(\mathbb{R}_x)} \, da$$

which is the desired estimate. 

5. Smoothing estimates for dispersive inhomogeneous equations

Let us consider the inhomogeneous equation

$$\begin{cases}
(i\partial_t + a(D_x)) u(t, x) = f(t, x) & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\
u(0, x) = 0 & \text{in } \mathbb{R}^n_x,
\end{cases}$$

where we always assume that function $a(\xi)$ is real-valued. Let the principal part $a_m(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$, be a positively homogeneous function of order $m$. Recall the dispersive conditions we used in Section 3:

\[(\mathbf{H}) \quad a(\xi) = a_m(\xi), \quad \forall a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0),\]

\[(\mathbf{L}) \quad a(\xi) \in C^\infty(\mathbb{R}^n), \quad \forall a(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n), \quad \forall a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0),
|\partial^\alpha(a(\xi) - a_m(\xi))| \leq C_\alpha |\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \geq 1.
\]

The following is a counterpart of Theorem 5 which treated homogeneous equations:
THEOREM 8. Assume (H). Suppose $m > 0$ and $s > 1/2$. Then we have

\[(20) \quad \left\| \langle x \rangle^{-s} \left| D_x \right|^{m-1} \int_0^t e^{i(t-\tau)\alpha(D_x)} f(\tau,x) \, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} \leq C \| \langle x \rangle^s f(t,x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} \]

in the case $n \geq 2$, and

\[(21) \quad \left\| \langle x \rangle^{-s} \left| D_x \right|^{(m-1)/2} \int_0^t e^{i(t-\tau)\alpha(D_x)} f(\tau,x) \, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \| \langle x \rangle^s f(t,x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \]

in the case $n = 1$.

Chihara [9] proved Theorem 8 with $m > 1$ under the assumption (H). As was pointed out in [9, p.1958], we cannot replace $\left| D_x \right|$ by $\left| D_x \right|^{m-1}$ in estimate (21) for the case $n = 1$, but there is another explanation for this obstacle. If we decompose $f(t,x) = \chi_x(D_x) f(t,x) + \chi_{\pm}(D_x) f(t,x)$, where $\chi_{\pm}(\xi)$ is a characteristic function of the set $\{ \xi \in \mathbb{R} : \pm \xi \geq 0 \}$, then we easily obtain

\[\left\| \langle x \rangle^{-s} \left| D_x \right|^{(m-1)/2} \int_0^t e^{i(t-\tau)\alpha(D_x)} f(\tau,x) \, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \left( \left\| \langle x \rangle^s |D_x|^{-1} f_+(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} + \left\| \langle x \rangle^s |D_x|^{-1} f_-(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \right)\]

from Theorem 8. But we cannot justify the estimate

\[\left\| \langle x \rangle^s |D_x|^{-1} f_{\pm}(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \left\| \langle x \rangle^s |D_x|^{-1} f(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)}\]

for $s > 1/2$ by Lemma 1 because it requires $s < n/2$ and it is impossible for $n = 1$.

As a counterpart of Theorem 6, we have

THEOREM 9. Assume (L). Suppose $n \geq 1$, $m > 0$, and $s > 1/2$. Then we have

\[(22) \quad \left\| \langle x \rangle^{-s} \left| D_x \right|^{m-1} \int_0^t e^{i(t-\tau)\alpha(D_x)} f(\tau,x) \, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} \leq C \| \langle x \rangle^s f(t,x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)}.\]

The following result is a straightforward consequence of Theorem 9 and the $L^2$–boundedness of $\left| D_x \right|^{-1/2} \left| D_x \right|^{-(m-1)/2}$ with $(1/2 <) s < n/2$ and $m \geq 1$ (which is assured by Lemma 1):

COROLLARY 3. Assume (L). Suppose $n \geq 2$, $m \geq 1$, and $s > 1/2$. Then we have

\[\left\| \langle x \rangle^{-s} \left| D_x \right|^{m-1} \int_0^t e^{i(t-\tau)\alpha(D_x)} f(\tau,x) \, d\tau \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} \leq C \| \langle x \rangle^s f(t,x) \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)}.\]
We remark that the same argument of canonical transformations as used for homogeneous equations in Section 3 works for inhomogeneous ones, as well. That is, the proofs of Theorems 8 and 9 are carried out by reducing them to model estimates in Corollary 2. We omit the details because the argument is essentially the same, but we just remark that we use Theorem 2 instead of Theorem 1.

The following is a counterpart of Theorem 7:

**THEOREM 10.** Assume (HL). Suppose \( n \geq 1, m > 0, s > 1/2, \) and \( T > 0. \) Then we have

\[
\int_0^T \left\| \frac{1}{\langle x \rangle^{-s}} (D_x)^{m-1} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) \, d\tau \right\|_{L^2(\mathbb{R}^n_x)}^2 \, dt \\
\leq C \int_0^T \| \langle x \rangle^s f(t, x) \|^2_{L^2(\mathbb{R}^n_x)} \, dt,
\]

where \( C > 0 \) is a constant depending on \( T > 0. \)

**Proof.** By multiplying \( \chi(D_x) \) and \((1 - \chi)(D_x)\) to \( f(t, x) \), we decompose it into the sum of low frequency part and high frequency part, where \( \chi(\xi) \) is an appropriate cut-off function. As in the proof of Theorem 6, the estimate for the high frequency part can be reduced to Corollary 2 by using Theorem 2 instead of Theorem 1, together with the boundedness result Theorem 3. Here we note that, for \( t \in [0, T] \),

\[
\int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) \, d\tau = \int_0^t e^{i(t-\tau)a(D_x)} \chi_{[0, T]}(\tau) f(\tau, x) \, d\tau,
\]

where \( \chi_{[0, T]} \) denotes the characteristic function of the interval \([0, T]\). The estimate for the low frequency part is trivial. In fact, if \( \text{supp} \mathcal{F}_x f(t, \xi) \subset [\xi; |\xi| \leq R] \), we have

\[
\int_0^T \left\| \frac{1}{\langle x \rangle^{-s}} (D_x)^{m-1} \int_0^t e^{i(t-\tau)a(D_x)} f(\tau, x) \, d\tau \right\|_{L^2(\mathbb{R}^n_x)}^2 \, dt \\
\leq \int_0^T \left( \int_0^T \left\| (D_x)^{m-1} e^{i(t-\tau)a(D_x)} f(\tau, x) \right\|_{L^2(\mathbb{R}^n_x)}^2 \, d\tau \right)^2 \, dt \\
\leq C T^2 \langle R \rangle^{2(m-1)} \int_0^T \| \langle x \rangle^s f(t, x) \|^2_{L^2(\mathbb{R}^n_x)} \, dt,
\]

by Plancherel’s theorem. \( \square \)

If we combine Theorem 8 with Theorem 5, we have a result for the equation

\[
\begin{cases}
(i \partial_t + a(D_x)) u(t, x) = f(t, x) & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\
u(0, x) = q(x) & \text{in } \mathbb{R}^n_x.
\end{cases}
\] (23)
**Corollary 4.** Assume (H). Suppose \( m > 0 \) and \( s > 1/2 \). Then the solution \( u \) to equation (23) satisfies
\[
\left\| \langle x \rangle^{-1} |D_x|^{-(m-1)/2} D_x u(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \left( \| \varphi \|_{L^2(\mathbb{R}_x)} + \left\| \langle x \rangle^{1/2} D_x u(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \right)
\]
in the case \( n = 1 \), and
\[
\left\| \langle x \rangle^{-1} |D_x|^{(m-1)/2} u(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \left( \| \varphi \|_{L^2(\mathbb{R}_x)} + \left\| \langle x \rangle^{1/2} |D_x|^{-(m-1)/2} f(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \right)
\]
in the case \( n \geq 2 \).

If we combine Theorem 9 with Theorem 6, we have the following:

**Corollary 5.** Assume (HL). Suppose \( n \geq 1, m > 0, \) and \( s > 1/2 \). Then the solution \( u \) to equation (23) satisfies
\[
\int_0^T \left\| \langle x \rangle^{-1} |D_x|^{(m-1)/2} u(t,x) \right\|^2_{L^2(\mathbb{R}_x^2)} dt \leq C \left( \| \varphi \|^2_{L^2(\mathbb{R}_x)} + \int_0^T \left\| \langle x \rangle^{1/2} |D_x|^{-(m-1)/2} f(t,x) \right\|^2_{L^2(\mathbb{R}_x^2)} dt \right),
\]
where \( C > 0 \) is a constant depending on \( T > 0 \).

