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ON LOCAL PROPERTIES OF SOME CLASSES OF INFINITELY DEGENERATE ELLIPTIC DIFFERENTIAL OPERATORS

Abstract. We give necessary and sufficient conditions for local solvability and hypoellipticity of some classes of infinitely degenerate elliptic differential operators.

1. Introduction

We deal with local properties of differential operators

$$G^c_{a,b} = X_2X_1 + i\left(c(x)|x|^{-4} + (a(x)-b(x))|x|^{-3}\right)e^{-\frac{1}{|x|}}\frac{\partial}{\partial y}, \tag{1}$$

where $(x, y) \in \mathbb{R}^2$, $i = \sqrt{-1}$, the functions $a(x), b(x), c(x)$ satisfy

$$\begin{aligned} a(x) &= \left\{ \begin{array}{l} a_+ \in \mathbb{C} \ \text{ if } \ x > 0 \\ a_- \in \mathbb{C} \ \text{ if } \ x < 0 \end{array} \right\} =: a, \\ b(x) &= \left\{ \begin{array}{l} b_+ \in \mathbb{C} \ \text{ if } \ x > 0 \\ b_- \in \mathbb{C} \ \text{ if } \ x < 0 \end{array} \right\} =: b, \\ c(x) &= \left\{ \begin{array}{l} c_+ \in \mathbb{C} \ \text{ if } \ x > 0 \\ c_- \in \mathbb{C} \ \text{ if } \ x < 0 \end{array} \right\} =: c \end{aligned}$$

and

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} - ib(x)\text{sign}(x)|x|^{-2}e^{-\frac{1}{|x|}}\frac{\partial}{\partial y}, \\ X_2 &= \frac{\partial}{\partial x} - ia(x)\text{sign}(x)|x|^{-2}e^{-\frac{1}{|x|}}\frac{\partial}{\partial y}. \end{aligned}$$

We will assume that $Re\, a_+ \cdot Re\, a_- \cdot Re\, b_+ \cdot Re\, b_- \neq 0$. The form (1) is motivated by [1], [2] (see also [3])where the authors studied the hypoellipticity and the local solvability of finitely degenerate differential operators. Hypoellipticity and local solvability of $G^c_{a,b}$ were investigated in [4] when $a_+ = a_- = -1, b_+ = b_- = 1, c_+ = c_-$, or $a_+ = a_- = -1, b_+ = b_- = 1, c_+ + c_- = 2$, and in [5] where $a_+ = a_- = -e^{i\varphi}$, $b_+ = b_- = e^{i\varphi}$, $c_+ = (1-\lambda_+)e^{i\varphi}$, $c_- = (1-\lambda_-)e^{i\varphi}$, where $\varphi \in [0, \pi/2)$, $\lambda_+, \lambda_- \in \mathbb{C}$. For those cases in [1], [2], [4] and [5], we have given another proof of non-hypoellipticity, recently in [6], [7], [8], by constructing explicit non-smooth solutions of homogeneous equations. In [9], by the same method we give the values of $a_+, a_-, b_+, b_-, c_+, c_-$ where $G^c_{a,b}$ is not hypoelliptic. Here we shall use the method of constructing parametrixes. Since $G^c_{a,b}$ is elliptic away from the line $x = 0$, we will consider only the case when $|x| \le 1$. Throughout the paper we denote by C a general positive constant which may vary from place to place. The paper is organized as follows. In section 2 we consider the case $Re\, a_+ < 0$, $Re\, a_- < 0$, $Re\, b_+ > 0$, $Re\, b_- > 0$, which directly generalizes the results of Hoshiro-Yagdjian. In section 3 we investigate the case $Re\, a_+ > 0$, $Re\, a_- > 0$, $Re\, b_+ > 0$, $Re\, b_- > 0$. In section 4 we deal with the case $Re\, a_+ > 0$, $Re\, a_- < 0$, $Re\, b_+ > 0$, $Re\, b_- < 0$. Finally, in section 5 we consider the case $Re\, a_+ < 0$, $Re\, a_- > 0$, $Re\, b_+ > 0$, $Re\, b_- > 0$. The results in sections 3-5 have completely new characteristics comparing with those considered in [4] and [5].

2. The case $Re\, a_+ < 0$, $Re\, a_- < 0$, $Re\, b_+ > 0$, $Re\, b_- > 0$

THEOREM 1. *Assume that $Re\, a_+ < 0$, $Re\, a_- < 0$, $Re\, b_+ > 0$, $Re\, b_- > 0$. Then $G^c_{a,b}$ is not hypoelliptic (nor local solvable) if and only if*

$$\frac{c_+}{a_+ - b_+} = k, \qquad \frac{c_-}{a_- - b_-} = l$$

or

$$\frac{c_+}{a_+ - b_+} = -k - 1, \qquad \frac{c_-}{a_- - b_-} = -l - 1$$

where k and l are non-negative integers.

