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DISPERSIVE ESTIMATES FOR T-DEPENDENT HYPERBOLIC SYSTEMS*

Abstract. This note is devoted to the study of time-dependent symmetric hyperbolic systems and the derivation of dispersive estimates for their solutions. It is based on a diagonalisation of the full symbol within adapted symbol classes.

We are going to consider the hyperbolic system

$$\mathrm{D}_t U = A(t,\mathrm{D})U, \qquad U(0,\cdot) = U_0, \tag{1}$$

where $A(t,\mathrm{D})$ denotes a smoothly time-dependent matrix Fourier multiplier with first order symbol

$$A(t,\xi) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{C}^{m\times m})$$

subject to certain (natural) assumptions which are described later on in detail. As usual we denote $\mathrm{D}_t = -\mathrm{i}\partial_t$.

Our approach is based on diagonalising the (full) symbol of the operator in order to get a representation of solutions in terms of Fourier integrals and later on to use these representations to deduce dispersive estimates for solutions.

1. Prerequisites and basic assumptions

1.1. Hyperbolic symbol classes

We make use of the implicitly defined function t_ξ from

$$(1+t_\xi)|\xi| = N \tag{2}$$

with a suitable constant N and define the zones

$$Z_{hyp}(N) = \{(t,\xi)|t \ge t_\xi\}, \qquad Z_{pd}(N) = \{(t,\xi)|0 \le t \le t_\xi\}. \tag{3}$$

In $Z_{hyp}(N,)$ we apply a diagonalisation procedure to the full symbol. The basic idea of this diagonalisation scheme comes from the treatment of degenerate hyperbolic problems and is closely related to the approach of [3].

DEFINITION 1. *The time-dependent Fourier multiplier $a(t,\xi)$ belongs to the hyperbolic symbol class $\mathcal{S}^{\ell_1,\ell_2}\{m_1,m_2\}$ if it satisfies the symbol estimates*

$$\left|\mathrm{D}_t^k \mathrm{D}_\xi^\alpha a(t,\xi)\right| \le C_{k,\alpha} |\xi|_{N,t}^{m_1-|\alpha|} \left(\frac{1}{1+t}\right)^{m_2+k} \tag{4}$$

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for all multi-indices $\alpha\in\mathbb{N}^n$ *with* $|\alpha|\le\ell_1$ *and all natural numbers* $k\le\ell_2$ *and with* $|\xi|_{N,t}=\max(|\xi|,N/(1+t))$. *We say it belongs to* $\mathcal{S}_N^{\ell_1,\ell_2}\{m_1,m_2\}$ *if the estimates are true within the hyperbolic zone* $Z_{hyp}(N)$.

EXAMPLE 1. A polynomial $p(t,\xi)=\sum_{|\alpha|=m}h_\alpha(t)\xi^\alpha$ with $t^k h_\alpha^{(k)}(t)\in L^\infty(\mathbb{R})$ for $k\le\ell$ belongs to $\mathcal{S}^{\infty,\ell}\{m,0\}$.

If the symbol estimates hold for all derivatives we write $\mathcal{S}_{(N)}\{m_1,m_2\}$ for $\mathcal{S}_{(N)}^{\infty,\infty}\{m_1,m_2\}$. Furthermore, the definition extents immediately to matrix-valued Fourier multiplier. The rules of the corresponding symbolic calculus are simple consequences of Definition 1 together with (2), (3) and collected in the following proposition.

PROPOSITION 1.

1. $\mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1,m_2\}$ *is a vector space.*
2. $\mathcal{S}_{(N')}^{\ell_1',\ell_2'}\{m_1-k,m_2+\ell\}\hookrightarrow\mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1,m_2\}$ *for all* $\ell\ge k\ge0$, $\ell_1'\ge\ell_1$, $\ell_2'\ge\ell_2$ *(and* $N'\le N$*).*
3. $\mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1,m_2\}\cdot\mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1',m_2'\}\hookrightarrow\mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1+m_1',m_2+m_2'\}$.
4. $\mathrm{D}_t^k\mathrm{D}_\xi^\alpha\mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1,m_2\}\hookrightarrow\mathcal{S}_{(N)}^{\ell_1-|\alpha|,\ell_2-k}\{m_1-|\alpha|,m_2+k\}$.
5. $\mathcal{S}_{(N)}^{0,0}\{-1,2\}\hookrightarrow L_\xi^\infty L_t^1(Z_{hyp}(N))$.