Corollary 6 is an extension of the result by Hoshiro [12], which treated the case that \( a(\xi) \) is a polynomial. The proof relied on Mourre’s method, which is known in spectral and scattering theories. Here we use the argument of canonical transformations, extending the result and simplifying the proof.
Smoothing properties of inhomogeneous equations

References


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Michael RUZHANSKY
Department of Mathematics Imperial College London
180 Queen’s Gate, London SW7 2AZ, UK
e-mail: m.ruzhansky@imperial.ac.uk

Mitsuru SUGIMOTO
Graduate School of Mathematics, Nagoya University
Furocho, Chikusa-ku, Nagoya 464-8602, JAPAN
e-mail: sugimoto@math.nagoya-u.ac.jp

J. Schmeelk

AN IMPULSIVE DIFFERENTIAL EQUATION
IN AN INFINITE DIMENSIONAL FOCK SPACE

Abstract. A scale of infinite dimensional Fock spaces, $\Gamma_p^{sB} = \bigcup_{s \geq 1} \Gamma_p^{sB}$, is introduced together with some of their fundamental properties. Each space $\Gamma_p^{sB}$ is a Frechet space described in [18, 23, 19]. An impulsive differential equation involving a generalized Laplacian [20] is then introduced where at the points, $t_i, 1 \leq i < \infty$ belonging to $R$ receives an impulse. Due to the generalization of this paper, the impulses take their values in the infinite dimensional Fock space, $\Gamma_p^{sB} = \bigcup_{s \geq 1} \Gamma_p^{sB}$. The components of the vector in the space, $\Gamma_p^{sB}$, are tempered distributions thus generalizing the classical Fock space having components in $L^p(R^n)$. An explicit algorithm computing the solution to the problem together with a uniqueness technique. The existence technique is in the spirit of an operational calculus.

1. Introduction

A system of impulsive differential equations [1, 2, 13] within the framework of a Euclidean space, $R^n$, can be described by the following system,

\begin{equation}
\dot{z}(t) = A(t)z(t) + f(t), t \neq t_i,
\end{equation}

where $t \in R, z(t) : R \rightarrow R^n, A(t) : R \rightarrow R^n \times R^n, f : R \rightarrow R^n$ together with the impulsive conditions,

\begin{equation}
\Delta z(t_i) = z(t_i^+) - z(t_i^-) = b_i, i = \pm 1, \pm 2, ...
\end{equation}

Moreover, $\{b_i\}_{i=\pm 1}^\infty$ is a sequence of n-dimensional constants. To illustrate these notions we included the following fundamental example [2]:

\begin{equation}
\dot{x}(t) = 1 + |x(t)|^2, t \neq \frac{k\pi}{4}, k = 1, 2, ...
\end{equation}

and

\begin{equation}
\Delta x(t_k) = x(t_k^+) - x(t_k^-) + 1 \text{ whenever } t_k = \frac{k\pi}{4}
\end{equation}

together with $x(0) = 0$. Equations (3) and (4) are fundamental examples corresponding to conditions (1) and (2). The solution to (3) and (4) is easily seen to be

\begin{equation}
x(t) = \begin{cases}
tan(t), & 0 < t < \frac{1}{4}
\tan(t - \frac{k\pi}{4}), & t \in (\frac{k\pi}{4}, \frac{(k+1)\pi}{4})
\end{cases}
\end{equation}
where \( x(t) \) is periodic with period \( \frac{\pi}{4} \). The graph of the solution is illustrated in Figure 1.

We note the classical corresponding differential equation (3) without the impulse condition (4) has the solution, \( x(t) = \tan(t) \) with an interval of existence, \((0, \frac{\pi}{2})\) since \( \lim_{t \to \frac{\pi}{2}} x(t) \to \infty \).

This paper generalizes conditions (1) and (2) whereby condition (1) contains a generalized Laplacian [20] and condition (2) becomes

\[
\Delta u(t_i) = I_i(u(t_i))
\]

where \( u(t_i) \in \Gamma^p_{\mathcal{B}} \) for some \( s \geq 1 \) and \( I_i : (\dot{u} \to \Gamma^p_{\mathcal{B}}) \to (\dot{u} \to \Gamma^{p',\mathcal{B}}) \), where \( s > s' > 1 \).

2. The scale of infinite dimensional fock spaces \( \Gamma^{pB} = \bigcup_{s \geq 1} \Gamma^{p,sB} \)

For each \( s \geq 1 \) the space, \( \Gamma^{p,sB} \{ (p > 1, B = \{ B_i \}_{i=0}^{\infty}, B_i > B_j, j > i) \} \), is called an infinite dimensional Fock space. The \( p \) and \( B_i, i \geq 0 \) are all real numbers. These spaces are topological spaces of real-valued functionals on \( S'(R^{3n}; R) \), the space of real-valued tempered distributions. The set of functionals belonging to the space \( \Gamma^{p,sB} \) are all \( C^\infty (S'(R^{3n}; R)) \). We also require if \( \Phi \in \Gamma^{p,sB} \), then

\[
(6) \quad \Phi(x) = \sum_{q=0}^{\infty} a_q[x,\ldots,x] = \sum_{q=0}^{\infty} a_q x^q
\]

where \( a_0 \in R \) and \( a_q, q \geq 1 \) are \( q \)-multilinear symmetric continuous functionals on \( S'(R^{3n}) \times \ldots \times S'(R^{3n}) \to R \). We identify for each \( \Phi \in \Gamma^{p,sB} \) the associated state vector,
Each multilinear functional, \( a_q, q \geq 1 \), has an infinite dimensional domain space. The functionals having representation given in expression (7) are members of our infinite dimensional Fock space. We equip our infinite dimensional Fock space with the following sequence of norms:

\[
\| \Phi \|_{sB^m_\infty} = \sup_q \frac{\|a_q\|_m q!^{\frac{1}{p}}}{(sB_m)^q} < \infty \quad m = 0, 1, \ldots
\]

where

\[
\|a_q\|_m = \sup_{\|x\|=-m} |a_q x^q| \quad m = 0, 1, \ldots, \quad x \in S'(\mathbb{R}^n)
\]

and

\[
\|x\|_{-m} = \sup_{\|\phi\|_m} |<x, \phi>| \quad m = 0, 1, \ldots, \quad \phi \in S(\mathbb{R}^n)
\]

and

\[
\|\phi\|_m = \sup_{\alpha_1 + \ldots + \alpha_n \leq m} \left| (1 + |\tau_1|^2) \ldots (1 + |\tau_n|^2)^m |\phi|^{\alpha_1 \ldots \alpha_n} (\tau_1, \ldots, \tau_n) \right|
\]

where

\[
\phi^{(0, \ldots, 0, \ldots, 0)}(\tau_1, \ldots, \tau_n) = \frac{\partial^{\alpha_i}}{\partial \tau_{i}^{\alpha_i}} \phi \quad 1 \leq i \leq n.
\]

The functions \( \phi \), are rapid descent test functions and the functionals, \( x \), are tempered distributions described in Constantinescu [6] and Zemanian [27]. The set of entire functionals belonging to the space, \( \Gamma^{p,s,R} \), equipped with the natural topology
induced by the sequence of norms, (8) is a Frechet space. We then consider \( 1 \leq s \leq s' \) where clearly \( \Gamma^{p,sB} \subset \Gamma^{p,s'B} \). Also the canonical injection, \( J_{s,s'} : \Gamma^{p,sB} \to \Gamma^{p,s'B} \), is continuous.

The multilinear symmetric functional \( a_q, q \geq 1 \), will have a square summing property analogous to classical Fock space sum ability in the following sense.

**Proposition 1.** The sequence of multilinear symmetric functionals \( \{a_q\}^\infty_{q=1} \), \( a_0 \in R \), described in expression (7) is square summing in each norm, \( \sum_{q=0}^\infty ||a_q||^2_{m} < \infty \), \( m = 0,1,2,\ldots \).

**Proof.** We select \( \Phi \in \Gamma^{p,sB} \), where \( \Phi(x)= \sum_{q=0}^\infty a_q x^q \). We consider a norm, \( ||\Phi||_{sB} = \sup_q \frac{||a||_{1/2}^{1/2} ||a||^{1/2}}{q^{1/2}} < C_m < \infty \) implying \( ||a||_{m} < \frac{C_m(sB_m)^q}{q^{1/2}} \) for every \( q \). From these statements and returning to square summing notion, we obtain

\[
\sum_{q=0}^\infty ||a||_{m}^2 \leq ||\Phi||_{sB} \sum_{q=0}^\infty \frac{||a||_{m}^{1/2} (sB_m)^q}{q^{1/2}} \leq ||\Phi||_{sB} C_m \sum_{q=0}^\infty \frac{1}{q^{1/3}} < \infty.
\]

We remark the kernel representation for each of the multilinear symmetric functional, \( a_q, q \geq 1 \), has the form of a rapid descent test function, \( \phi_q, q \geq 1 \). This representation then gives the association,

\[
\Phi \leftrightarrow \phi_0 \quad \phi_1 \quad \ldots \quad \phi_q \quad \ldots
\]

where \( \phi_0 = a_0 \) and \( \phi_q(x_1,\ldots,x_q) = a_q(\delta(t_1 - x_1),\ldots,\delta(t_q - x_q)), q \geq 1 \). The \( \delta(t_i - x_i), 1 \leq i \leq q \) are translates of the Dirac delta functional. Details regarding the topological properties of this association can be found in reference [24]. We conclude this section by indicating the vector given in expression (13) also enjoys the so-called square summing property.

### 3. Infinite dimensional laplacian operator

We briefly review cylinder functionals developed by K.O. Friedrichs and H.N. Shapiro [10]. Cylinder functionals have \( p \)-variables where each variable takes its value from a
set of functions containing the classical piecewise constant functions. The $p$ variable functions can be written as

(14) \[ \phi(t) \leftrightarrow \{\phi_1, \ldots, \phi_p\}, \]

where each $t \in \mathbb{R}^3$. A cylinder functional can be written as

(15) \[ f_p(\phi(t)) \leftrightarrow f_p(\phi_1(t), \ldots, \phi_p(t)), \]

where the subscript $p$ denotes the number of components within its representation.