Proof. I) In this part we prove the hypoellipticity and the local solvability by constructing right and left parametrices. Let us consider the Fouruer transform of u with respect to y

$$\hat{u}(x, \eta) = \int_{-\infty}^{+\infty} e^{-iy\eta} u(x, y) dy.$$

By this transform $G^c_{a,b}$ become

$$\begin{aligned}
\hat{G}^c_{a,b}\left(x, \frac{d}{dx}, \eta\right) &= \left(\frac{d}{dx} + ax^{-2}\mathrm{sign}(x)e^{-\frac{1}{|x|}}\eta\right)\left(\frac{d}{dx} + bx^{-2}\mathrm{sign}(x)e^{-\frac{1}{|x|}}\eta\right) \\
&- (c|x|^{-4} + (a-b)|x|^{-3})e^{-\frac{1}{|x|}}\eta = \\
&= \frac{d^2}{dx^2} + (a+b)x^{-2}\mathrm{sign}(x)e^{-\frac{1}{|x|}}\eta\frac{d}{dx} \\
&+ abx^{-4}e^{-\frac{2}{|x|}}\eta^2 + (b-c)|x|^{-4}e^{-\frac{1}{|x|}}\eta - (a+b)|x|^{-3}e^{-\frac{1}{|x|}}\eta.
\end{aligned}$$

We would like to study solutions of the equation

$$\hat{G}^c_{a,b}\left(x, \frac{d}{dx}, \eta\right)\hat{u}(x, \eta) = 0. \tag{2}$$

Put $\hat{u}(x,\eta) = xe^{-be^{-\frac{1}{|x|}}\eta} f(z)$, where $z = (b-a)e^{-\frac{1}{|x|}}\eta$. Then we will have the following confluent hypergeometric equation for $f(z)$

$$z\frac{d^2 f(z)}{dz^2} + (1-z)\frac{df(z)}{dz} - \frac{c}{b-a} f(z) = 0.$$

The equation has two linearly independent solutions

$$f_1(z) = \Psi\left(\frac{c}{b-a}, 1, z\right), \qquad f_2(z) = e^z \Psi\left(1 - \frac{c}{b-a}, 1, -z\right)$$

where $\Psi(\alpha, \gamma, z)$ is the Tricomi function (see [10], p. 255). Therefore we have the following pair of solutions of the equation (2)

$$\text{if } x \geq 0: \qquad \hat{u}_1^+(x,\eta) := xe^{-b_+\eta e^{-\frac{1}{|x|}}} \Psi\left(\frac{c_+}{b_+ - a_+}, 1, (b_+ - a_+)\eta e^{-\frac{1}{|x|}}\right),$$

$$u_2^+(x,\eta) := xe^{-a_+\eta e^{-\frac{1}{|x|}}} \Psi\left(1 - \frac{c_+}{b_+ - a_+}, 1, (a_+ - b_+)\eta e^{-\frac{1}{|x|}}\right),$$

(3)

$$\text{if } x \leq 0: \qquad \hat{u}_1^-(x,\eta) := xe^{-b_-\eta e^{-\frac{1}{|x|}}} \Psi\left(\frac{c_-}{b_- - a_-}, 1, (b_- - a_-)\eta e^{-\frac{1}{|x|}}\right),$$

$$\hat{u}_2^-(x,\eta) := xe^{-a_-\eta e^{-\frac{1}{|x|}}} \Psi\left(1 - \frac{c_-}{b_- - a_-}, 1, (a_- - b_-)\eta e^{-\frac{1}{|x|}}\right).$$

As $x \to 0$ we have the following asymptotics for $\hat{u}_1^{\pm}(x,\eta)$, $\hat{u}_2^{\pm}(x,\eta)$

$$\hat{u}_1^+(x,\eta) \approx \frac{1 - x\,(\log[(b_+ - a_+)\eta] + \Phi(c_+/(b_+ - a_+)) - 2\Phi(1))}{\Gamma\,(c_+/(b_+ - a_+))},$$

$$\hat{u}_2^+(x,\eta) \approx \frac{1 - x\,(\log[(a_+ - b_+)\eta] + \Phi(1 - c_+/(b_+ - a_+)) - 2\Phi(1))}{\Gamma\,(1 - c_+/(b_+ - a_+))},$$

(4)

$$\hat{u}_1^-(x,\eta) \approx \frac{-1 - x\,(\log[(b_- - a_-)\eta] + \Phi(c_-/(b_- - a_-)) - 2\Phi(1))}{\Gamma\,(c_-/(b_- - a_-))},$$

$$\hat{u}_2^-(x,\eta) \approx \frac{-1 - x\,(\log[(a_- - b_-)\eta] + \Phi(1 - c_-/(b_- - a_-)) - 2\Phi(1))}{\Gamma\,(1 - c_-/(b_- - a_-))},$$

where $\Phi(z)$ is the Gauss polygramma function ($\Phi(z) = \Gamma(z)/\,\Gamma'(z)$). Define $D^+(x,\eta)$ (respectively $D^-(x,\eta)$) as the Wronskian of $\hat{u}_1^+(x,\eta)$, $\hat{u}_2^+(x,\eta)$ (resp. $\hat{u}_1^-(x,\eta)$, $\hat{u}_2^-(x,\eta)$). Similarly (as in [4], Proposition 2) it is not difficult to see that

$$D^+(0,\eta) \quad = \quad \begin{cases} -\left(\operatorname{ctg}\frac{\pi c_+}{b_+ - a_+} - i\right)\sin\frac{\pi c_+}{b_+ - a_+} & \text{if } \arg(b_+ - a_+)\eta \in (0,\pi] \\ -\left(\operatorname{ctg}\frac{\pi c_+}{b_+ - a_+} + i\right)\sin\frac{\pi c_+}{b_+ - a_+} & \text{if } \arg(b_+ - a_+)\eta \in (-\pi,0] \end{cases}$$

$$D^-(0,\eta) \quad = \quad \begin{cases} -\left(\operatorname{ctg}\frac{\pi c_-}{b_- - a_-} - i\right)\sin\frac{\pi c_-}{b_- - a_-} & \text{if } \arg(b_- - a_-)\eta \in (0,\pi] \\ -\left(\operatorname{ctg}\frac{\pi c_-}{b_- - a_-} + i\right)\sin\frac{\pi c_-}{b_- - a_-} & \text{if } \arg(b_- - a_-)\eta \in (-\pi,0]. \end{cases}$$