Of particular importance are the embedding relations of point 2 with $k=\ell$. They constitute a symbolic hierarchy, which is used in the diagonalisation scheme, cf. Section 2.1. We define the residual symbol classes

$$\mathcal{H}_{(N)}^{\ell_1,\ell_2}\{m\}=\bigcap_{k\in\mathbb{Z}}\mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m-k,k\}.$$

1.2. Basic assumptions

We collect our assumptions on the symbol $A(t,\xi)$. Throughout this note we require
(A1)$_{\ell_1,\ell_2}$ *Operator of first order with bounded coefficients.* We assume that the matrix operator $A(t,\mathrm{D})$ has a smooth symbol satisfying

$$A(t,\xi)\in\mathcal{S}^{\ell_1,\ell_2}\{1,0\}.$$

Furthermore, we assume that there exists a ξ-homogeneous matrix $A_0(t,\xi)$ with $A(t,\xi)-A_0(t,\xi)\in\mathcal{S}_N^{\ell_1,\ell_2}\{0,1\}$. We will always denote $\omega=\xi/|\xi|\in\mathbb{S}^{n-1}$.The symbol $A_0(t,\xi)$ is determined by its values $A_0(t,\omega)$ on the cylinder $\mathbb{R}_+\times\mathbb{S}^{n-1}$.
(A2) *Uniform strict hyperbolicity up to* $t=\infty$. We assume that the characterisitic roots (eigenvalues) of the symbol $A_0(t,\xi)$ are real and distinct for all t and $\xi\neq0$. In ascend-

ing order we denote them as $\lambda_1(t,\xi),\dots,\lambda_m(t,\xi)$. Furthermore, we assume that

$$\liminf_{t\to\infty}\min_{\omega\in\mathbb{S}^{n-1}}|\lambda_i(t,\omega)-\lambda_j(t,\omega)|>0$$

for all $i\neq j$.

PROPOSITION 2. *Assume (A1)$_{\ell_1,\ell_2}$ and (A2). For all $j=1,\dots,m$ the characteristic roots satisfy $\lambda_j(t,\xi)\in\mathcal{S}_N^{\infty,\ell_2}\{1,0\}$ and for all $i\neq j$ their difference satisfies $(\lambda_i(t,\xi)-\lambda_j(t,\xi))^{-1}\in\mathcal{S}_N^{\infty,\ell_2}\{-1,0\}$. Furthermore, the eigenprojection $P_j(t,\xi)$ corresponding to $\lambda_j(t,\xi)$ satisfies $P_j(t,\xi)\in\mathcal{S}_N^{\infty,\ell_2}\{0,0\}$.*

Sketch of proof. The properties of the characteristic roots follow from the spectral estimate $|\lambda_j(t,\omega)|\le||A(t,\omega)||$ together with the obvious symbol properties of the coefficients of the characteristic polynomial and the uniform strict hyperbolicity. The eigenprojections can be expressed in terms of the characteristic roots

$$P_j(t,\xi)=\prod_{i\neq j}\frac{A(t,\xi)-\lambda_i(t,\xi)}{\lambda_j(t,\xi)-\lambda_i(t,\xi)}$$

and again the symbolic calculus yields the desired result. □

PROPOSITION 3. *Assume (A1)$_{\ell_1,\ell_2}$ and (A2). There exists an invertible matrix $M(t,\omega)\in\mathcal{S}_N\{0,0\}$ which diagonalises the symbol $A(t,\omega)$,*

$$A(t,\omega)M(t,\omega)=M(t,\omega)\mathcal{D}(t,\omega),\qquad \mathcal{D}(t,\omega)=\operatorname{diag}\big(\lambda_1(t,\omega),\dots,\lambda_m(t,\omega)\big).$$

Furthermore, its inverse satisfies $M^{-1}(t,\omega)\in\mathcal{S}_N^{\infty,\ell_2}\{0,0\}$.

We require two more assumptions.