More specifically, when the quantum theory of fields is introduced in Chapter II [10] each $t$ varies in a “cell” contained within $\mathbb{R}^3$. We consider $\phi_\gamma(t) = 0$, $1 \leq \gamma \leq p$, if $t$ does not belong to any $n$-cell. We then define as in Friedrich and Shapiro [10]

\[ \delta f_p(\phi(t)) = 0. \]

If $t \in \gamma^{th}$ cell as in Friedrich and Shapiro [10], we set

\[ \frac{\delta}{\delta \phi(t)dt} f_p(\phi(t)) = \frac{1}{\Delta t} \frac{\partial}{\partial \phi(t)} f_p(\phi_1(t), \ldots, \phi_p(t)). \]

The $\Delta_n$ is the “volume” of the $\gamma^{th}$ cell. Schiff [17] requires the “volume of the cell” to tend to zero. Then Schiff enjoys the presence of the Dirac Delta functional. Friedrich and Shapiro [10] select the following quadratic functional,

\[ f_2[\phi] = \int \int b(x', x'') \phi(x') \phi(x'') dx'dx'', \]

and its generalized Laplacian becomes

\[ L f_2[t] = 2 \int b(x, x) dx. \]

To avoid computational difficulties we select a $\phi(t, t') \in S(\mathbb{R}^2)$, the space of $\mathbb{R}^2$ rapid descent test functions. We then define $h$ as

(16) \[ <h, \phi> \equiv 2 \int \phi(x, x) dx, \]

proving $h$ to be a tempered distribution. In applications the independent variable, $x$, are from $\mathbb{R}^3$.

**Proposition 2.** The functional, $h$, defined in expression (16) is a tempered distribution.

**Proof.** The linearity of $h$ is obvious and we prove the continuity by proving boundedness of $h$. We select any rapid descent test function, $\phi(t, t') \phi$ and compute the following
\[ | < h, \phi(t,t') > | = 2 \int |\phi(t,t) dt| = 2 \int \frac{|\phi(t,t)(1+t^2)^4}{(1+t^2)^4} dt \]

\[ \leq 2 \sup_{|\alpha|<2, t \in \mathbb{R}} |(1+t^2)^4 \phi^{(\alpha)}(t,t)| \pi < 2\pi |\phi(t_1,t_2)|_2, \]

(17)

If we select a special \( \Phi \in \Gamma^{p, \mathbb{A}} \) where its Fock representation (7) is given as

\[
\Phi \leftrightarrow \left| \begin{array}{c}
a_0 \\
a_1 \\
. \\
a_q \\
. \\
. \\
0 \\
\end{array} \right|
\]

(18)

and apply two generalized differentiations, then

\[
D^2_\mathbb{A} \Phi \leftrightarrow \left| \begin{array}{c}
2a_2[h,h] \\
0 \\
. \\
. \\
0 \\
\end{array} \right|
\]

(19)

Selecting, \( h = \delta(t-t') \) and implementing an integral operator on (19) results in

\[
\Delta \Phi \leftrightarrow \left| \begin{array}{c}
2 \int a_2[\delta(t-t'), \delta(t-t')] dt \\
0 \\
. \\
. \\
\end{array} \right|
\]

(20)

Noting that \( a_2[\delta(t-t'), \delta(t-t')] = \phi(t,t) \) is a member of \( S(\mathbb{R}^2) \) as shown in Schmeelk [2], we have an analogy to expression (3.3). Our Laplacian is mathematically developed in Schmeelk [20] but here we have its Fock representation. If \( \Phi \in \Gamma^{p, \mathbb{A}} \) where \( \Phi \) has its representation given in expression (18), then its generalized Laplacian becomes

\[
\left| \begin{array}{c}
2 \int a_2[\delta(t-t'), \delta(t-t')] dt \\
2 \times 3 \int a_3[\delta(t-t'), \delta(t-t')] dt \\
. \\
. \\
(q+1)(q+2) \int a_{q+2}[\delta(t-t'), \delta(t-t'), \ldots] dt \\
\end{array} \right|
\]

(21)
We observe each $a_q$ is a multilinear symmetric functional so for convenience we insert the pair of translated Dirac delta functionals in the first two arguments. We can now develop an infinite dimensional impulsive differential equation within this setting.

4. Impulsive differential equation solutions in $\Gamma^{pB} = \bigcup_{\lambda \geq 1} \Gamma^{p,\lambda B}$

Let $u(t) : (\alpha, \beta) \rightarrow \Gamma^{pB}, t \neq t_i$ where $(\alpha, \beta)$ is an interval in $R$. Moreover for $t \neq t_i$ the vector, $u(t) \in \Gamma^{p,\lambda B}$, and thus must satisfy

$$\sup_{q \in (\alpha, \beta)} \frac{||a_q(t)||m_q^{1/p}}{(sB_m)^q} < M_m < \infty, m = 1, 2, ...$$

(22)

For $t = t_i$ we have the impulse component,

$$\Delta u(t_i) = u(t_i^+) - u(t_i^-) = \tilde{u}(t_i), i = 1, 2, ...$$

The vectors, $u(t_i^+)$ and $u(t_i^-)$ together with the vector, $\tilde{u}(t_i)$, all must satisfy condition (22).

We now introduce the transformation,

$$I_i : ((\alpha, \beta) \rightarrow \Gamma^{p,\lambda B}) \rightarrow ((\alpha, \beta) \rightarrow \Gamma^{p,\lambda B})$$

where $I_i(u(t_i)) \equiv \Delta(u(t_i)), i = 1, 2, ...$. The functional, $u(t) \in \Gamma^{p,\lambda B}, t \neq t_i$ is said to be strongly continuous if and only if whenever, $\varepsilon > 0 \exists \varepsilon \in N_+$ (positive reals) then there $\exists\delta_{\varepsilon, \lambda} > 0$ such that whenever

$$|t - t'| < \delta_{\varepsilon, \lambda} \Rightarrow ||u(t) - u(t'||sB_m < \varepsilon.$$

Strongly differentiable has a somewhat similar requirement whereby

$$||\frac{t - u(t') - u(t)}{t' - t} - v(t')||sB_m < \varepsilon$$

whenever $0 < |t - t'|, \delta_{\varepsilon, \lambda}$ and $v(t') \in \Gamma^{p,\lambda B}$. The impulsive differential equation in $\Gamma^{p,\lambda B}$ is now defined as

$$\frac{\partial}{\partial t} u(t)(x) = -P(\Delta^2)u(t, x), t \neq t_i$$

(27)

where $u(t) \in ((\alpha, \beta) \rightarrow \Gamma^{p,\lambda B}, x \in S'(\mathbb{R}^n))$ and $P(\Delta^2)$ is an infinite dimensional Laplacian polynomial operator of order $n$ where $P(\Delta^2) = d_n(\Delta^2) + d_{n-1}(\Delta^2) + ..., d_0$, where $d_i \in (R), (0 \leq i \leq n)$. The impulsive conditions are given by

$$u(t_i^+) = u(t_i) + \Delta(u(t_i)) = u(t_i) + I_i(u(t_i)),$$

$$\frac{\partial}{\partial t} u(t_i^+) = \frac{\partial}{\partial t} u(t_i) + \Delta \frac{\partial}{\partial t} u(t_i) + I_i(\frac{\partial}{\partial t} u(t_i)),$$

$$\frac{\partial^i}{\partial t^i} u(t_i^-) = \frac{\partial^i}{\partial t^i} u(t_i) + \Delta(\frac{\partial^i}{\partial t^i} u(t_i)) = \frac{\partial^i}{\partial t^i} u(t_i) + I_i(\frac{\partial^i}{\partial t^i} u(t_i))$$

(28)
We also require for \( t = t_0^+ \in R \) the conditions,

\[
\frac{\partial}{\partial t} u(t)(\bullet)|_{t=t_0^+} \in \Gamma_{p,sB}
\]

are also satisfied. We now define the following sets in \( R \).

\[
\{ G_k = t \in R : t_{k-1} < t < t_k, k \in N \}
\]

\[
\{ D_k = t \in R : t_{k-1} < t \leq t_k, k \in N \}
\]

\[
\{ F_k = t \in R : t_{k-1} \leq t < t_k, k \in N \}
\]

We then view equation (27) in operational form,

\[
\left[ \frac{\partial}{\partial t} + P(\Delta^2) \right] u(t)(x) = 0
\]

and introduce the standard integral operator,

\[
(\mathcal{Q}_{t'}^t) (u(\tau)(x)) = \int_{t'}^t u(\tau)(x) d\tau
\]

for \( t, t' \in G_k \). Applying the operator (34) to equation (33) will formally give us

\[
[I - (\mathcal{Q}_{t'}^t) P_1(\Delta^2) + (\mathcal{Q}_{t'}^t)^2 P_2(\Delta^2) + ... + (\mathcal{Q}_{t'}^t)^i P_i(\Delta^2)]
\]

We now develop the mathematical formulation for the inverse operator given in (34) and its generalization given in (35).