Note that $D^+(0,\eta)\cdot D^-(0,\eta)\neq 0$ when $\eta\neq 0$. Therefor $\hat{u}_1^+(x,\eta), \hat{u}_2^+(x,\eta)$ are linearly independent solutions of (2) when $x\geq 0,\ \eta\neq 0$, and $\hat{u}_1^-(x,\eta), \hat{u}_2^-(x,\eta)$ are linearly independent solutions of (2) when $x\leq 0,\ \eta\neq 0$. Next for $\eta>0$ we define the following pair of solutions of (2)

$$u^+(x,\eta)=\begin{cases}\hat{u}_1^+(x,\eta)\ \text{ if }\ x\geq 0\\ c_{1,-}^{u^+}(\eta)\hat{u}_1^-(x,\eta)+c_{2,-}^{u^+}(\eta)\hat{u}_2^-(x,\eta)\ \text{ if }\ x\leq 0\end{cases}\tag{5}$$

$$v^+(x,\eta)=\begin{cases}c_{1,+}^{v^+}(\eta)\hat{u}_1^+(x,\eta)+c_{2,+}^{v^+}(\eta)\hat{u}_2^+(x,\eta)\ \text{ if }\ x\geq 0\\ \hat{u}_1^-(x,\eta)\ \text{ if }\ x\leq 0\end{cases}\tag{6}$$

where coefficients $c_{1,-}^{u^+}, c_{2,-}^{u^+}, c_{1,+}^{v^+}, c_{2,+}^{v^+}$ are chosen such that $u^+(x,\eta), v^+(x,\eta)$ are continously differentiable at $x=0$

$$\begin{aligned}c_{1,-}^{u^+}(\eta)\ &=\ \frac{-\log\left[(a_--b_-)\eta\right]-\log\left[(b_+-a_+)\eta\right]-\Phi\left(1-\frac{c_-}{b_--a_-}\right)}{\Gamma\left(\frac{c_+}{b_+-a_+}\right)\Gamma\left(1-\frac{c_-}{b_--a_-}\right)D^-(0,\eta)}\\ &+\ \frac{-\Phi\left(\frac{c_+}{b_+-a_+}\right)+4\Phi(1)}{\Gamma\left(\frac{c_+}{b_+-a_+}\right)\Gamma\left(1-\frac{c_-}{b_--a_-}\right)D^-(0,\eta)},\end{aligned}$$

$$\begin{aligned}c_{2,-}^{u^+}(\eta)\ &=\ \frac{\log\left[(b_+-a_+)\eta\right]+\log\left[(b_--a_-)\eta\right]+\Phi\left(\frac{c_+}{b_+-a_+}\right)}{\Gamma\left(\frac{c_-}{b_--a_-}\right)\Gamma\left(\frac{c_+}{b_+-a_+}\right)D^-(0,\eta)}\\ &+\ \frac{\Phi\left(\frac{c_-}{b_--a_-}\right)-4\Phi(1)}{\Gamma\left(\frac{c_-}{b_--a_-}\right)\Gamma\left(\frac{c_+}{b_+-a_+}\right)D^-(0,\eta)},\end{aligned}$$

$$\begin{aligned}c_{1,+}^{v^+}(\eta)\ &=\ \frac{\log\left[(a_+-b_+)\eta\right]+\log\left[(b_--a_-)\eta\right]+\Phi\left(1-\frac{c_+}{b_+-a_+}\right)}{\Gamma\left(\frac{c_-}{b_--a_-}\right)\Gamma\left(1-\frac{c_+}{b_+-a_+}\right)D^+(0,\eta)}\\ &+\ \frac{\Phi\left(\frac{c_-}{b_--a_-}\right)-4\Phi(1)}{\Gamma\left(\frac{c_-}{b_--a_-}\right)\Gamma\left(1-\frac{c_+}{b_+-a_+}\right)D^+(0,\eta)},\end{aligned}$$

$$\begin{aligned}c_{2,+}^{v^+}(\eta)\ &=\ \frac{-\log\left[(b_+-a_+)\eta\right]-\log\left[(b_--a_-)\eta\right]-\Phi\left(\frac{c_+}{b_+-a_+}\right)}{\Gamma\left(\frac{c_-}{b_--a_-}\right)\Gamma\left(\frac{c_+}{b_+-a_+}\right)D^+(0,\eta)}\\ &+\ \frac{-\Phi\left(\frac{c_-}{b_--a_-}\right)+4\Phi(1)}{\Gamma\left(\frac{c_-}{b_--a_-}\right)\Gamma\left(\frac{c_+}{b_+-a_+}\right)D^+(0,\eta)}.\end{aligned}$$

For $\eta < 0$ we define the following pair of solutions of (2)

$$u^-(x,\eta) = \begin{cases} \hat{u}_2^+(x,\eta) \ \text{if} \ x \geq 0 \\ c_{1,-}^{u^-}(\eta)\hat{u}_1^-(x,\eta) + c_{2,-}^{u^-}(\eta)\hat{u}_2^-(x,\eta) \ \text{if} \ x \leq 0 \end{cases} \tag{7}$$

$$v^-(x,\eta) = \begin{cases} c_{1,+}^{v^-}(\eta)\hat{u}_1^+(x,\eta) + c_{2,+}^{v^-}(\eta)\hat{u}_2^+(x,\eta) \ \text{if} \ x \geq 0 \\ \hat{u}_2^-(x,\eta) \ \text{if} \ x \leq 0 \end{cases} \tag{8}$$

where the coefficients $c_{1,-}^{u^-}, c_{2,-}^{u^-}, c_{1,+}^{v^-}, c_{2,+}^{v^-}$ are chosen such that $u^-(x,\eta), v^-(x,\eta)$ are continously differentiable at $x = 0$