(A3) The matrix $F^{(0)}=\operatorname{diag}\big((\mathrm{D}_tM^{-1})M+M^{-1}(A-A_0)M\big)$ satisfies

$$\sup_{(s,\xi),(t,\xi)\in Z_{hyp}(N)}\left\|\int_s^t\operatorname{Im}F^{(0)}(\theta,\xi)\mathrm{d}\theta\right\|<\infty. \tag{5}$$

This assumption is independent of the choice of the diagonaliser $M(t,\xi)$ in Proposition 3 and trivially satisfied when $A(t,\xi)$ is symmetric and homogeneous.

(A4) The imaginary part $\operatorname{Im}A(t,\xi)=\frac{1}{2\mathrm{i}}(A(t,\xi)-A^*(t,\xi)$ satisfies the estimate

$$\operatorname{Im}A(t,\xi)+c|\xi|\mathrm{I}\ge 0$$

within $Z_{pd}(N)$ for sufficiently large N and some constant c.

2. Representation of solutions

Using the partial Fourier transform $\mathscr{F}$ with respect to the spatial variables we can reduce the system (1) into a system of ordinary differential equations. Our first objective is to represent its fundamental solution

(6) $$\mathrm{D}_t \mathcal{E}(t,s,\xi) = A(t,\xi)\mathcal{E}(t,s,\xi), \qquad \mathcal{E}(s,s,\xi) = \mathrm{I}$$

within the hyperbolic zone $(t,\xi),(s,\xi) \in Z_{hyp}(N)$.

2.1. Diagonalisation scheme

We follow the treatment of [3] to construct the fundamental solution to (6). To avoid unnecessary repetitions we just give the corresponding statements.

LEMMA 1. *Let $M(t,\xi)$ be the diagonaliser from Proposition 3. Then $\mathcal{E}_0(t,s,\xi) = M^{-1}(t,\xi)\mathcal{E}(t,s,\xi)M(s,\xi)$ satisfies*

(7) $$\mathrm{D}_t \mathcal{E}_0(t,s,\xi) = \big(\mathcal{D}(t,\xi) + R_0(t,\xi)\big)\mathcal{E}_0(t,s,\xi), \qquad \mathcal{E}_0(s,s,\xi) = \mathrm{I}$$

with $R_0(t,\xi) = \big(\mathrm{D}_t M^{-1}\big)M + M^{-1}(A - A_0)M \in \mathcal{S}_N^{\ell_1,\ell_2-1}\{0,1\}$.

LEMMA 2. *For each $1 \le k \le \ell_2 - 1$ there exists a zone constant N and matrix valued symbols*

- $N_k(t,\xi) = \mathrm{I} + \sum_{\mu=1}^{k} N^{(\mu)}(t,\xi)$, $N^{(\mu)}(t,\xi) \in \mathcal{S}_N^{\ell_1,\ell_2-\mu}\{-\mu,\mu\}$, *invertible for all $(t,\xi) \in Z_{hyp}(N)$ and with inverse satisfying $N_k^{-1}(t,\xi) \in \mathcal{S}_N\{0,0\}$*
- $F_{k-1}(t,\xi) = \sum_{\mu=0}^{k-1} F^{(\mu)}(t,\xi)$, $F^{(\mu)}(t,\xi) \in \mathcal{S}_N^{\ell_1,\ell_2-\mu-1}\{-\mu,\mu+1\}$, *diagonal,*
- $R_k(t,\xi) \in \mathcal{S}_N^{\ell_1,\ell_2-k-1}\{-k,k+1\}$,

such that $\mathcal{E}_k(t,s,\xi) = N_k^{-1}(t,\xi)\mathcal{E}_0(t,s,\xi)N_k(s,\xi)$ satisfies

(8) $$\mathrm{D}_t \mathcal{E}_k(t,s,\xi) = \big(\mathcal{D}(t,\xi) + F_{k-1}(t,\xi) + R_k(t,\xi)\big)\mathcal{E}_k(t,s,\xi), \qquad \mathcal{E}_k(s,s,\xi) = \mathrm{I}$$

for all $(t,\xi),(s,\xi) \in Z_{hyp}(N)$.

REMARK 1. For $k = 1$ we have in particular $F^{(0)}(t,\xi) = \operatorname{diag} R_0(t,\xi)$.