5. Inverse operators

We introduce the usual inverse to \( (\frac{\partial}{\partial t})^\lambda \), \( \lambda \) a positive integer, on the space, \( C^0((\alpha, \beta) ; \Gamma_{p,sB}) \).

The operator, \( (\mathcal{Q}_{t_0}^t)^{-\lambda} \), is defined as \( (\mathcal{Q}_{t_0}^t)^{-\lambda} u(\tau)(x) = \int_{t_0}^t \frac{(t-\tau)^{\lambda-1}}{(\lambda-1)!} u(\tau)(x) d\tau \) where \( |t_0, t| \subset (\alpha, \beta) \subset R \).

This section requires the use of multinomial coefficients, specialized Euclidean n-space points, specialized factorials and Euclidean n space summands. To enhance the readability of this section and in particular theorem 1, we introduce the following compact notation:
Proof. (i) a real valued scalar function, \( B(\xi) = \sum_{|\lambda| = 0}^{\infty} b_{|\lambda|}(\xi)^{\lambda} \), with real coefficients, \( b_{\lambda} \), and nonzero radius of convergence of \( p \); (ii) a polynomial transformation of the form, \( P(Q_0^t, \Delta^2) = \sum_{|\lambda| = 1}^{\infty} (Q_0^t)^{\lambda} P_\lambda(\Delta^2) \), where \( P_\lambda, 1 \leq \lambda \leq l \) are real valued polynomials of degree, \( n_\lambda, 1 \leq \lambda \leq l \) and where we formally replace the independent real variable with the infinite dimensional Laplacian operator of section 3, i.e. \( P_\lambda(\Delta^2) = \sum_{i=0}^{n_\lambda} d_i^\lambda(\Delta^2)^i \), \( d_i^\lambda \in R, 0 \leq i \leq n_\lambda, 1 \leq \lambda \leq l \). (iv) that \( p \) satisfies 1 < \( p < \frac{1}{1-\mu} \) if \( \mu_\lambda \neq 1 \) and \( p < \infty \) if \( \mu_\lambda = 1 \) where we define the order of \( P(Q_0^t, \Delta^2) \) to be \( \mu_\lambda = \min_{1 \leq \lambda \leq l} \left( \frac{1}{n_\lambda} \right) \).

Then the series \( B(Q_0^t, \Delta^2) = \sum_{i=0}^{\infty} b_i (P_\lambda(Q_0^t, \Delta^2))^i \), obtained by substituting \( P(Q_0^t, \Delta^2) \) for \( \xi \) into \( B(\xi) \) and formally expanding into a series of \( (Q_0^t, \Delta^2) \) monomials is applied to \( u(t) \). This process gives another \( v(t) \in \mathcal{C}^{sl}B \) continuous for \( t \in [0,T] \subset (\alpha,\beta) \) where \( s' \geq (l+1)s \).

\[ \sum_{|\lambda| = 0}^{\infty} b_{|\lambda|}(\xi)^{\lambda} \]
This then gives us

\[ \||B_n(P(Q_0^T\Delta^2))u(t)||_{\mathcal{F}B} \leq \sup_q \sum_{i=0}^n b_{2i}(\frac{|i|}{t}) (Q_0^T\Delta^2)^{2i} \]

\[ |\langle d_{n+1}\rangle (q + 2n \bullet i), t_q^{-1} \rangle| \leq \sum_{i=0}^n b_{2i}(\frac{|i|}{t}) (Q_0^T\Delta^2)^{2i} \]

Introducing \((1 + \tau_{\alpha})^2 \cdots (1 + \tau_{\alpha+2})^2\) into the numerator and denominator of the integrand in series (38) majorizes (38) with

\[ M_n \sup_q \sum_{i=0}^n b_{2i}(\frac{|i|}{t}) \frac{|t - t_0|^{2i}}{(\bullet i)!} \cdot \left[ K^{\frac{i}{2}} (q + 2n \bullet i)^{-1} \frac{(sB_m)^{(q + 2n)} \psi - 1}{(q + 2n \bullet i)!} \right] \]

Note the following inequalities:

\[ 2n \bullet i < 2 \bullet 2^{2i} \cdot \left( \frac{|i|}{t} \right) < i^{2i}, |b_{2i}| < K^{\frac{i}{2}}, (q + 2n \bullet i)! \leq q! (2n \bullet i)^{(q + 2n)} \psi, \]

and

\[ ((\bullet i)!)^{-1} \leq ((\bullet i)!)^{-1}. \]

Using inequalities (39) and (40) in the majorized form of expression (38), we obtain

\[ 2K^2 M_n \sup_q \sum_{i=0}^n \frac{|t - t_0|^{2i}}{(\bullet i)!} \cdot \left[ K^{\frac{i}{2}} (2n \bullet i)^{(q + 2n)} \psi - 1 \right] \]

\[ 2K^2 M_n \sup_q \frac{(sB_m)^{(q + 2n)} \psi - 1}{(\bullet i)!} \cdot \left[ (2n \bullet i)^{(q + 2n)} \psi - 1 \right] \]

Employing inequality,

\[ \frac{n^n}{e^n} \leq n! \leq n^n \]

in expression (41), we obtain

\[ 2K^2 M_n \sup_q \frac{(sB_m)^{(q + 2n)} \psi - 1}{(\bullet i)!} \cdot \left[ (2n \bullet i)^{(q + 2n)} \psi - 1 \right] \]

Since hypothesis (iv) in the order, \(\mu_\alpha\) implies \(1 - \mu_\alpha + \varepsilon = \frac{1}{n}\) for \(\varepsilon > 0\), we have \(n_1(1 - \frac{1}{n}) \leq 1 - n_1\). Implementing this observation in expression (43) we have
(44)  
\[ 2K'M_m \sup_{t} \frac{u(l+1)}{(x^2)^p} \sum_{i=0}^{n} (\frac{K_i e}{P})^i |t-t_0|^i (2 \bullet (l+1)^2 (sB_m)^2 (2n_1)^{2n_1} \ldots) \cdot \]

We let \( n \to \infty \) in expression (44). Consequently each series converges by the root test. If we select \( s' \geq (l+1)s \), is bounded. As a result when \( \varepsilon > 0, n \in N_+ \), there exists an \( N \in N_+ \), such that \( ||B_n(P(Q_m^\varepsilon \Lambda^2)) - B_m(P(Q_m^\varepsilon \Lambda^2))|| ||sB_n \leq \varepsilon \) for all \( \varepsilon > 0, n, m > N \).

Therefore we have a Cauchy sequence in a Frechet space, \( \Gamma^{P,S,d,B} \). It is clear that any rearrangement of the series converges to the same limit, \( v(t) \in \Gamma^{P,S,d,B} \). The continuity of \( v(t) \) follows for our infinite dimensional Laplacian operator in much the same manner as it did for the operator \( D_h \) in reference [13].

6. Impulsive differential equation solutions in \( \Gamma^{PB} \)

We generalize the impulsive differential equation, (1) , to be

(45)  
\[ \frac{\partial^j u(t)(x)}{\partial t^j} = \frac{\partial^{j-1}}{\partial t^{j-1}} P_1(\Lambda^2)u(t)(x) + \ldots + P_l(\Lambda^2)u(t)(x) \]

The \( \Lambda^2 \) in expression (45) is the infinite dimensional Laplacian operator developed in section III.

The \( P_1(\Lambda^2), 1 \leq \lambda \leq l \) are polynomials having the independent variable formally replaced with the infinite dimensional Laplacian operator. We equip the solution of equation (45) to satisfy the initial impulsive state conditions, (29) and (30) where both the initial and impulsive conditions are in \( \Gamma^{PB} \). A direct computation reveals the solution to be \( v(t) = u(t_0^l)(x) + [I - Q_m^\varepsilon P_1(\Lambda^2) + \ldots + Q_m^\varepsilon P_l(\Lambda^2)]u(t)(x) + \sum_{n_0 < l_0 \leq l} I_{k_0}(\tau_{k_0}) \) for \( t \in J^+ \) as in reference [13]. Similarly for \( t \in J^- \).

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John SCHMEELK
Virginia Commonwealth University
Department of Mathematics Emeritus
School of the Arts in Qatar
PO Box 8095
Doha, QATAR
c-mail: john.schmeelk@yahoo.com

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V. Valmorin

A GLOBAL CONSTRUCTION OF ALGEBRAS OF GENERALIZED FUNCTIONS

Abstract. An original method for the construction of algebras of generalized functions is given, which covers both Colombeau simplified algebras and Rosinger algebras cases, the main used such algebras. Examples are given showing how this method works for the most known algebras in this area.