$$\begin{aligned} c_{1,-}^{u^-}(\eta) &= \frac{-\log\left[(a_- - b_-)\eta\right] - \log\left[(a_+ - b_+)\eta\right] - \Phi\left(1 - \frac{c_-}{b_- - a_-}\right)}{\Gamma\left(1 - \frac{c_+}{b_+ - a_+}\right)\Gamma\left(1 - \frac{c_-}{b_- - a_-}\right)D^-(0,\eta)} \\ &+ \frac{-\Phi\left(1 - \frac{c_+}{b_+ - a_+}\right) + 4\Phi(1)}{\Gamma\left(1 - \frac{c_+}{b_+ - a_+}\right)\Gamma\left(1 - \frac{c_-}{b_- - a_-}\right)D^-(0,\eta)}, \end{aligned}$$

$$\begin{aligned} c_{2,-}^{u^-}(\eta) &= \frac{\log\left[(a_+ - b_+)\eta\right] + \log\left[(b_- - a_-)\eta\right] + \Phi\left(1 - \frac{c_+}{b_+ - a_+}\right)}{\Gamma\left(\frac{c_-}{b_- - a_-}\right)\Gamma\left(1 - \frac{c_+}{b_+ - a_+}\right)D^-(0,\eta)} \\ &+ \frac{\Phi\left(\frac{c_-}{b_- - a_-}\right) - 4\Phi(1)}{\Gamma\left(\frac{c_-}{b_- - a_-}\right)\Gamma\left(1 - \frac{c_+}{b_+ - a_+}\right)D^-(0,\eta)}, \end{aligned}$$

$$\begin{aligned} c_{1,+}^{v^-}(\eta) &= \frac{\log\left[(a_+ - b_+)\eta\right] + \log\left[(a_- - b_-)\eta\right] + \Phi\left(1 - \frac{c_+}{b_+ - a_+}\right)}{\Gamma\left(1 - \frac{c_-}{b_- - a_-}\right)\Gamma\left(1 - \frac{c_+}{b_+ - a_+}\right)D^+(0,\eta)} \\ &+ \frac{\Phi\left(1 - \frac{c_-}{b_- - a_-}\right) - 4\Phi(1)}{\Gamma\left(1 - \frac{c_-}{b_- - a_-}\right)\Gamma\left(1 - \frac{c_+}{b_+ - a_+}\right)D^+(0,\eta)}, \end{aligned}$$

$$\begin{aligned} c_{2,+}^{v^-}(\eta) &= \frac{-\log\left[(b_+ - a_+)\eta\right] - \log\left[(a_- - b_-)\eta\right] - \Phi\left(\frac{c_+}{b_+ - a_+}\right)}{\Gamma\left(1 - \frac{c_-}{b_- - a_-}\right)\Gamma\left(\frac{c_+}{b_+ - a_+}\right)D^+(0,\eta)} \\ &+ \frac{-\Phi\left(1 - \frac{c_-}{b_- - a_-}\right) + 4\Phi(1)}{\Gamma\left(1 - \frac{c_-}{b_- - a_-}\right)\Gamma\left(\frac{c_+}{b_+ - a_+}\right)D^+(0,\eta)}. \end{aligned}$$

Put $W^{\pm}(0,\eta) = u^{\pm}(0,\eta)v_x^{\pm}(0,\eta) - v^{\pm}(0,\eta)u_x^{\pm}(0,\eta)$. From (5), (6), (7), (8) we deduce that

$$W^{+}(0,\eta) = c_{2,+}^{v^+}(\eta)D^{+}(0,\eta),$$
$$W^{-}(0,\eta) = c_{1,-}^{u^-}(\eta)D^{-}(0,\eta).$$

We see that $W^{\pm}(0,\eta) = 0$ for $|\eta| > C$ if and only if $\frac{c_+}{a_+-b_+} = k$, $\frac{c_-}{a_--b_-} = l$ or $\frac{c_+}{a_+-b_+} = -k-1$, $\frac{c_-}{a_--b_-} = -l-1$ where k and l are non-negative integers. Hence $u^{\pm}(x,\eta)$, $v^{\pm}(x,\eta)$ are two linearly independent solutions of (2) for $|\eta| > C$ if and only if $\frac{c_+}{a_+-b_+} \neq k$, $\frac{c_-}{a_--b_-} \neq l$ or $\frac{c_+}{a_+-b_+} \neq -k-1$, $\frac{c_-}{a_--b_-} \neq -l-1$ where k and l are non-negative integers. Now if we denote the Wronskian of $u^{\pm}(x,\eta)$, $v^{\pm}(x,\eta)$ by $W^{\pm}(x,\eta)$ then by the Liouville theorem we have

$$W^{\pm}(x,\eta) = W^{\pm}(0,\eta)\exp\left[-\int_0^x p(s,\eta)ds\right],$$

where

$$p(s,\eta) = \begin{cases} (a_+ + b_+)x^{-2}sign(x)e^{-1/|x|}\eta \text{ if } x \geq 0, \\ (a_- + b_-)x^{-2}sign(x)e^{-1/|x|}\eta \text{ if } x \leq 0. \end{cases}$$

Hence

$$W^{+}(x,\eta) = \begin{cases} W^{+}(0,\eta)e^{-\eta(a_++b_+)e^{-1/|x|}} = c_{2,+}^{v^+}(\eta)D^{+}(0,\eta)e^{-\eta(a_++b_+)e^{-1/|x|}} \\ \qquad \text{if } x \geq 0, \\ W^{+}(0,\eta)e^{-\eta(a_-+b_-)e^{-1/|x|}} = c_{2,+}^{v^+}(\eta)D^{+}(0,\eta)e^{-\eta(a_-+b_-)e^{-1/|x|}} \\ \qquad \text{if } x \leq 0, \end{cases}$$

$$W^{-}(x,\eta) = \begin{cases} W^{-}(0,\eta)e^{-\eta(a_++b_+)e^{-1/|x|}} = c_{1,-}^{u^-}(\eta)D^{-}(0,\eta)e^{-\eta(a_++b_+)e^{-1/|x|}} \\ \qquad \text{if } x \geq 0, \\ W^{-}(0,\eta)e^{-\eta(a_-+b_-)e^{-1/|x|}} = c_{1,-}^{u^-}(\eta)D^{-}(0,\eta)e^{-\eta(a_-+b_-)e^{-1/|x|}} \\ \qquad \text{if } x \leq 0. \end{cases}$$