REMARK 2. The proof of this statement is analogous to the corresponding statement from [3] and applies the standard diagonalisation scheme from [11], [4], etc. Under $(\mathrm{A1})_{\ell_1,\infty}$ we can form the asymptotic sums $N(t,\xi) \sim \sum N^{(\mu)}(t,\xi) \in \mathcal{S}_N^{\ell_1,\infty}\{0,0\}$ and $F(t,\xi) \sim \sum F^{(\mu)}(t,\xi) \in \mathcal{S}_N^{\ell_1,\infty}\{0,1\}$ and the statement can be understood as perfect diagonalisation modulo $\mathcal{H}_N^{\ell_1,\infty}\{1\}$,

$$\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - R_0(t,\xi)\big)N(t,\xi) = N(t,\xi)\big(\mathrm{D}_t - F(t,\xi)\big) \mod \mathcal{H}_N^{\ell_1,\infty}\{1\}.$$

2.2. Estimates of the fundamental solution

We construct the fundamental solution $\mathcal{E}_k(t,s,\xi)$ within $Z_{hyp}(N)$.

THEOREM 1. *Assume (A1)$_{k-1,2k}$ for some $k \geq 1$. There exists a matrix family $\mathcal{Q}_k(t,s,\xi)$, uniformly bounded and invertible and satisfying*

$$\|\mathrm{D}_\xi^\alpha \mathcal{Q}_k(t,s,\xi)\| \leq C|\xi|^{-|\alpha|}, \tag{9}$$

$$\|\mathrm{D}_\xi^\alpha \mathcal{Q}_k(t,t_\xi,\xi)\| \leq C|\xi|^{-|\alpha|}, \qquad ,|\xi| \leq N, \tag{10}$$

for all $|\alpha| \leq k-1$, such that for all $(t,\xi),(s,\xi) \in Z_{hyp}(N)$

$$\mathcal{E}_k(t,s,\xi) = \exp\left(\mathrm{i}\int_s^t \big(\mathcal{D}(\tau,\xi) + F_{k-1}(\tau,\xi)\big)\mathrm{d}\tau\right)\mathcal{Q}_k(t,s,\xi). \tag{11}$$

Proof. We sketch the main steps of the proof. We denote the exponential in (11) by $\widetilde{\mathcal{E}}_k(t,s,\xi)$. Assumption (A3) implies

$$\|\widetilde{\mathcal{E}}_k(t,s,\xi)\| \lesssim 1 \tag{12}$$

uniformly in $(t,\xi),(s,\xi) \in Z_{hyp}(N)$ regardless of the order of s and t, because $F_{k-1}(t,\xi)$ $-F^{(0)}(t,\xi) \in \mathcal{S}_N^{0,0}\{-1,2\}$ and $\mathcal{D}(t,\xi)$ is real. Furthermore, the transformed equation (8) implies for $\mathcal{Q}_k(t,s,\xi)$ the system

$$\mathrm{D}_t \mathcal{Q}_k(t,s,\xi) = \mathcal{R}_k(t,s,\xi)\mathcal{Q}_k(t,s,\xi), \qquad \mathcal{Q}_k(s,s,\xi) = \mathrm{I}$$

with $\mathcal{R}_k(t,s,\xi) = \widetilde{\mathcal{E}}_k(s,t,\xi)R_k(t,\xi)\widetilde{\mathcal{E}}_k(t,s,\xi)$. This system can be solved by means of the Peano-Baker series

$$\mathcal{Q}_k(t,s,\xi) = \mathrm{I} + \sum_{j=1}^\infty \mathrm{i}^j \int_s^t \mathcal{R}_k(t_1,s,\xi)\int_s^{t_2}\mathcal{R}_k(t_2,s,\xi) \cdots \int_s^{t_{j-1}} \mathcal{R}_k(t_j,s,\xi)\mathrm{d}t_j \cdots \mathrm{d}t_2\mathrm{d}t_1. \tag{13}$$