1. Introduction

There is no general method for the construction of algebras of generalized functions covering all the main used until now. The more frequently used algebras of generalized functions are those of Colombeau [1, 2] and Rosinger [9, 10] in that order. Colombeau algebras exist in two versions: The simplified type algebra and the full type algebra (see eg. [8, 5, 12]). Presently we are concerned with the simplified one. In [6] an analysis of the structure of Colombeau simplified algebras has been elaborated in [7] leading to the concept of the so-called \((\mathcal{C}, \mathcal{E}, \mathcal{P})\) – algebras which generates a large range of algebras of generalized functions. The more known of them are those of Colombeau and Egorov [4]. Unfortunately Rosinger algebras of generalized functions are not covered by that framework. In fact from the view point of their construction, Colombeau and Rosinger algebras appear to be quite different. In the present work is given an new method for the construction of algebras of generalized functions that covers all the above mentioned algebras. To achieve this goal the new concept of \((\mathcal{M}, \mathcal{N}(V_P))\) – algebra is introduced. This concept is based on the idea that algebras of generalized functions may be represented as a factor of moderate elements by the corresponding null ones as it is formulated in Colombeau’s construction. It is shown that this method covers \((\mathcal{C}, \mathcal{E}, \mathcal{P})\) and the Rosinger type algebras. Nevertheless explicit constructions for Colombeau and Egorov algebras are also given. In the above notations \(V_P\) denotes an associative and commutative algebra \(V\) with \(\mathcal{P}\) a family of families of maps defined in \(V\) and valued in a given ordered set \(F\), \(\mathcal{M}\) and \(\mathcal{N}\) are sets of \(F\) – valued maps defined in a filtered set \(E\).

2. Bounds spaces

2.1. Definitions

Let \(E\) denote a set equipped with a filter \(\mathcal{F}\). Consider an ordered set \((F, \leq)\) where \(\leq\) is not necessarily total, equipped with a multiplication, an addition and the multiplication by a nonnegative number. We suppose that the same rules as the ones of an associative and commutative algebra are fulfilled when the field is replaced by \(\mathbb{R}_+^* = (0, \infty)\). In particular if \(a \in F\) and \(\lambda, \mu \in \mathbb{R}_+^*\), then \(\lambda(\mu a) = (\lambda \mu)a\). Furthermore we assume the
compatibility properties with respect to the order relation are satisfied: For all \((a,b,x,y) \in F^4\) and \((\lambda,\mu) \in \mathbb{R}_+^* \times \mathbb{R}_+^*\), one has:

\[
\begin{align*}
(a \leq b, \lambda \leq \mu) &\Rightarrow (\lambda a \leq \mu b) \\
(a \leq b, x \leq y) &\Rightarrow (a+x \leq b+y) \\
(a \leq b, x \leq y) &\Rightarrow (ax \leq by)
\end{align*}
\]

As examples \(E\) may be one of the sets \((0,1), (0,\infty), \mathbb{N}\) equipped with usual filters converging to 0 or \(\infty\). In the sequel we consider essentially the case \(F = \mathbb{R}_+\). Nevertheless we may consider others cases: Let \(F\) denote the set of polynomials with positive real coefficients. If \(P = \sum a_n x^n\) and \(Q = \sum b_n x^n\) are two elements of \(F\), let \(P \leq Q\) if for all \(n \in \mathbb{N}\), one has \(a_n \leq b_n\). One can verify that all the above necessary conditions are fulfilled.

We denote by \(\mathcal{A}(E,F)\) the set of all maps from \(E\) to \(F\). We define a binary relation \(\prec\) between two elements \(f\) and \(g\) of \(\mathcal{A}(E,F)\) by:

\[f \prec g \iff \exists x \in F, \forall x \in X, f(x) \leq g(x).\]

Let \(\mathcal{M}\) denote a subset of \(\mathcal{A}(E,F)\) such that :

1. \(\forall (\phi_1, \phi_2) \in \mathcal{M}^2, \exists \psi \in \mathcal{M}, \phi_1 + \phi_2 \prec \psi.\)
2. \(\forall (\phi_1, \phi_2) \in \mathcal{M}^2, \exists \psi \in \mathcal{M}, \phi_1 \phi_2 \prec \psi.\)
3. \(\exists \nu \in F, \exists \psi_0 \in \mathcal{M}, \nu \prec \psi_0 (\nu\) is considered as a constant map).

We denote by \(\mathcal{N}\) a subset of \(\mathcal{A}(E,F)\) such that :

4. \(\forall \psi \in \mathcal{M}, \forall \phi \in \mathcal{N}, \exists \phi_1 \in \mathcal{N}, \phi \phi_1 \prec \phi.\)

We call \((\mathcal{M}, \mathcal{N})\) a couple of bounds spaces, that will be justified in section 3.3

### 2.2. Examples

In all the following examples we take \(F = \mathbb{R}_+\). For the four first ones we take \(E = (0,1]\) and a basis of \(\mathcal{F}\) is the set of all open intervals \((0,\eta)\) with \(\eta \in E\).

1) \(\mathcal{M} = \{e \mapsto a, a > 1\}, \mathcal{N} = \{e \mapsto a, 0 < a < 1\}.
2) \(\mathcal{M} = \{e \mapsto e^p, p \in \mathbb{Z}\}, \mathcal{N} = \{e \mapsto e^p, p \in \mathbb{Z}_+\}\) where \(\mathbb{Z}_+\) denotes the set of positive integers.

That correspond to the simplified Colombeau algebras with entire powers of the parameter.

3) \(\mathcal{M} = \{e_0 > 0, \forall a > 1, \phi(e) < a^{1/s}\}, \mathcal{N} = \{e \mapsto a^{1/s}, 0 < a < 1\}.

These sets correspond to algebras of generalized hyperfunctions on the circle \([11, 13]\).

4) \(\mathcal{M} = \{a_n : n \in \mathbb{Z}_-\}, \mathcal{N} = \{a_n : n \in \mathbb{Z}_+\}\) where \((a_n)_{n \in \mathbb{Z}}\) is an asymptotic scale over \(E = (0,1]\) endowed with its Fréchet filter. An asymptotic scale consists of functions
defined on $E$ and satisfying some asymptotic properties. This concept was introduced for the construction of the so-called asymptotic algebras [3].

5) We take $E = \mathbb{N}$ endowed with is Fréchet filter. Let $\mathcal{M}$ be the set of all nonnegative sequences and $\mathcal{N}$ the subset of $\mathcal{M}$ formed by the finite support sequences. Here we are in the setting of Egorov’s algebra [4].

**Proposition 1.** Assume that the following condition

$$(*) \forall r \in F, \exists n \in \mathbb{N}^*, r \prec n$$

is fulfilled with $\nu$ as given in (3). Then, we have:

(i) $\forall r \in F, \exists \psi \in \mathcal{M}, r \prec \psi$

(ii) $\forall (r, s) \in F^2, \exists \psi_1, \psi_2 \in \mathcal{M}, r\psi_1 + s\psi_2 \prec \psi$

(iii) $\forall r \in F, \forall \psi_1, \psi \in \mathcal{N}, r\psi_1 \prec \psi$

(iv) moreover if there exist $a \in F$ and $\psi_0 \in \mathcal{N}$ such that $\psi_0 \prec a$, then we have : $\forall r \in F, \exists \psi \in \mathcal{N}, r\psi \prec a$.

**Proof.** (i) Let $r \in F$. Because of $(*)$, there is $n \in \mathbb{N}^*$ such that $r \prec 2^n\nu$. Using the property (1) of the definition of $\mathcal{M}$, we find by induction $\psi \in \mathcal{M}$ such that $2^n\nu \prec \psi$. From (3), one has $2^n\nu \prec 2^n\nu_0$. Hence, it follows that $r \prec \psi$.

(ii) let $(r, s) \in F^2$ and $(\psi_1, \psi_2) \in \mathcal{M}^2$. From (i), there is $(\xi_1, \xi_2) \in \mathcal{M}^2$ such that $r \prec \xi_1$ and $s \prec \xi_2$; it follows that $r\psi_1 + s\psi_2 \prec \xi_1\psi_1 + \xi_2\psi_2$. Using (2), we find $(\xi_1, \xi_2) \in \mathcal{M}^2$ such that $\xi_1\psi_1 \prec \xi_1$ and $\xi_2\psi_2 \prec \xi_2$. Property (1) gives $\psi \in \mathcal{M}$ such that $\xi_1 + \xi_2 \prec \psi$. Hence we have $r\psi_1 + s\psi_2 \prec \psi$.

(iii) let $r \in F$ and $\psi \in \mathcal{N}$. By (i) we have $r \prec \psi$ for some $\psi \in \mathcal{M}$. Now, from (4), there exists $\psi_1 \in \mathcal{N}$ such that $\psi \prec \psi_1$; hence $r\psi_1 \prec \psi_1 \prec \psi$. Hence we have $r\psi_1 \prec \psi_1 \prec \psi_1$.

(iv) let the hypothesis fulfilled. From (iii) there exists $\psi_1 \in \mathcal{N}$ such that $r\psi_1 \prec \psi_0$; hence $r\psi_1 \prec a$.

**Remark.** (a) note that if $\mathcal{M} \cap \mathcal{N} \neq \emptyset$ and $\psi \in \mathcal{M} \cap \mathcal{N}$ one has $\varphi \prec \psi$ with $(\varphi, \psi) \in \mathcal{M} \times \mathcal{N}$. Now, if there is $(\varphi, \psi) \in \mathcal{M} \times \mathcal{N}$ such that $\psi \prec \psi$, then from (4) there is $\psi_1 \in \mathcal{N}$ such that $\psi\psi_1 \prec \psi\psi_1$. Hence we have $\psi\psi_1 \prec \psi$. In particular, if $F = \mathbb{R}_+$, we obtain that $\psi_1 \prec 1$. Hence from (i) it follows : $\forall r > 0, \exists \psi \in \mathcal{N}, \psi \prec r$.