Since $Re\, a_+ < 0$, $Re\, a_- < 0$, $Re\, b_+ > 0$, $Re\, b_- > 0$, if $\frac{c_+}{a_+-b_+} \neq k$, $\frac{c_-}{a_--b_-} \neq l$ or $\frac{c_+}{a_+-b_+} \neq -k-1$, $\frac{c_-}{a_--b_-} \neq -l-1$ where k and l are non-negative integers, then $u^{+}(-1,\eta)$, $v^{+}(1,\eta)$ exponentially increase when $\eta \to +\infty$, and $u^{-}(-1,\eta)$, $v^{-}(1,\eta)$ exponentially increase when $\eta \to -\infty$. Hence we construct the Green function as follows

$$G(x,x',\eta) = \begin{cases} G^{+}(x,x',\eta) \text{ if } \eta \geq C, \\ G^{-}(x,x',\eta) \text{ if } \eta \leq -C, \end{cases}$$

where

$$\text{for } \eta \geq C: \quad G^{+}(x,x',\eta) = \begin{cases} \frac{v^+(x,\eta)u^+(x',\eta)}{W^+(x',\eta)} \text{ if } x \leq x', \\ \frac{v^+(x',\eta)u^+(x,\eta)}{W^+(x',\eta)} \text{ if } x' \leq x, \end{cases}$$

$$\text{for } \eta \leq -C: \quad G^{-}(x,x',\eta) = \begin{cases} \frac{v^-(x,\eta)u^-(x',\eta)}{W^-(x',\eta)} \text{ if } x \leq x', \\ \frac{v^-(x',\eta)u^-(x,\eta)}{W^-(x',\eta)} \text{ if } x' \leq x. \end{cases}$$

By noting that $|c^{v^+}_{2,+}(\eta)/W^+(0,\eta)| < C$ and using the asymptotic behaviors of $\Psi(\alpha,\gamma,z)$ at zero and at infinity we can show that (in the similar way as in [4])

$$\int_{-1}^{1}|G^+(x,x',\eta)|dx \le C\,; \quad \int_{-1}^{1}|G^+(x,x',\eta)|dx' \le C\,,$$

$$\int_{-1}^{1}|G^-(x,x',\eta)|dx \le C\,; \quad \int_{-1}^{1}|G^-(x,x',\eta)|dx' \le C\,,$$

$$\int_{-1}^{1}\Big|\frac{\partial G^+(x,x',\eta)}{\partial x}\Big|dx \le C\,; \quad \int_{-1}^{1}\Big|\frac{\partial G^+(x,x',\eta)}{\partial x}\Big|dx' \le C\,,$$

$$\int_{-1}^{1}\Big|\frac{\partial G^-(x,x',\eta)}{\partial x}\Big|dx \le C\,; \quad \int_{-1}^{1}\Big|\frac{\partial G^-(x,x',\eta)}{\partial x}\Big|dx' \le C.$$

More general, for an arbitrary natural number n we will have

$$\int_{-1}^{1}|D^n_\eta G^+(x,x',\eta)|dx \le C\eta^{-n}\,; \quad \int_{-1}^{1}|D^n_\eta G^+(x,x',\eta)|dx' \le C\eta^{-n},$$

$$\int_{-1}^{1}|D^n_\eta G^-(x,x',\eta)|dx \le C|\eta|^{-n}\,; \quad \int_{-1}^{1}|D^n_\eta G^-(x,x',\eta)|dx' \le C|\eta|^{-n}.$$

Finally if we define the operator Q

$$Qu(x,y) = \int_{-1}^{1}\int_{-1}^{1}\int_{-\infty}^{\infty} e^{i\eta(y-y')}\phi(\eta)G(x,x',\eta)u(x',y')dy'dx'd\eta,$$

where $\phi(\eta)$ is a cut-off function $\phi(\eta) \in C^\infty(\mathbb{R})$, $\phi(\eta) = 0$ if $|\eta| \le C$, $\phi(\eta) = 1$ if $|\eta| \ge 2C$, then Q will serve a right parametrix for $G^c_{a,b}$, and Q^* will serve a left parametrix for $G^{c*}_{a,b} = -G^{\bar{c}}_{\bar{b},\bar{a}}$. Hence the hypoellipticity and local solvability follow.

II) In this part we will prove the theorem in the non-hypoellipticity and non-local solvable cases. We argue only the cases when $\frac{c_+}{a_+-b_+} = k$, $\frac{c_-}{a_--b_-} = l$, where k, l are non-negative numbers. The other case can be treated similarly. By the theorem of Hörmander if $G^c_{a,b}$ is local solvable at the origin for some its neighborhood ω, there exist constants C, m such that

(9) $$\Big|\int fv\,dxdy\Big| \le C \sup \sum_{\alpha+\beta\le m}|D^\alpha_x D^\beta_y f| \sum_{\alpha+\beta\le m}|D^\alpha_x D^\beta_y G^{c*}_{a,b}v|$$

for all $f, v \in C^\infty_0(\omega)$. Functions which violate this inequality will be constructed. For large λ let $f_\lambda = F(\lambda^2 x, \lambda^2 y)\lambda^5$, where function $F(x,y) \in C^\infty_0(\mathbb{R})$ and