Using (12) it follows that $\mathcal{R}_k(t,s,\xi)$ satisfies uniform in s the same bounds as $R_k(t,\xi)$ and hence for $k \geq 1$ all integrands are uniformly integrable over the hyperbolic zone. This implies that $\mathcal{Q}_k(t,s,\xi)$ is uniformly bounded,

$$\|\mathcal{Q}_k(t,s,\xi)\| \lesssim \exp\left(\int_s^t R_k(\tau,\xi)\mathrm{d}\tau\right) \lesssim 1,$$

and converges locally uniform in $(s,\xi) \in Z_{hyp}(N)$ to a limit $\mathcal{Q}_k(\infty,s,\xi)$. Furthermore by Liouville theorem,

$$\det \mathcal{Q}_k(t,s,\xi) = \exp\left(\int_s^t \operatorname{trace} R_k(\tau,\xi)\mathrm{d}\tau\right) \simeq 1,$$

and all matrices $\mathcal{Q}_k(t,s,\xi)$ are uniformly invertible over the $Z_{hyp}(N)$.

It remains to obtain symbol type estimates for derivatives of $\mathcal{Q}_k(t,s,\xi)$ with respect to ξ. They are achieved by differentiating (13) term by term using the symbol estimate of $R_k(t,\xi) \in \mathcal{S}_N^{k-1,k-1}\{-k,k+1\}$ in combination with

$$\widetilde{\mathcal{E}}_k(t,s,\xi)R_k(t,\xi)\widetilde{\mathcal{E}}_k(t,s,\xi) \in \mathcal{S}_N^{k-1,k-1}\{-1,2\} \quad \text{uniform in } s$$

and $|\mathrm{D}_\xi^\alpha t_\xi| \le C_\alpha |\xi|^{-1-|\alpha|}$. See [3], [11] or [10] for a more detailed argument. □

REMARK 3. The benefit of applying k steps of diagonalisation is that we obtain symbol type estimates for $k-1$ derivatives of the amplitude $\mathcal{Q}_k(t,s,\xi)$ (provided that we assume sufficient smoothness of $A(t,\xi)$ in t and ξ). If we are satisfied with uniform bounds—which are enough to prove energy estimates—, one step of diagonalisation (i.e., $k=1$ and $(\mathrm{A1})_{0,2}$) is enough.

The following theorem clarifies the rôle of assumption (A3), provided we have knowledge about arbitrary many derivatives.

THEOREM 2. *Assume $(A1)_{0,\infty}$ and (A2). Then assumption (A3) is equivalent to the existence of constants c and C such that*

$$c\|V\| \le \|\mathcal{E}(t,s,\xi)V\| \le C\|V\|, \qquad V \in \mathbb{C}^m,$$

holds true uniformly in $(t,\xi),(s,\xi) \in Z_{hyp}(N)$ for a sufficiently big N.

Sketch of proof. Theorem 1 gives the uniform bound under (A3). Without (A3) equation (12) has to be replaced by a polynomial bound

$$\|\widetilde{\mathcal{E}}_k(t,s,\xi)\|, \|\widetilde{\mathcal{E}}_k(s,t,\xi)\| \le C_k \left(\frac{1+t}{1+s}\right)^K, \qquad t \ge s,$$

where the constant K is independent of k. Similarly, we obtain with the same exponent

$$\|\mathcal{E}_k(t,s,\xi)\| \le \exp\left(\int_s^t \|\mathrm{Im}\,(F_{k-1}(\tau,\xi)+R_k(\tau,\xi))\|\mathrm{d}\tau\right) \le C_k' \left(\frac{1+t}{1+s}\right)^K$$

for all $t \ge s$. Choosing k big enough, the polynomial decay of the remainder $R_k(t,\xi)$ becomes strong enough to compensate all increasing terms and we obtain

$$\mathcal{E}_k(t,s,\xi) = \widetilde{\mathcal{E}}_k(t,s,\xi)\mathcal{Z}_k(s,\xi) - \mathrm{i}\int_t^\infty \widetilde{\mathcal{E}}_k(t,\theta,\xi)R_k(\theta,\xi)\mathcal{E}_k(\theta,s,\xi)\mathrm{d}\theta \tag{14}$$

with

$$\mathcal{Z}_k(s,\xi) = \mathrm{I} + \mathrm{i}\int_s^\infty \widetilde{\mathcal{E}}_k(t,\theta,\xi)R_k(\theta,\xi)\mathcal{E}_k(\theta,s,\xi)\mathrm{d}\theta \lesssim 1.$$

The integral in (14) is bounded by $(1+s)^{K-1}(1+t)^{-K}$, while the first term has the lower bound $(1+s)^K(1+t)^{-K}$. Chosing s big enough implies that $\mathcal{E}_k(t,s,\xi)$ is a small perturbation of $\widetilde{\mathcal{E}}_k(t,s,\xi)$.