(b) Note that the ordered set $(F, \leq)$ of polynomials with positive coefficients does not verify the condition $(*)$ of Proposition 1. Nevertheless if $\mathcal{M}$ has a constant function then (i) becomes a straightforward consequence of (1). It follows that (ii) – (iv) are also valid.

3. Generalized algebras

3.1. Definitions

Here and in the sequel, we suppose that the hypotheses of Proposition 1 are fulfilled. Let $V$ denote a commutative and associative algebra over $K$ where $K = \mathbb{R}$ or $\mathbb{C}$. Consider the set $\mathcal{A}(V, F)$ of all maps from $V$ to $F$. We suppose that there is a set $I$ with a
basis of filter \((I_t)_{t \in \Gamma}\), that is satisfying
\begin{equation}
\forall \gamma \in \Gamma, \ I_t \neq \emptyset \text{ and } \forall (\gamma_1, \gamma_2) \in \mathbb{R}^2, \ \exists \gamma \in \Gamma, \ I_t \subset I_{\gamma_1} \cap I_{\gamma_2}
\end{equation}
and to each \(\gamma \in \Gamma\) are associated a family \(\mathcal{P}_\gamma = (p_{t,i})_{i \in I_t}\) of elements of \(\mathcal{A}(V, F)\) and a family of maps \((\alpha_\gamma)_{\gamma \in \Gamma}\) from \(K\) to \(F\) satisfying the following conditions :
\begin{equation}
\forall (\gamma_1, \gamma_2) \in \mathbb{R}^2, (I_{\gamma_1} \subset I_{\gamma_2}) \Rightarrow (\forall i \in I_{\gamma_1}, p_{t,i} \leq p_{t,\gamma_2}) \tag{6}
\end{equation}
\begin{equation}
\forall \gamma \in \Gamma, \forall i \in I_{\gamma}, \exists \zeta \in V^2, \forall (\lambda, \mu) \in \mathbb{K}^2, \ p_{t,i}(\lambda \xi + \mu \zeta) \leq \alpha_\gamma(\lambda) p_{t,i}(\xi) + \alpha_\gamma(\mu) p_{t,i}(\zeta) \tag{7}
\end{equation}
\begin{equation}
\forall \gamma \in \Gamma, \forall i \in I_{\gamma}, \exists (j, k) \in I^2, \exists \zeta \in F, \forall (\xi, \zeta) \in V^2, \ p_{t,i}(\xi \zeta) \leq C p_{t,j}(\xi) p_{t,k}(\zeta) \tag{8}
\end{equation}
Note that if \(V\) has a family \(\mathcal{P} = (p_i)_{i \in I}\) of seminorms then (5)-(8) are satisfied with \((I_t)_{t \in \Gamma}\) reduced to \(I\) and \(\alpha_\gamma = |.|\).

### 3.2. An example

Set \(F = \mathbb{R}_+\) and let \(I\) denote the set of all polynomials of degree \(\geq 1\) with positive coefficients except the 0-th coefficient which equals zero. For \(p, q \in I\), \(p \leq q\) means that \(\deg p \leq \deg q\) and \(a_{p,m} \leq a_{q,m}, 1 \leq m \leq \deg p\) where \(a_{p,m}\) (resp. \(a_{q,m}\)) is the \(m\)-th coefficient of \(p\) (resp. \(q\)). We denote by \(\Gamma\) a subset of \(I\) with the following property
\begin{equation}
\forall (p, q) \in I^2, \exists \gamma \in \Gamma, \gamma \leq \inf\{p, q\} \tag{9}
\end{equation}
Such a subset \(\Gamma\) exists because if \(\Gamma\) is the set of terms of a sequence in \(I\) decreasing to zero, we can take \(\gamma \in \Gamma\) such that \(a_{\gamma,m} \leq \inf\{a_{p,m}, a_{q,m}\}, 1 \leq m \leq \inf\{\deg p, \deg q\}\). We set
\[I_\gamma = \{p \in I, \ p \leq \gamma\}\] and \(\alpha_\gamma(v) = 2^{\deg v - 1} \sup\{|v|, |v|^{\deg v}\}, v \in \mathbb{K}\).
It follows straightforwardly from (9) that \((I_t)_{t \in \Gamma}\) satisfies (5).

Now let \(V\) be a \(\mathbb{K}\)-algebra with a given family of semi-norms \((p_i)_{i \in I}\) (indexed by \(I\)) such that
\begin{equation}
\forall i \in I, \exists (j, k) \in I^2, \exists D > 0, p_i((\xi \zeta)) \leq D p_j(\xi) p_k(\zeta), (\xi, \zeta) \in V^2. \tag{10}
\end{equation}
We define \(p_{t,i}\) by
\[p_{t,i} = \sup_{q \in I_t} q \circ p_i.
\]
For nonnegative numbers \(\alpha, \beta, r, s\) we have
\[(\alpha r + \beta s)^k \leq 2^{k-1}|(\alpha r)^k + (\beta s)^k|, k \in \mathbb{N}.\]
Then if \( q \in \Gamma \) with \( \deg q = n \) it follows that

\[
q(\alpha r + \beta s) \leq 2^{n-1}[\sup\{\alpha, \alpha^n\}q(r) + \sup\{\beta, \beta^n\}q(s)].
\]

which means

\[(11) \quad q(\alpha r + \beta s) \leq \sigma_q(\alpha)q(r) + \sigma_q(\beta)q(s).\]

Set \( p_i = q \circ p_i \). Let \( x, y \in V \) and \( \lambda, \mu \in \mathbb{K} \). Then we have

\[
p_i(\lambda x + \mu y) = q[p_i(\lambda x + \mu y)] \leq q(\lambda|p_i(x) + |\mu|p_i(y))
\]

Invoking (11) yields

\[(12) \quad p_i(\lambda x + \mu y) \leq \sigma_q(\lambda)p_i(x) + \sigma_q(\mu)p_i(y).\]

Furthermore if \( q = a_0X^n + \ldots + a_1X \) and \( C = \sup\{D^m a^{-1}_m, a_m \neq 0, 1 \leq m \leq n\} \) we have

\[
a_m D^m r^m s^m \leq C a_m r^m a_s^m, 1 \leq m \leq n.
\]

It follows that \( q(\alpha r s) \leq C q(r)q(s) \) and then

\[
q(p_i(xy)) \leq q(Dp_j(x)p_k(y)) \leq Cq(p_j(x)q(p_k(y))
\]

which means that

\[(13) \quad p_i(xy) \leq C p_j(x)p_k(y).\]

Hence (6), (7), (8) are satisfied.

### 3.3. Generalized algebra of type \((\mathcal{M}, \mathcal{N}, V, \mathcal{F})\)

We keep notations of section 3.1 and we denote by \( X(V) \) the set of all families \((u_i)_e\) with \( e \in E \) and \( u_e \in V \). If \( u = (u_i)_e \in X(V) \) and \( i \in I_\gamma \), we set

\[
\tilde{p}_{\gamma,i}[u](e) = p_{\gamma,i}(u_e).
\]

Hence \( \tilde{p}_{\gamma,i}[u] \) is a well defined map from \( E \) to \( F \). We define :

\[
X\mathcal{M}(V) = \{ u = (u_i)_e \in X(V), \exists \gamma \in \Gamma, \forall i \in I_\gamma, \exists \varphi \in \mathcal{M}, \tilde{p}_{\gamma,i}[u] < \varphi \}
\]

\[
X\mathcal{N}(V) = \{ u = (u_i)_e \in X(V), \exists \gamma \in \Gamma, \forall i \in I_\gamma, \forall \psi \in \mathcal{N}, \tilde{p}_{\gamma,i}[u] < \psi \}
\]

**THEOREM 1.** Suppose that the following condition holds:

\[
(**) \quad \forall \psi \in \mathcal{N}, \exists \varphi \in \mathcal{M}, \psi < \varphi.
\]

Then \( X\mathcal{M}(V) \) is a subalgebra of \( X(V) \) and \( X\mathcal{N}(V) \) is an ideal of \( X\mathcal{M}(V) \).
Proof. From the hypothesis we have straightforwardly $X_{\mathcal{N}}(V) \subset X_M(V)$. Let $u, v \in X_M(V)$ and $\lambda, \mu \in \mathbb{K}$. From the definition of $X_M(V)$ there exist $(\gamma_1, \gamma_2) \in \Gamma^2$ and $(\psi_1, \psi_2) \in M^2$ such that

$$\tilde{p}_{\gamma_1, j}[u] \prec \psi_1 \text{ and } \tilde{p}_{\gamma_2, k}[v] \prec \psi_2, \ (j, k) \in I_{\gamma_1} \times I_{\gamma_2}.$$  

According to (5) we may choose $\gamma \in \Gamma$ such that $I_\gamma \subset I_{\gamma_1} \cap I_{\gamma_2}$. Hence for all $i \in I_\gamma$, we have $\tilde{p}_{\gamma, i}[u] \prec \psi_1$ and $\tilde{p}_{\gamma, i}[v] \prec \psi_2$. It follows that

$$\tilde{p}_{\gamma, i}[\lambda u + \mu v] \leq \alpha_\gamma(\lambda) \tilde{p}_{\gamma, i}[u] + \alpha_\gamma(\mu) \tilde{p}_{\gamma, i}[v].$$

Hence we have

$$\tilde{p}_{\gamma, i}[\lambda u + \mu v] \leq \alpha_\gamma(\lambda) \psi_1 + \alpha_\gamma(\mu) \psi_2.$$  

Using condition (2) of the definition of $M$ and [(i), Proposition 1] gives $\psi \in M$ such that $\tilde{p}_{\gamma, i}[\lambda u + \mu v] \prec \psi$. Hence $\lambda u + \mu v \in X_M(V)$.  