(10) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)dxdy = 1.$$

Next it is easy to see that the function

$$U(x,\eta) = \begin{cases} xe^{-\bar{b}_+\eta e^{-1/|x|}} L_k^0\left((\bar{b}_+ - \bar{a}_+)\eta e^{-1/|x|}\right) & \text{if } x \geq 0, \\ xe^{-\bar{b}_-\eta e^{-1/|x|}} L_l^0\left((\bar{b}_- - \bar{a}_-)\eta e^{-1/|x|}\right) & \text{if } x \leq 0, \end{cases}$$

where $L_k^0(z) = \frac{1}{n!}e^z D_z^n(e^{-z}z^n)$ are the Laguerre polynomials, solve the equation (2). Put

$$v_\lambda = \chi(x)\chi(y)\int_0^\infty g(\lambda\rho)e^{i\lambda^2 y\rho}U(x,\lambda^2\rho)d\rho,$$

where $g \in C_0^\infty(0,\infty),\ \chi \in C_0^\infty(-\infty,\infty), \int g(\rho)d\rho = 1,\ \chi(x) = 1$, when $|x| \leq \epsilon$, with a fixed small enough positive number ϵ. Now we will show that the inequality (9) does not hold for $f = \partial_x f_\lambda$ and $v = v_\lambda$ with the parameter λ large positive enough. Indeed, for λ large enough, $f_\lambda \in C_0^\infty(\omega)$ and

$$-\int\int \frac{\partial f_\lambda}{\partial x} v_\lambda(x,y)dxdy = \int\int \frac{\partial v_\lambda}{\partial x} f_\lambda(x,y)dxdy = A + B, \tag{11}$$

where

$$A = \int\int\int f_\lambda(x,y)\chi(y)\chi_x(x)g(\lambda\rho)U(x,\lambda^2\rho)e^{i\lambda^2 y\rho}d\rho dxdy,$$

$$B = \int\int\int f_\lambda(x,y)\chi(y)\chi(x)g(\lambda\rho)U_x(x,\lambda^2\rho)e^{i\lambda^2 y\rho}d\rho dxdy.$$

It is not difficult to show that $\lim_{\lambda\to\infty} B = 1$ and for every positive number N, there exists a number C_N such that $|A| \leq C_N(1+\lambda)^{-N}$. Next it is easy to check that the function

$$w_\lambda(x,y) = \int_0^\infty g(\lambda\rho)e^{i\lambda^2 y\rho}U(x,\lambda^2\rho)d\rho$$

solves the equation $G_{\bar{b},\bar{a}}^{\bar{c}} w_\lambda(x,y) = 0$. Hence

$$G_{a,b}^{c*}\chi(x)\chi(y)w_\lambda(x,y) = Q(x,y,D_x,D_y)w_\lambda(x,y)$$

where $Q(x,y,D_x,D_y)$ is a first order differential operator with coefficients vanishing in $S_\epsilon = [-\epsilon,\epsilon]\times[-\epsilon,\epsilon]$. Next by similar argument in [2], [5] it is shown that

$$|D_x^\alpha D_y^\beta w_\lambda(x,y)| \leq C_{N,m}(1+\lambda)^{-N}$$

for any $(x,y) \notin S_\epsilon$. Therefore we deduce that

$$\sup \sum_{\alpha+\beta\leq m} |D_x^\alpha D_y^\beta G_{a,b}^{c*} v_\lambda| \leq C_n(1+\lambda)^{-N}. \tag{12}$$

Finally we see that (10), (11), (12) contradict (9).

□

REMARK 1. The following cases $Re\, a_+ > 0$, $Re\, a_- < 0$, $Re\, b_+ < 0$, $Re\, b_- > 0$; $Re\, a_+ > 0$, $Re\, a_- > 0$, $Re\, b_+ < 0$, $Re\, b_- < 0$; $Re\, a_+ < 0$, $Re\, a_- > 0$, $Re\, b_+ > 0$, $Re\, b_- < 0$ can be considered analogously.

3. The case $Re\, a_+ > 0$, $Re\, a_- > 0$, $Re\, b_+ > 0$, $Re\, b_- > 0$

THEOREM 2. *Assume that $Re\, a_+ > 0$, $Re\, a_- > 0$, $Re\, b_+ > 0$, $Re\, b_- > 0$. Then $G^c_{a,b}$ is not hypoelliptic nor local solvable at the origin.*

Proof. We separate the proof into some cases
I) The non-resonance case $a_+ \neq b_+, a_- \neq b_-$. We retain all notations used previously. For $\eta > 0$ we define the following solution of (2) $V(x,\eta) = u^+(x,\eta)$ is defined as in (5). Note that when $x \leq 0$ the solutions $\hat{u}_1^-(x,\eta), \hat{u}_2^-(x,\eta)$ exponentially decrease when $\eta \to +\infty$.
A) If $\frac{c_+}{b_+-a_+} \notin \mathbb{Z}_- \cup 0$ then we set $f = f_\lambda, v = v_\lambda$ as in section 2, with $U(x,\eta)$ replaced by $V(x,\eta)$. B) If $\frac{c_+}{b_+-a_+} \in \mathbb{Z}_- \cup 0$ then we set $f = \partial_x f_\lambda, v = v_\lambda$ as in section 2, with $U(x,\eta)$ replaced by $V(x,\eta)$.
Then in a similar way as in section 2 we can contradict (9) by using f, v with large enough λ.
II) The resonance case $a_+ = b_+, a_- = b_-$. A) When $c_+ \neq 0, c_- \neq 0$ by taking the limit when $a_+ \to b_+, a_- \to b_-$ in (3) it is easy to see that the following pair is solutions of (2) (see [10], p. 266)