Assume now that (A3) is violated. Then we find sequences $t_\mu \to \infty$, s_μ, and ξ_μ such that one matrix entry of the integral in (5) tends to either ∞ or $-\infty$. We consider the $+\infty$ case, and assume w.l.o.g. that $s_\mu > s$ for sufficiently big s and that the matrix entry corresponds to the first diagonal element. Then $\widetilde{\mathcal{E}}_k(t_\mu, s_\mu, \xi_\mu)e_1 \to \infty$ and therefore also $\mathcal{E}(t_\mu, s_\mu, \xi_\mu)N_k(s_\mu, \xi_\mu)M(s_\mu, \xi_\mu)e_1 \to \infty$ which contradicts to the uniform upper bound. Similarly, the $-\infty$ case contradicts to the lower bound and the statement is proven. □

The estimate in the pseudo-differential zone is based on (A4).

LEMMA 3. *Assume (A4). Then the fundamental solution to* (6) *satisfies*

$$\|\mathcal{E}(t,0,\xi)\| \lesssim 1$$

uniform in $(t,\xi) \in Z_{pd}(N)$.

Proof. We fix ξ. Let $V(t)$ be the solution to $\mathrm{D}_t V = A(t,\xi)V$, $V(0) = V_0$. Then with $(\cdot,\cdot)$ the Euclidean inner product on $\mathbb{C}^m$ we obtain from (A4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|^2 = -2(\operatorname{Im} AV, V) \le 2c|\xi|\|V(t)\|^2$$

for all t with $(t,\xi) \in Z_{pd}(N)$. Hence, by applying Gronwall inequality we obtain

$$\|V(t)\|^2 \le C\|V_0\|^2 \exp(2ct|\xi|) \lesssim \|V_0\|^2.$$

□

Symbol-like estimates for derivatives follow by an inductive argument as used in [3], [11] or [10].

LEMMA 4. *Assume (A1)*$_{\ell_1,\ell_2}$*, (A4). Then the estimate*

$$\|\mathrm{D}_\xi^\alpha \mathcal{E}(t_\xi, 0, \xi)\| \le C|\xi|^{-|\alpha|}, \qquad |\xi| \le N$$

holds true for any $|\alpha| \le \min(\ell_1, \ell_2 + 1)$.

3. Generalised energy conservation

The results of the previous section with $k = 1$ allow to conclude upper and lower bounds for the energy. We only state the result.

THEOREM 3. *Assume (A1)*$_{0,2}$*—(A4). Then the solution* $U = U(t,x)$ *of* (1) *satisfies*

$$\|U(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le C\|U_0\|_{L^2(\mathbb{R}^n)}.$$

Furthermore, $\lim_{t\to\infty} \|U(t,\cdot)\|_{L^2(\mathbb{R}^n)} = 0$ *implies* $U_0 = 0$.

4. Dispersive estimates

We want to explain how to use the information derived in Section 2 to derive dispersive estimates for solutions. We note first, that interesting estimates depend only on the hyperbolic zone. Let for this $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function, $\chi(\xi) = 1$ for $|\xi| \le 1$, and denote $\chi_{pd}(t,\xi) = \chi((1+t)|\xi|/N)$ and $\chi_{hyp}(t,\xi) = 1 - \chi_{pd}(t,\xi)$.

LEMMA 5. *Assume (A4). Then solution $U = U(t,x)$ to* (1) *satisfies*

$$\|\mathscr{F}^{-1}[\chi_{pd}(t,\xi)\hat{U}(t,\xi)]\|_{L^\infty(\mathbb{R}^n)} \le C(1+t)^{-n}\|U_0\|_{L^1(\mathbb{R}^n)}$$

localised to the pseudo-differential zone $Z_{pd}(N)$ (for any choice of N).