From (6), there exist $C \in F$ and $(j, k) I_j^2$ such that

$$\tilde{p}_{\gamma, j}[uv] \leq C \tilde{p}_{\gamma, j}[u] \tilde{p}_{\gamma, k}[v] \prec C \psi_1 \psi_2.$$  

Using condition (2) of the definition of $M$ and [[(i), Proposition 1]] gives $\psi \in M$ such that $\tilde{p}_{\gamma, j}[uv] \prec \psi$, showing that $uv \in X_M(V)$. Hence $X_M(V)$ is a subalgebra of $X(V)$.  

Let $u, v \in X_M(V)$ and $\lambda, \mu \in \mathbb{K}$. From the definition of $X_M(V)$, there are $(\gamma_1, \gamma_2) \in \Gamma^2$ such that for all $(j, k) \in I_{\gamma_1} \times I_{\gamma_2}$ and all $\psi_1, \psi_2 \in \mathcal{N}$, we have

$$\tilde{p}_{\gamma_1, j}[u] \prec \psi_1 \text{ and } \tilde{p}_{\gamma_2, k}[v] \prec \psi_2.$$  

As above take $I_\gamma \subset I_{\gamma_1} \cap I_{\gamma_2}$ and $\psi \in \mathcal{N}$. From [[(iii), Proposition 1]] there are $\psi_1, \psi_2 \in \mathcal{N}$ such that

$$2\alpha_\gamma(\lambda) \psi_1 \prec \psi \text{ and } 2\alpha_\gamma(\mu) \psi_2 \prec \psi.$$  

Since $I_\gamma \subset I_{\gamma_1} \cap I_{\gamma_2}$, then for all $i \in I_\gamma$, we have $\tilde{p}_{\gamma, i}[u] \prec \psi_1$ and $\tilde{p}_{\gamma, i}[v] \prec \psi_2$. It follows that

$$2\tilde{p}_{\gamma, i}[\lambda u + \mu v] \prec 2(\alpha_\gamma(\lambda) \psi_1 + \alpha_\gamma(\mu) \psi_2) \prec 2 \psi.$$  

Hence, multiplying by $1/2$ gives $\tilde{p}_{\gamma, i}[\lambda u + \mu v] \prec \psi$, that is $\lambda u + \mu v \in X_{\mathcal{N}}(V)$.  

Let $u \in X_{\mathcal{N}}(V)$ and $v \in X_M(V)$. There is $\gamma_1 \in \Gamma$ such that for all $\psi_1 \in \mathcal{N}$, we have $\tilde{p}_{\gamma_1, j}[u] \prec \psi_1$ for all $j \in I_{\gamma_1}$. In the same way, there are $\gamma_2 \in \Gamma$ and $\psi_2 \in M$ such that if $k \in I_{\gamma_2}$, then $\tilde{p}_{\gamma_2, k}[v] \prec \psi_2$. Now, let $\gamma \in \Gamma$ such that $I_\gamma \subset I_{\gamma_1} \cap I_{\gamma_2}$ and $\psi \in \mathcal{N}$. Let $i \in I_\gamma$. There exist $C \in F$ and $(j, k) I_j^2$ such that

$$\tilde{p}_{\gamma, j}[uv] \prec C \tilde{p}_{\gamma, j}[u] \tilde{p}_{\gamma, k}[v].$$  

Let $\psi \in \mathcal{N}$ and choose $\psi_1 \in \mathcal{N}$ such that $C \psi_1 \psi_1 \prec \psi$. The above inequality shows that $\tilde{p}_{\gamma, j}[uv] \prec \psi$ that is $uv \in X_{\mathcal{N}}(V)$. Hence $X_{\mathcal{N}}(V)$ is an ideal of $X_M(V)$.  

\[\Box\]

\textbf{Definition 1.} Assume that [(+), Proposition 1] and [+(+), Theorem 1] are satisfied. Then, the generalized algebra of type $(M, \mathcal{N}, V_\psi)$ is the factor algebra:

$$\mathcal{G}_{M, \mathcal{N}, V_\psi}(V) = X_M(V)/X_{\mathcal{N}}(V).$$
4. Usual algebras as \((\mathcal{M}, \mathcal{N}, V_p)\)–algebras

In this section \(\Omega\) denotes an open set of \(\mathbb{R}^n\) and \(\mathcal{E}(\Omega)\) the space of smooth functions in \(\Omega\). We denote by \((K_i)\) an increasing sequence of compact sets exhausting \(\Omega\) with \(K_0 \neq \emptyset\).

4.1. Simplified Colombeau algebra

We consider on \(\mathcal{E}(\Omega)\) the sequence of seminorms

\[
\mu_l(f) = \sup\{|\partial^\alpha f(x)|, \alpha \in \mathbb{N}^n, |\alpha| \leq l, x \in K_l\}, l \in \mathbb{N}.
\]

We note here and for the sequel that \(\mu_l(fg) \leq 2^l \mu_l(f)\mu_l(g), f, g \in \mathcal{E}(\Omega)\). The set \(\mathcal{E}_M(\Omega)\) of moderate nets consists of nets \((f_\varepsilon)_{\varepsilon} \in \mathcal{E}(\Omega)^{(0,l]}\) with the properties

\[
\forall l \in \mathbb{N}, \exists \varepsilon \in \mathbb{R}, \exists \eta > 0, \mu_l(f_\varepsilon) \leq \varepsilon^l, 0 < \varepsilon < \eta
\]

and the set \(\mathcal{N}(\Omega)\) of null nets consists of nets \((f_\varepsilon)_{\varepsilon} \in \mathcal{E}(\Omega)^{(0,1]}\) with the properties

\[
\forall l \in \mathbb{N}, \forall q \in \mathbb{R}, \exists \eta > 0, \mu_l(f_\varepsilon) \leq \varepsilon^q, 0 < \varepsilon < \eta.
\]

These spaces are both algebras and \(\mathcal{N}(\Omega)\) is an ideal of \(\mathcal{E}_M(\Omega)\). The simplified Colombeau algebra \(\mathcal{G}^s(\Omega)\) is defined as the factor

\[
\mathcal{G}^s(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega).
\]

Let \(E = (0, 1]\) endowed with its Fréchet filter, \(F = \mathbb{R}_+\) with its usual order \(\leq\), set

\[
\mathcal{M} = \{\varepsilon \mapsto \varepsilon^p, p \in \mathbb{R}\} \quad \text{and} \quad \mathcal{N} = \{\varepsilon \mapsto \varepsilon^p, p \in \mathbb{R}_+\}.
\]

Since \(\mathcal{N} \subset \mathcal{M}\) it is easily seen that (1)-(4) and condition (⋆) of Proposition 1 are satisfied, we may take \(\nu = 1\) in (3). Condition (⋆⋆) is also trivially satisfied. Taking \(V = \mathcal{E}(\Omega), I = \mathbb{N}\) with the basis of filter \(\{\mathbb{N}\}\) and \(\mathcal{P} = \{(\mu_l)_{l \in \mathbb{N}}\}\) we find

\[
X_\mathcal{M}(V) = \mathcal{E}_M(\Omega) \quad \text{and} \quad X_\mathcal{N}(V) = \mathcal{N}(\Omega)
\]

showing that \(\mathcal{G}^s(\Omega) = G_{\mathcal{M}, \mathcal{N}, \mathcal{P}}(V)\).

4.2. Egorov algebra

Denote by \(\mathcal{N}_E(\Omega)\) the subset of \(\mathcal{E}(\Omega)^{\mathbb{N}}\) whose elements \((u_n)\) satisfy the properties

\[
\forall K \text{ compact set} \subset \Omega, \exists n_0 \in \mathbb{N}, u_n(x) = 0, x \in K, n \geq n_0.
\]

It is seen that \(\mathcal{N}_E(\Omega)\) is an ideal of \(\mathcal{E}(\Omega)^{\mathbb{N}}\). The Egorov algebra \(\mathcal{G}_E(\Omega)\) is defined as the factor

\[
\mathcal{G}_E(\Omega) = \mathcal{E}(\Omega)^{\mathbb{N}}/\mathcal{N}_E(\Omega).
\]
We show that $\mathcal{G}_E(\Omega)$ is a $(\mathcal{M}, \mathcal{N}, V_\varepsilon)$-type algebra. For let $I$ denote the set of compacts subsets of $\Omega$. Let $V = \mathcal{E}(\Omega)$ and $(K_i)_i$ be as in the previous section. If $I \in \mathbb{N}$ and $u \in V$ we set $p_i(u) = \sup \{|u(x)|, x \in K_i\}$ and $\mathcal{P} = (p_i)_i$. We take $E = \mathbb{N}$, it follows that $\mathcal{X}(V) = \mathcal{E}(\Omega)^\mathbb{N}$. Let $\mathcal{M}$ be the set of all sequences of nonnegative real numbers and $\mathcal{N}$ the subset of $\mathcal{M}$ consisting of all sequences which finite support. If $(u_n)_n \in \mathcal{X}(V)$ and $l \in \mathbb{N}$ we have $(p_l(u_n))_n \in \mathcal{M}$, it follows that $\mathcal{X}_\mathcal{M}(V) = \mathcal{X}(V) = \mathcal{E}(\Omega)^\mathbb{N}$. It is easily seen that (1)-(8) with (*) and (**) are satisfied. From the definition of $\mathcal{N}$ it follows that $(\mathcal{N}_l)_l \in \mathcal{N}$ if and only if for each $l \in \mathbb{N}$, $(u_n|K_l)_n$ have a finite support. Hence the corresponding $(\mathcal{M}, \mathcal{N}, V_\varepsilon)$-algebra is Egorov’s algebra.