$$
\begin{aligned}
\text{when } x \geq 0: \quad & \tilde{u}_1^+(x,\eta) := x e^{-a_+\eta e^{-\frac{1}{|x|}} - 2(c_+\eta e^{-\frac{1}{|x|}})^{\frac12}} \Psi\Big(\tfrac12, 1, 4\big(c_+\eta e^{-\frac{1}{|x|}}\big)^{\frac12}\Big),\\
& \tilde{u}_2^+(x,\eta) := x e^{-a_+\eta e^{-\frac{1}{|x|}} + 2(c_+\eta e^{-\frac{1}{|x|}})^{\frac12}} \Psi\Big(\tfrac12, 1, -4\big(c_+\eta e^{-\frac{1}{|x|}}\big)^{\frac12}\Big),\\
\text{when } x \leq 0: \quad & \tilde{u}_1^-(x,\eta) := x e^{-a_-\eta e^{-\frac{1}{|x|}} - 2(c_-\eta e^{-\frac{1}{|x|}})^{\frac12}} \Psi\Big(\tfrac12, 1, 4\big(c_-\eta e^{-\frac{1}{|x|}}\big)^{\frac12}\Big),\\
& \tilde{u}_2^-(x,\eta) := x e^{-a_-\eta e^{-\frac{1}{|x|}} + 2(c_-\eta e^{-\frac{1}{|x|}})^{\frac12}} \Psi\Big(\tfrac12, 1, -4\big(c_-\eta e^{-\frac{1}{|x|}}\big)^{\frac12}\Big).
\end{aligned}
$$

Next for $\eta > 0$ let us define the following solution of (2)

$$
\tilde{V}(x,\eta) = \begin{cases} \tilde{u}_1^+(x,\eta) \ \text{ if } \ x \geq 0,\\ c^{\tilde{V}}_{1,-}(\eta)\tilde{u}_1^-(x,\eta) + c^{\tilde{V}}_{2,-}(\eta)\tilde{u}_2^-(x,\eta) \ \text{ if } \ x \leq 0, \end{cases}
$$

where $c^{\tilde{V}}_{1,-}(\eta), c^{\tilde{V}}_{2,-}(\eta)$ are chosen as in section 2. Now we can repeat the proof in section 2.
B) When $c_+ \neq 0, c_- = 0$ then we have solutions $\tilde{u}_1^+(x,\eta), \tilde{u}_2^+(x,\eta)$ when $x \geq 0$, and $e^{-a_-\eta e^{\frac{1}{|x|}}}, x e^{-a_-\eta e^{\frac{1}{|x|}}}$ when $x \leq 0$.
C) When $c_+ = 0, c_- \neq 0$ then we have solutions $e^{-a_+\eta e^{\frac{1}{|x|}}}, x e^{-a_+\eta e^{\frac{1}{|x|}}}$ when $x \geq 0$, and $\tilde{u}_1^-(x,\eta), \tilde{u}_2^-(x,\eta)$ when $x \leq 0$.
D) When $c_+ = 0, c_- = 0$ then we have solutions $e^{-a_+\eta e^{\frac{1}{|x|}}}, x e^{-a_+\eta e^{\frac{1}{|x|}}}$ when $x \geq 0$, and $e^{-a_-\eta e^{\frac{1}{|x|}}}, x e^{-a_-\eta e^{\frac{1}{|x|}}}$ when $x \leq 0$.
III) The half-resonance case $a_+ \neq b_+, a_- = b_-$. This case can be treated by using the solutions in part I) when $x \geq 0$, and the solutions in part II) when $x \leq 0$.

IV) The half-resonance case $a_+ = b_+, a_- \neq b_-$. This case can be treated by using the solutions in part II) when $x \geq 0$, and the solutions in part I) when $x \leq 0$.

□

REMARK 2. The following case $Re\, a_+ < 0$, $Re\, a_- < 0$, $Re\, b_+ < 0$, $Re\, b_- < 0$ can be considered analogously.

4. The case $Re\, a_+ > 0$, $Re\, a_- < 0$, $Re\, b_+ > 0$, $Re\, b_- < 0$

THEOREM 3. *Assume that* $Re\, a_+ > 0, Re\, a_- < 0, Re\, b_+ > 0, Re\, b_- < 0$. *Then* $G^c_{a,b}$ *is always hypoelliptic and local solvable at the origin.*

Proof. We consider only the non-resonance case $a_+ \neq b_+, a_- \neq b_-$. The other cases (resonance and half-resonance) can be treated analogously. Next for $\eta > 0$ we define the following pair of solutions of (2)

$$U^+(x,\eta) = \begin{cases} \hat{u}_1^+(x,\eta) \text{ if } x \geq 0, \\ c^{U^+}_{1,-}(\eta)\hat{u}_1^-(x,\eta) + c^{U^+}_{2,-}(\eta)\hat{u}_2^-(x,\eta) \text{ if } x \leq 0, \end{cases} \tag{13}$$

$$V^+(x,\eta) = \begin{cases} \hat{u}_2^+(x,\eta) \text{ if } x \geq 0, \\ c^{V^+}_{1,-}(\eta)\hat{u}_1^-(x,\eta) + c^{V^+}_{2,-}(\eta)\hat{u}_2^-(x,\eta) \text{ if } x \leq 0, \end{cases} \tag{14}$$

where the coefficients $c^{U^+}_{1,-}, c^{U^+}_{2,-}, c^{V^+}_{1,-}, c^{V^+}_{2,-}$ are chosen as in section 2 such that $U^+(x,\eta)$, $V^+(x,\eta)$ are continuously differentiable at $x = 0$.
For $\eta < 0$ we define the following pair of solutions of (2)