Proof. Based on $\mathscr{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ and Hölder inequality it is sufficient to estimate $\|\mathcal{E}(t,0,\xi)\chi_{pd}(t,\xi)\|_{L^1(\mathbb{R}^n)} \le \|\mathcal{E}(t,0,\xi)\|_{L^\infty(|\xi|\le\xi_t)}\|\chi_{pd}\|_{L^1(\mathbb{R}^n)}$ and the estimate follows from Lemma 3 and the geometry of the zone. □

This estimate is much stronger than any estimate we could expect for the solution $U(t) = \mathscr{F}^{-1}[\mathcal{E}(t,0,\xi)\mathscr{F}U_0]$ itself. Therefore, we concentrate on the remaining hyperbolic zone. By Theorem 1 we know that solutions are represented as Fourier integrals of a particular form,

$$\mathscr{F}^{-1}[\chi_{hyp}(t,\xi)\hat{U}(t,\xi)] = \sum_{j=1}^{m} \int \mathrm{e}^{\mathrm{i}(x\cdot\xi + t\vartheta_j(t,\xi))} B_j(t,\xi)\hat{U}_0(\xi)\mathrm{d}\xi, \tag{15}$$

where the matrix-valued symbol $B_j(t,\xi)$ contains all contributions from the matrices $\mathcal{Q}_k(t,t_\xi,\xi)$, $\mathcal{E}(t_\xi,0,\xi)$, $N_k(t_\xi,\xi)M(t_\xi,\xi)$, $M^{-1}(t,\xi)N^{-1}(t,\xi)$ and $F_{k-1}(t,\xi)$ and is supported within $Z_{hyp}(N)$. Under $(\mathrm{A1})_{k-1,2k}$–(A4) it satisfies

$$\|\mathrm{D}_\xi^\alpha B_j(t,\xi)\| \le C|\xi|^{-|\alpha|}, \qquad |\alpha| \le k-1,$$

k the number of diagonalisation steps used in the construction. The phase function is real, homogeneous in ξ and given by

$$\vartheta_j(t,\xi) = \frac{1}{t}\int_0^t \lambda_j(\theta,\xi)\mathrm{d}\theta.$$

Fourier integrals of this type can be estimated generalising ideas of Sugimoto, [8], [9]. He introduced for a closed surface Σ two indices

$$\gamma_0(\Sigma) = \sup_{p\in\Sigma}\inf_{\eta\in T_p\Sigma}\gamma(\Sigma;p,\eta), \qquad \gamma(\Sigma) = \sup_{p\in\Sigma}\sup_{\eta\in T_p\Sigma}\gamma(\Sigma;p,\eta),$$

where for any tangent vector η on the surface the number $\gamma(\Sigma;p,\eta)$ denotes the order of contact between the tangent $p+\eta\mathbb{R}$ and $\Sigma\cap(p+\eta\mathbb{R}\oplus \mathrm{N}_p\Sigma)$. We will give two estimates related to the statements of [8], [9], taking into account the improvements of [5].

THEOREM 4. *Let $\Sigma \subset \mathbb{R}^n$ be a smooth closed surface of codimension* 1.

1. *Let $\gamma_0 = \gamma_0(\Sigma)$. Then it holds for all $f \in C^1(\Sigma)$*
$$\left| \int_\Sigma e^{ix\cdot\xi} f(\xi) d\xi \right| \le C \langle x \rangle^{-\frac{1}{\gamma_0}} \|f\|_{C^1}.$$
2. *Assume Σ is convex. Then with $\gamma = \gamma(\Sigma)$ and $r = \lceil (n-1)/\gamma \rceil + 1$ the estimate*
$$\left| \int_\Sigma e^{ix\cdot\xi} f(\xi) d\xi \right| \le C \langle x \rangle^{-\frac{n-1}{\gamma}} \|f\|_{C^r}$$
holds true for all $f \in C^r(\Sigma)$.

REMARK 4. It is enough to have $\Sigma \in C^{\gamma+1}$ in order to prove these statements. The original proof of Sugimoto for part 2, [8], uses real analyticity of the surface Σ, which was improved by [7], [5].

In order to derive dispersive estimates for the expressions in (15), we introduce the t-dependent family of level sets
$$\Sigma_t^{(j)} = \{\xi \in \mathbb{R}^n \mid \vartheta_j(t,\xi) = 1\}.$$
We restrict for the sake of simplicity to the case of convex surfaces. Then our estimates are based on the following assumption:

(B) The surfaces $\Sigma_t^{(j)}$ are strictly convex for all $t \ge t_0$ and converge in C^{γ_j+1} to a surface $\Sigma^{(j)}$ with $\gamma(\Sigma^{(j)}) = \gamma_j$.