4.3. $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras

The $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ type algebras are constructed as follows. If $r \in \mathbb{K}^{[0,1]}$, we set $r = (r_\varepsilon)_\varepsilon$ and $|r| = (|r_\varepsilon|)_\varepsilon$. Moreover if $r, t \in \mathbb{R}^{(0,1)}$, $t \leq r$ means $t_\varepsilon \leq r_\varepsilon, \varepsilon \in (0,1]$. If $T \subset \mathbb{K}^{(0,1)}$ one defines

$$T^+ = \{t \in T, t_\varepsilon \geq 0, \varepsilon \in (0,1]\} \text{ and } |T| = \{ |t|, t \in T \}$$

Consider the following two conditions for $T \subset \mathbb{K}^{(0,1)}$:

$$\forall t \in T^+, \forall r \in \mathbb{K}^{(0,1)}, |r| \leq t \Rightarrow r \in T \quad (S)$$

and

$$T^+ = |T|. \quad (MS)$$

Here (S) stands for solidity and (MS) for modulus stability. Note that if (S) is satisfied then (MS) can be replaced by the condition $|S| \subset S$.

Now let $A$ denote a subring of $\mathbb{K}^{(0,1)}$ and $I_A$ an ideal of $A$ satisfying both (S) and (MS). Let $\mathcal{E}$ be a topological $\mathbb{K}$-algebra endowed with a family $\mathcal{P} = (p_I)_I \subset L$ of seminorms such that

$$\forall I \in L, \exists (j,k) \in L^2, \exists C \in \mathbb{R}_+, p_I(fg) \leq Cp_I(f)p_I(g), f,g \in \mathcal{E}.$$ 

Then one defines

$$\mathcal{H}_{A,\mathcal{E}_\mathcal{P}} = \left\{ (u_\varepsilon)_\varepsilon \in \mathcal{E}^{(0,1)}, \forall I \in L, (p_I(u_\varepsilon))_\varepsilon \in A^+ \right\};$$

$$\mathcal{J}_{I_A,\mathcal{E}_\mathcal{P}} = \left\{ (u_\varepsilon)_\varepsilon \in \mathcal{E}^{(0,1)}, \forall I \in L, (p_I(u_\varepsilon))_\varepsilon \in I_A^+ \right\};$$

$$C = A/I_A.$$

It is shown that both spaces $\mathcal{H}_{A,\mathcal{E}_\mathcal{P}}$ and $\mathcal{J}_{I_A,\mathcal{E}_\mathcal{P}}$ are algebras, the latter being an ideal of the former. The associated $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra is the algebra $\mathcal{A}$ defined by

$$\mathcal{A} = \mathcal{H}_{A,\mathcal{E}_\mathcal{P}}/\mathcal{J}_{I_A,\mathcal{E}_\mathcal{P}}.$$ 

It is easily seen that $\mathcal{H}_{(A,K_i)} = A$ and $\mathcal{J}_{(A,K_i)} = I_A$, then the corresponding quotient gives the ring $C$.

We now show that every $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra is a $(\mathcal{M}, \mathcal{N}, E_\mathcal{P})$-algebra. For let $E = (0,1]$ with its Fréchet filter, $F = \mathbb{R}_+$ equipped with the order $\leq$ and $I = W$ with $\{I\}$ as basis of filter. If $a \in |A|$ we define $f_a : E \rightarrow F$ by $f_a(\varepsilon) = |a_\varepsilon|$ and we set

$$\mathcal{M} = \{ f_a, a \in |A| \} \text{ and } \mathcal{N} = \{ f_a, a \in |I_A| \}.$$
We note that $\mathcal{N} \subset \mathcal{M}$, then $(\star \star)$ is satisfied. We get (3) with $v = 1$ and $q_0 = f_1$ where $I$ is the unit of $\mathbb{H}^{[0,1]}$. Then $(\star)$ is satisfied with any integer $n \geq r + 1$. From (S) and (MS) it is seen that (1), (2) are verified and finally we have (4) with $\psi_1 = 0$.

From (S) and (MS), if $r \in \mathbb{R}_{+,1}$, then $r \in |A|$ (resp. $r \in |I|$) if and only if there exists $a \in |A|$ (resp. $a \in |I|$) such that $r \leq a$. It follows that

$$X_{\mathcal{M}}(E) = \mathcal{H}_{(A,\mathcal{E})}$$

and

$$X_{\mathcal{I}}(E) = \mathcal{J}_{(I,\mathcal{I})}$$

proving that $\mathcal{A} = \mathcal{H}_{(A,\mathcal{E})}/\mathcal{J}_{(I,\mathcal{I})} = \mathcal{G}_{\mathcal{M},\mathcal{N},\mathcal{P}}(E)$.

### 4.4. Rosinger algebra

Rosinger’s nowhere dense ideal $I_{\text{rad}}(\Omega)$ is the set of all $u = (u_e)_{e} \in \mathcal{E}(\Omega)^{(0,\infty)}$ such that there is a nowhere dense closed set $\gamma \subset \Omega$ such that for all $x \in \Omega \setminus \gamma$ there exist $\eta > 0$ and an open neighborhood $W_x$ of $x$ in $\Omega \setminus \gamma$ such that $u_e(y) = 0$ for all $y \in W_x$ and $\varepsilon < \eta$. Rosinger’s algebra is

$$\mathcal{R}(\Omega) = \mathcal{E}(\Omega)^{(0,\infty)}/I_{\text{rad}}(\Omega).$$

Let $u \in I_{\text{rad}}(\Omega)$ and choose $\gamma$ as above defined. Let $\omega$ denote a relatively compact open set in $\Omega \setminus \gamma$. For each $x \in \omega$ there are $\eta_x > 0, W_x$ open set of $\Omega \setminus \gamma$ such that $x \in W_x$ and $u|_{W_x} \equiv 0$, $0 < \varepsilon < \eta_x$. Using classical compactness arguments gives $\eta > 0$ such that $u_e|_{\omega} \equiv 0$ for $0 < \varepsilon < \eta$. It follows that $u = (u_e)_{e} \in I_{\text{rad}}(\Omega)$ if and only if there is a nowhere dense closed set $\gamma \subset \Omega$ such that for all relatively compact open set of $\Omega \setminus \gamma$, there is $\eta > 0$ such that $u_e|_{\omega} \equiv 0$ for $0 < \varepsilon < \eta$.

Let $E = (0,\infty)$ endowed with its Fréchet filter converging to 0 and let $V = \mathcal{E}(\Omega)$. We define $\Gamma$ as the set of all nowhere dense closed sets in $\Omega$ and $I$ the set of all relatively compact open sets in $\Omega$. For each $\gamma \in \Gamma$, we denote by $I_{\gamma}$ the set of $\omega \in I$ such that $\omega \cap \gamma \neq \emptyset$ and we set $\alpha_{\gamma} = |\cdot|$. Note that $I_{\gamma} \neq \emptyset$ and if $(\gamma_1, \gamma_2) \in \Gamma^2$, then $\gamma = \gamma_1 \cup \gamma_2 \in \Gamma$ and $I_\gamma = I_{\gamma_1} \cap I_{\gamma_2}$. It follows that $(I_{\gamma})_{\gamma \in \Gamma}$ satisfies (5). Let $\zeta \in V$. If $\gamma \in \Gamma$ and $\omega \in I_{\gamma}$ we set $p_{\gamma,\omega} = \sup \{|\zeta(x)|, x \in \omega\}$. Let $F = \mathbb{R}_+$, and $\mathcal{N} = \{0\}$. It is easily seen that $u = (u_e)_{e} \in I_{\text{rad}}(\Omega)$ if and only if there is $\gamma \in \Gamma$ such that for any $\omega \in I_{\gamma}$ one has $p_{\gamma,\omega}[u] < 0$. That is $I_{\text{rad}}(\Omega) = X_{\mathcal{N}}(\mathcal{V})$. If $\mathcal{M}$ is the set of all positive maps from $E$ to $\mathbb{R}_+$, we obtain

$$\mathcal{R}(\Omega) = \mathcal{G}_{\mathcal{M},\mathcal{N},\mathcal{P}}(E).$$

### References


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Vincent VALMORIN,
Laboratoire CEREGMIA, Université des Antilles et de la Guyane
Campus de Fouilloye 97159 Pointe à Pitre Cedex, FRANCE
e-mail: vincent.valmorin@univ-ag.fr

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