$$U^-(x,\eta) = \begin{cases} c^{U^-}_{1,+}(\eta)\hat{u}_1^+(x,\eta) + c^{U^-}_{2,+}(\eta)\hat{u}_2^+(x,\eta) \text{ if } x \geq 0, \\ \hat{u}_1^-(x,\eta) \text{ if } x \leq 0, \end{cases} \tag{15}$$

$$V^-(x,\eta) = \begin{cases} c^{V^-}_{1,+}(\eta)\hat{u}_1^+(x,\eta) + c^{V^-}_{2,+}(\eta)\hat{u}_2^+(x,\eta) \text{ if } x \geq 0, \\ \hat{u}_2^-(x,\eta) \text{ if } x \leq 0, \end{cases} \tag{16}$$

where the coefficients $c^{U^-}_{1,+}, c^{U^-}_{2,+}, c^{V^-}_{1,+}, c^{V^-}_{2,+}$ are chosen such that $U^-(x,\eta)$, $V^-(x,\eta)$ are continuously differentiable at $x = 0$. Since $\mathrm{Re}\, a_+ > 0, \mathrm{Re}\, a_- < 0, \mathrm{Re}\, b_+ > 0, \mathrm{Re}\, b_- < 0$, then $U^+(-1,\eta), V^+(-1,\eta)$ exponentially increase when $\eta \to +\infty$, and $U^-(1,\eta), V^-(1,\eta)$ exponentially increase when $\eta \to -\infty$. Therefore we construct the Green function as follows

$$G(x,x',\eta) = \begin{cases} G^+(x,x',\eta) \text{ if } \eta \geq C, \\ G^-(x,x',\eta) \text{ if } \eta \leq -C, \end{cases}$$

where, for $\eta \geq C$:

$$G^{+}(x,x',\eta)=\begin{cases}\dfrac{V^{+}(x,\eta)U^{+}(x',\eta)-V^{+}(x',\eta)U^{+}(x,\eta)}{D^{+}(x',\eta)} \text{ if } x\leq x',\\ 0 \text{ if } x'\leq x,\end{cases}$$

and for $\eta \leq -C$:

$$G^{-}(x,x',\eta)=\begin{cases}\dfrac{V^{-}(x,\eta)U^{-}(x',\eta)-V^{-}(x',\eta)U^{-}(x,\eta)}{D^{-}(x',\eta)} \text{ if } x\leq x',\\ 0 \text{ if } x'\leq x.\end{cases}$$

Now the rest of the proof goes through as in section 2. We leave the details to the readers.

□

REMARK 3. The following case $\mathrm{Re}\, a_{+} < 0, \mathrm{Re}\, a_{-} > 0, \mathrm{Re}\, b_{+} < 0, \mathrm{Re}\, b_{-} > 0$ can be treated analogously.

5. The case $Re\, a_{+} < 0$, $Re\, a_{-} > 0$, $Re\, b_{+} > 0$, $Re\, b_{-} > 0$

THEOREM 4. *Assume that* $Re\, a_{+} < 0, Re\, a_{-} > 0, Re\, b_{+} > 0, Re\, b_{-} > 0$. *Then* $G^{c}_{a,b}$ *is not hypoelliptic nor local solvable at the origin.*

Proof. We argue only for the non-resonance case $a_{+} \neq b_{+}, a_{-} \neq b_{-}$. The half-resonance case can be treated analogously. Put

$$\tilde{U}(x,\eta)=\begin{cases}\tilde{u}_{1}^{+}(x,\eta) \text{ if } x\geq 0,\\ c^{\tilde{U}}_{1,-}(\eta)\tilde{u}_{1}^{-}(x,\eta)+c^{\tilde{U}}_{2,-}(\eta)\tilde{u}_{2}^{-}(x,\eta) \text{ if } x\leq 0,\end{cases}$$

where $c^{\tilde{U}}_{1,-}(\eta), c^{\tilde{U}}_{2,-}(\eta)$ are chosen as in section 2. Since $\mathrm{Re}\, b_{+} > 0, \mathrm{Re}\, a_{-} > 0, \mathrm{Re}\, b_{-} > 0$ the solution $\tilde{U}(x,\eta)$ exponentially decreases when $\eta \to +\infty$. Now the rest of the proof goes as in section 2.

□

REMARK 4. The seven following cases $\mathrm{Re}\, a_{+} < 0, \mathrm{Re}\, a_{-} < 0, \mathrm{Re}\, b_{+} > 0, \mathrm{Re}\, b_{-} < 0$; $\mathrm{Re}\, a_{+} > 0, \mathrm{Re}\, a_{-} < 0, \mathrm{Re}\, b_{+} > 0, \mathrm{Re}\, b_{-} > 0$; $\mathrm{Re}\, a_{+} > 0, \mathrm{Re}\, a_{-} > 0, \mathrm{Re}\, b_{+} > 0, \mathrm{Re}\, b_{-} < 0$; $\mathrm{Re}\, a_{+} < 0, \mathrm{Re}\, a_{-} < 0, \mathrm{Re}\, b_{+} < 0, \mathrm{Re}\, b_{-} > 0$; $\mathrm{Re}\, a_{+} > 0, \mathrm{Re}\, a_{-} > 0, \mathrm{Re}\, b_{+} < 0, \mathrm{Re}\, b_{-} > 0$; $\mathrm{Re}\, a_{+} > 0, \mathrm{Re}\, a_{-} < 0, \mathrm{Re}\, b_{+} < 0, \mathrm{Re}\, b_{-} < 0$; $\mathrm{Re}\, a_{+} < 0, \mathrm{Re}\, a_{-} > 0, \mathrm{Re}\, b_{+} < 0, \mathrm{Re}\, b_{-} < 0$; can be considered analogously.

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