THEOREM 5. *Assume (A1)$_{\ell,2k}$–(A4) in combination with (B) and let $\gamma_{\max} = \max_j \gamma(\Sigma^{(j)})$. If $\ell \ge k-1 \ge \frac{n-1}{\gamma_{\max}} + 1$, $\ell \ge \gamma_{\max} + 1$ then the dispersive estimate*
$$\|U(t,\cdot)\|_{L^\infty} \le C(1+t)^{-\frac{n-1}{\gamma_{\max}}(\frac{1}{p}-\frac{1}{q})} \|U_0\|_{H^{r,p}(\mathbb{R}^n)}$$
holds true for any solution $U = U(t,x)$ of (1) *where $p \in [1,2]$, $pq = p+q$ and $r > n(1/p - 1/q)$.*

REMARK 5. The stabilisation assumption (B) can be weakened to a uniformity assumption, in such a sense that for sufficiently big $t \ge t_0$ the constants appearing in the corresponding estimates of Theorem 4 are uniform in t.

REMARK 6. The corresponding result for non-convex surfaces holds true, but gives a much weaker decay rate.

5. Concluding remarks

1 If $A_0(t,\xi)$ is symmetric, the diagonaliser $M(t,\xi)$ can be chosen unitary and therefore $(D_t M^{-1})M$ is self-adjoint. If in addition $A(t,\xi) = A_0(t,\xi)$ is assumed to be homogeneous in ξ assumptions (A3) and (A4) are satisfied.

If we assume that $A(t,\mathrm{D})$ is a differential operator —which is a very restrictive assumption here—, we have a representation $A(t,\xi)=A_0(t,\xi)+A_1(t)$ and (A4) is equivalent to dissipativity, $\mathrm{Im} A_1(t)\geq 0$. If $A_0(t,\xi)$ is symmetric, (A3) reduces to the integrability of $\mathrm{Im}\,\mathrm{diag}\,(M^{-1}(t,\xi)A_1(t)M(t,\xi))\geq 0$.

2 The results apply to hyperbolic equations of higher order. We consider a homogeneous equation of order m,

$$\mathrm{D}_t^m u+\sum_{k=0}^{m-1}\sum_{|\alpha|=m-k} a_{k,\alpha}(t)\mathrm{D}_t^k\mathrm{D}_x^\alpha u=0, \qquad \mathrm{D}_t^k u(0,\cdot)=u_k, \tag{16}$$

with $a_{k,\alpha}\in\mathcal{S}^{*,\ell}\{0,0\}$ and assume uniform strict hyperbolicity. We rewrite it as a system in companion form, its eigenvalues $\lambda_j(t,\xi)$ are given by the (real) characteristic roots associated to (16). Assumption (A4) follows from homogeneity, assumption (A3) is equivalent to

$$\max_{j=1,\dots,m}\sup_{T>0,\omega\in\mathbb{S}^{n-1}}\left|\int_0^T\sum_{k\neq j}\frac{\partial_t\lambda_j(t,\omega)}{\lambda_j(t,\omega)-\lambda_k(t,\omega)}\mathrm{d}t\right|<\infty. \tag{17}$$

This assumption is necessary to have a generalised energy conservation for (16) under the symbol assumption $a_{k,\alpha}\in\mathcal{S}^{*,\infty}\{0,0\}$. In the treatment of [2] the condition $\partial_t a_{k,\alpha}(t)\in L^1(\mathbb{R}_+)$ implies (17).

Equations of higher order with arbitrary lower order terms but constant coefficients were considered in [6] and [7].

3 Most of the considerations transfer to problems bearing fast oscillations in the classification of Reissig-Yagdjian [3], [4]. The only major difference is that the corresponding statement of Theorem 2 is no longer valid.

It is an interesting question whether one can generalise the approach of [1] to higher order equations and larger systems. In this case the estimates for time-derivatives are weakened to an improvement of the form $(1+t)^{-p}$, $p<1$ instead of $p=1$ from Definition 1, but accompanied with a so-called stabilisation condition to treat an extended pseudo-differential zone.

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